Lecture 8: The Laplacian on $\mathbb{R}^d$

Sources: ·) Weidmann: Linear operators in Hilbert spaces, Chap. 4/5/10

·) Complementary reading: Langesen-notes, Chap. 15/17/18

Goal: spectral properties of $-\Delta$ on $\mathbb{R}^d$

Crash course I: Unbounded self-adjoint operators

On an complex Hilbert space, consider a linear operator

$$T: \mathcal{H} \supset \underbrace{\mathrm{dom}\, T}_{\substack{\text{domain of } T,\\ \text{not necess.} = \mathcal{H}}} \to \mathcal{H}$$

E.g. $\mathcal{H} = L^2(\mathbb{R}^d)$, $\mathrm{dom}\, T = H^2(\mathbb{R}^d)$, $Tu = -\Delta u$

·) T closed $:\Leftrightarrow$ its graph $\{(f, Tf) : f \in \mathrm{dom}\, T\}$ closed subspace of $\mathcal{H} \times \mathcal{H}$.

$\left(\|(u,v)\|_{\mathcal{X}\times\mathcal{X}} := \left(\|u\|_{\mathcal{X}}^2 + \|v\|_{\mathcal{X}}^2 \right)^{1/2} . \right)$

·) <u>Closed graph thm</u> : Any two of the following imply the third:

- $\operatorname{dom} T$ closed in $\mathcal{X}$
- T closed op.
- T bdd., i.e. $\exists\, C>0 : \|Tf\| \le C\|f\| \ \forall f \in \operatorname{dom} T$

·) Resolvent set of T closed: $\rho(T) = \{ \lambda \in \mathbb{C} : (T-\lambda)^{-1}$ bdd. lin. op. $\mathcal{X} \to \mathcal{X}$, $\operatorname{dom}(\ )^{-1} = \mathcal{X} \}$

Spectrum of T: $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

·) $\rho(T)$ open, $\sigma(T)$ closed in $\mathbb{C}$, $\lambda \mapsto (T-\lambda)^{-1}$ is an operator valued analytic function. (E.g. $\lambda \mapsto ((T-\lambda)^{-1} f, f)_{\mathcal{X}}$ analytic)

·) Divide $\sigma(T)$ in disjoint sets:

- $\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(T-\lambda) \neq \{0\}\}$ pointspectrum / EV
- $\sigma_c(T) = \{\lambda \in \mathbb{C} : \ker(T-\lambda) = \{0\}, \overline{\operatorname{ran}(T-\lambda)} = \mathcal{H}, \operatorname{ran}(T-\lambda) \neq \mathcal{H}\}$

  continuous spectrum,
- $\sigma_r(T) = \{\lambda \in \mathbb{C} : \ker(T-\lambda) = \{0\}, \overline{\operatorname{ran}(T-\lambda)} \neq \mathcal{H}\}$

  residual spectrum.

•) For $\lambda \in \sigma_c(T)$, $(T-\lambda)^{-1} : \mathcal{H} \supset \operatorname{ran}(T-\lambda) \to \mathcal{H}$ ex. and is unbounded. (By closed graph thm.)

•) Define adjoint T^* by

$$\operatorname{dom} T^* := \{g \in \mathcal{H} : \exists g' \in \mathcal{H} : (Tf, g)_{\mathcal{H}} = (f, g')_{\mathcal{H}} \ \forall f \in \operatorname{dom} T\},$$

$$T^* g := g'$$

(well-defined if $\operatorname{dom} T$ is dense in $\mathcal{H}$: if $(Tf,g)_{\mathcal{H}} = (f,g')_{\mathcal{H}} = (f,g'')_{\mathcal{H}}$ then $(f, g'-g'')_{\mathcal{H}} = 0 \ \forall f \in \operatorname{dom} T \Rightarrow g'-g'' = 0$.)

·) T^* closed even if T not closed.

·) T symmetric $:\Leftrightarrow$ $T \subset T^*$, i.e. $\operatorname{dom} T \subset \operatorname{dom} T^*$; $T^* f = Tf$ $\forall f \in \operatorname{dom} T$. (Consequence: T symm. $\Leftrightarrow$ $(Tf, g)_{\mathcal{H}} = (f, Tg)_{\mathcal{H}}$ $\forall f, g \in \operatorname{dom} T$)

$\Leftrightarrow$ $(Tf, f)_{\mathcal{H}} \in \mathbb{R}$ $\forall f \in \operatorname{dom} T$.

