Spectral Theory of PDEs VT 2022

1. problem sheet. Each problem gives max. 4 points.
Submission via e-mail before 25 February, 1pm.

Problem 1
Compute the eigenvalues of $-\Delta$ with Dirichlet boundary conditions on the triangle in $\mathbb{R}^2$ with corners at the origin, in $(\pi, 0)$ and in $(0, \pi)$.

RECIPE: Argue that each eigenfunction can by (odd) reflection be extended to an eigenfunction on the square $(0, \pi)^2$ and use your knowledge on the eigenfunctions and eigenvalues of the square.

Problem 2
Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and $B : \mathcal{H} \to \mathcal{H}$ a bounded, linear operator which is compact and self-adjoint. Assume, moreover, that $\{u_j : j \in \mathbb{N}\}$ is an orthonormal basis of $\mathcal{H}$ and γ_j are real numbers such that

$$Bu_j = \gamma_j u_j, \qquad j \in \mathbb{N}.$$

Prove directly, without referring to the spectral theorem for compact, self-adjoint operators, that the sequence $\{\gamma_j\}$ converges to zero.

Problem 3
Let $\Omega \subset \mathbb{R}^d$ be a domain (= open, connected set). Prove the following statements:

(i) $H^1(\Omega)$ is complete. HINT: Use the completeness of $L^2(\Omega)$.

(ii) $H^1_0(\Omega)$ is dense in $L^2(\Omega)$ (and so is $H^1(\Omega)$).

Problem 4

(i) Find a function $f \in L^2(-1, 1)$ such that $f \notin H^1(-1, 1)$ (and verify that this is true).

(ii) Let $g \in L^2(a, b)$, where $-\infty < a < b < +\infty$, and let

$$f(x) := \int_a^x g(t)\mathrm{d}t, \qquad x \in (a, b).$$

Show that f belongs to $H^1(a, b)$ and $f' = g$ in the sense of weak derivatives.

(iii) Conversely, let $f \in H^1(a, b)$. Show that there exists a constant c such that $f(x) = c + \int_a^x f'(t)\mathrm{d}t$ for almost all $x \in (a, b)$.

Problem 5
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with sufficiently regular boundary, e.g. C^1-smooth. For $\sigma \in \mathbb{R}$, $\sigma > 0$, consider the Laplacian eigenvalue problem $-\Delta u = \xi u$ with a *Robin boundary condition*

$$\frac{\partial u}{\partial \nu} + \sigma u = 0 \qquad \text{on } \partial\Omega.$$

Show that this problem implies[1]

$$\int_\Omega \nabla u \cdot \overline{\nabla v}\mathrm{d}x + \sigma \int_{\partial\Omega} u\overline{v}\mathrm{d}S = \xi \int_\Omega u\overline{v}\mathrm{d}x \qquad \text{for all } v \in H^1(\Omega).$$

(Here $\mathrm{d}S$ indicates integration with respect to the natural surface measure on $\partial\Omega$ and u and v in the boundary integral refer to the traces of u and v, respectively.) Moreover, use the discrete spectral theorem to show that the above problem has an infinite sequence of positive eigenvalues converging to $+\infty$ and a corresponding orthonormal basis of eigenfunctions.

[1] They are actually equivalent.