·) T self-adjoint $:\Leftrightarrow$ $T = T^*$ $\Leftrightarrow$ T symmetric plus "maximality condition" $\operatorname{ran}(T-\lambda) = \mathcal{H} = \operatorname{ran}(T-\bar{\lambda})$ $\forall \lambda \in \mathbb{C} \setminus \mathbb{R}$.

·) $T = T^* \Rightarrow \sigma(T) \subset \mathbb{R}$ and $\sigma_r(T) = \emptyset$, i.e.

$$\sigma(T) = \sigma_p(T) \,\dot{\cup}\, \sigma_c(T).$$

Ex. (Multiplication operator) For $f : \mathbb{R}^d \to \mathbb{R}$ continuous let

$$M_f\, g := f g, \quad \operatorname{dom} M_f = \{ g \in L^2(\mathbb{R}^d) : fg \in L^2(\mathbb{R}^d)\}.$$

Claim: (i) $M_f = M_f^*$, (ii) $\sigma(M_f) = \overline{\{f(x) : x \in \mathbb{R}^d\}}$,

(iii) $\sigma_p(M_f) = \{\mu \in \mathbb{R} : \underbrace{|f^{-1}(\{\mu\})|}_{\text{Leb. measure}} > 0\}$

Proof (i) As $C_0^\infty(\mathbb{R}^d) \subset \operatorname{dom} M_f$, M_f^* exists. For $g, h \in \operatorname{dom} M_f$:

$$(M_f g, h)_{L^2} = \int_{\mathbb{R}^d} fg\overline{h}\,dx = \int_{\mathbb{R}^d} g\,\overline{fh}\,dx = (g, fh)_{L^2}$$

$$= (g, M_f h)_{L^2} \quad \Rightarrow M_f \text{ symmetric.}$$

Let $\lambda \in \mathbb{C} \setminus \overline{\{f(x) : x \in \mathbb{R}^d\}}$. Then

·) $M_f - \lambda$ is injective: $0 = (M_f - \lambda) g = (f - \lambda) g \Rightarrow g = 0$ a.e.

·) $M_f - \lambda$ is surjective: Let $h \in L^2(\mathbb{R}^d)$ and $g(x) := \dfrac{h(x)}{f(x) - \lambda}$.

Then $(M_f - \lambda) g = (f - \lambda)\dfrac{h}{f - \lambda} = h$, and

$$\int_{\mathbb{R}^d} |g(x)|^2 dx \leq \underbrace{\frac{1}{\left(\operatorname{dist}(\lambda, \{f(x) : x \in \mathbb{R}^d\})\right)^2}}_{>0} \int_{\mathbb{R}^d} |h(x)|^2 dx < \infty.$$

$\rightsquigarrow g \in L^2(\mathbb{R}^d)$.

$\Rightarrow M_f = M_f^*$ and $\mathbb{C} \setminus \overline{\{f(x) : x \in \mathbb{R}^d\}} \subset \rho(M_f)$.

(ii) Let $\lambda \notin \sigma(M_f)$ and assume $\lambda \in \overline{\{f(x) : x \in \mathbb{R}^d\}}$.

Then $\alpha := \|(M_f - \lambda)^{-1}\|$. If $f \equiv \lambda$ identically then

$\mathbb{1}_{(0,1)^d} \in \ker(M_f - \lambda) \Rightarrow \lambda \in \sigma(M_f)$. ↯

Otherwise there ex. $\Omega \subset \mathbb{R}^d$ bdd open set with $0 < |f(x) - \lambda| < \frac{1}{\alpha}\ \forall x \in \Omega$.

Then $\mathbb{1}_\Omega \in L^2(\mathbb{R}^d)$ and

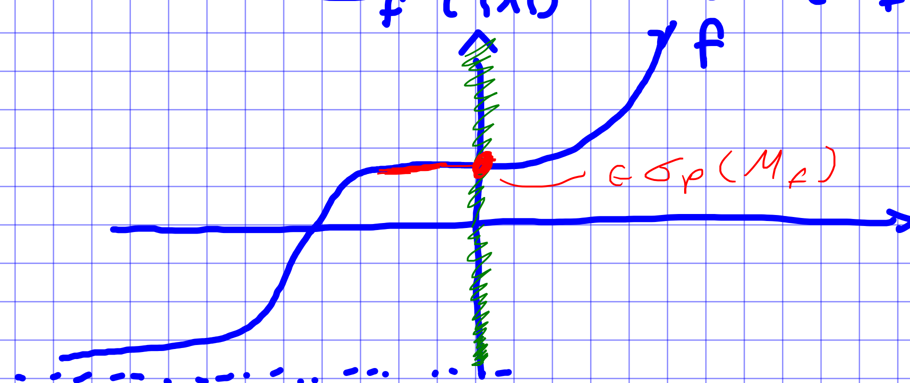
$$\alpha^2 \|\mathbb{1}_\Omega\|^2 \geq \|(M_f - \lambda)^{-1} \mathbb{1}_\Omega\|^2 = \int_\Omega \frac{1}{|f(x)-\lambda|^2}\, dx > \alpha^2 \|\mathbb{1}_\Omega\|^2.$$

↯

$\Rightarrow \sigma(M_f) = \overline{\{f(x) : x \in \mathbb{R}^d\}}$.

(iii) If $|f^{-1}(\{\lambda\})| > 0$ then $\mathbb{1}_{f^{-1}(\{\lambda\})} \in \ker(M_f - \lambda)$.

Converse: exercise.



Crash course II: Fourier transform

- For $u \in L^1(\mathbb{R}^d)$, define $\hat{u}(x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \underbrace{e^{-ix\cdot y}}_{|\cdot|=1} u(y)\,dy$

  Fourier transform, $x \cdot y = x_1 y_1 + \dots + x_d y_d = \langle x, y \rangle$

- Riemann-Lebesgue lemma: $\hat{u} \in C_0(\mathbb{R}^d) := \{ v \in C(\mathbb{R}^d) : \lim_{|x|\to\infty} v(x) = 0 \}$

- Plancherel thm: There ex. a unitary operator $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ s.t. $\mathcal{F}u = \hat{u}$ holds for all $u \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. In part., the Parseval identity

$$(\mathcal{F}u, \mathcal{F}v)_{L^2} = (u, v)_{L^2} \quad \forall\, u, v \in L^2(\mathbb{R}^d)$$

holds. Moreover, $(\mathcal{F}^{-1}u)(x) = (\mathcal{F}u)(-x)$ $\forall x \in \mathbb{R}^d$, $u \in L^2(\mathbb{R}^d)$.

[Reminder: unitary = isometric plus surjective, $\Leftrightarrow \mathcal{F}^{-1} = \mathcal{F}^*$]

[Notation for $\alpha \in \mathbb{N}_0^d$: $|\alpha| = \sum_{j=1}^d \alpha_j$, $x^\alpha := \prod_{j=1}^d x_j^{\alpha_j}$, $D^\alpha := \prod_{j=1}^d \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}}$

E.g. for $d = 5$, $\alpha = (2,1,0,3,0)^T$: $|\alpha| = 6$, $x^\alpha = x_1^2 x_2 x_4^3$, $D^\alpha \varphi = \frac{\partial^6 \varphi}{\partial x_1^2 \partial x_2 \partial x_4^3}$]

Lemma Let $k \in \mathbb{N}_0$ and $u \in H^k(\mathbb{R}^d)$. Then

$$(\mathcal{F}(D^\alpha u))(x) = i^{|\alpha|} x^\alpha (\mathcal{F}u)(x) \quad \text{a.e.},$$

for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$.

Proof: For $u \in C_0^\infty(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}_0^d$:

$$(\mathcal{F}(D^\alpha u))(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot y} (D^\alpha u)(y)\, dy$$

$$\overset{\text{by parts}}{=} (-1)^{|\alpha|} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (-i)^{|\alpha|} x^\alpha e^{-ix\cdot y} u(y)\, dy$$

$$= i^{|\alpha|} x^\alpha (\mathcal{F}u)(x).$$

For $u \in H^k(\mathbb{R}^d)$ by approximation. $\square$

Lemma For $k \in \mathbb{N}_0$, $H^k(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1+|x|^2)^k |(\mathcal{F}u)(x)|^2 dx < \infty \right\}$

Moreover, there ex. $c, C > 0$: $c \|u\|_{H^k}^2 \leq \int_{\mathbb{R}^d} (1+|x|^2)^k |(\mathcal{F}u)(x)|^2 dx \leq C \|u\|_{H^k}^2$ $\forall u \in H^k(\mathbb{R}^d)$.

Proof .) Note: there ex. $c_k, C_k > 0$ s.t.

$$c_k \sum_{|\alpha| \le k} |x^\alpha|^2 \le (1+|x|^2)^k \le C_k \sum_{|\alpha|\le k} |x^\alpha|^2 \quad \forall x \in \mathbb{R}^d. \qquad (*)$$

By Parseval's identity, for $u \in H^k(\mathbb{R}^d)$,

$$\|u\|_{H^k}^2 = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^2}^2 = \sum_{|\alpha|\le k} \|\mathcal{F}(D^\alpha u)\|_{L^2}^2 = \sum_{|\alpha|\le k} \int_{\mathbb{R}^d} |x^\alpha (\mathcal{F}u)(x)|^2 dx$$

$$\overset{(*)}{\ge} \int_{\mathbb{R}^d} (1+|x|^2)^k \, |(\mathcal{F}u)(x)|^2 dx,$$

in part., the last integral is finite. If conversely, $u \in L^2(\mathbb{R}^d)$ satisfies $\int_{\mathbb{R}^d} (1+|x|^2)^k |(\mathcal{F}u)(x)|^2 dx < \infty$, then by $(*)$ the fcts.

$$x \mapsto x^\alpha (\mathcal{F}u)(x) \quad \text{belong to } L^2(\mathbb{R}^d) \text{ for } |\alpha| \le k.$$

In particular, for each $1 \le |\alpha| \le k$, there ex. $u_\alpha \in L^2(\mathbb{R}^d)$ s.t.

$$i^{|\alpha|} x^\alpha (\mathcal{F}u)(x) = (\mathcal{F}u_\alpha)(x) \qquad \text{a.a. } x \in \mathbb{R}^d.$$

For $\varphi \in C_0^\infty(\mathbb{R}^d)$:

$$(-1)^{|\alpha|} \int_{\mathbb{R}^d} u(x) (D^\alpha \varphi)(x)\, dx = (-1)^{|\alpha|} \left(u, \overline{D^\alpha \varphi}\right)_{L^2}$$

$$= (-1)^{|\alpha|} \left(\mathcal{F} u, \mathcal{F} \overline{D^\alpha \varphi}\right)_{L^2} = (-1)^{|\alpha|} \int_{\mathbb{R}^d} (\mathcal{F} u)(x)\, \overline{i^{|\alpha|} x^\alpha (\mathcal{F} \bar{\varphi})}\, dx$$

$$= \left(\mathcal{F} u_\alpha, \mathcal{F} \bar{\varphi}\right)_{L^2} = \left(u_\alpha, \bar{\varphi}\right)_{L^2} = \int_{\mathbb{R}^d} u_\alpha(x)\, \varphi(x)\, dx.$$

$\Rightarrow D^\alpha u = u_\alpha$. $\square$

The Laplacian on $\mathbb{R}^d$

Theorem The Laplacian

$$-\Delta : L^2(\mathbb{R}^d) \supset H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d),$$

$$-\Delta u := -\sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2},$$

is self-adjoint in $L^2(\mathbb{R}^d)$. Its spectrum is $\sigma(-\Delta) = \sigma_c(-\Delta) = [0,\infty)$.

Proof Let $f(x) = 1 + |x|^2$, $x \in \mathbb{R}^d$. We show $I - \Delta = \mathcal{F}^* M_f \mathcal{F}$.

For $u \in H^2(\mathbb{R}^d)$, $x \mapsto (1+|x|^2)(\mathcal{F}u)(x)$ belongs to $L^2(\mathbb{R}^d)$, i.e. $\mathcal{F}u \in \operatorname{dom} M_f$.

$$M_f(\mathcal{F}u)(x) = (1+|x|^2)(\mathcal{F}u)(x) = (\mathcal{F}((1-\Delta)u))(x), \quad x \in \mathbb{R}^d.$$

$$\Rightarrow \quad M_f \mathcal{F} = \mathcal{F}(I - \Delta).$$

As $\mathcal{F}$ is unitary and M_f is self-adjoint, $I - \Delta$ is self.

$\Rightarrow$ $-\Delta$ self-adj.

$$\sigma(-\Delta) = \{ |x|^2 : x \in \mathbb{R}^d \} = [0, \infty),$$

$$\sigma_p(-\Delta) = \{ \mu \in \mathbb{R} : | \underbrace{\tilde{f}^{-1}}_{\tilde{f}(x) = |x|^2}(\{\mu\}) | > 0 \} = \emptyset.$$

$\square$

Note: $-\Delta u = \lambda u$ has solutions, e.g. for $d = 1$: $u(x) = e^{\pm ikx}$, $\lambda = k^2$. But these do not belong to $L^2(\mathbb{R})$. "generalized EF"