Lecture 7: Can one hear the shape of a drum?

Sources:
- M. Kac: "Can one hear the shape of a drum?" (1966)
- W. Arendt, A. F. M. ter Elst, J. Kennedy: "Analytical aspects of isospectral drums" (2014)

Ex. sheet 2, problem 3: $\lambda_{\underline{kN}}(\tilde{\Omega}) \le \lambda_k(\Omega) \;\forall k.$

The sound of a drum

Vibration of a membrane of shape $\Omega \subset \mathbb{R}^2$ described by wave eq.

$$\left\{\begin{array}{lll} \dfrac{\partial^2 u}{\partial t^2} = \Delta_x u & \text{in } \Omega & \\ u(0,x) = f(x) & \multirow{2}{*}{\}\text{ in } \Omega} & \\ \dfrac{\partial u}{\partial t}(0,x) = g(x) & & \\ u = 0 & \text{on } \partial\Omega & \end{array}\right\}$$

Assume $u(t,x) = X(t)\varphi(x)$

Wave eq.: $X''(t)\varphi(x) = X(t)(\Delta\varphi)(x)$

$$\Leftrightarrow \quad \frac{X''(t)}{X(t)} = \frac{(\Delta\varphi)(x)}{\varphi(x)} = -\lambda, \quad \lambda > 0$$

Solving $X''(t) = -\lambda X(t)$ yields $X(t) = c e^{i\sqrt{\lambda} t} + d e^{-i\sqrt{\lambda} t}$.

Let λ_k EV of $-\Delta$ with Dirichlet BC, $\{\varphi_k\}$ ON-basis of corr. EF

$$\Rightarrow u(t,x) = \sum_{k=1}^{\infty} \left(c_k e^{i\sqrt{\lambda_k} t} + d_k e^{-i\sqrt{\lambda_k} t} \right) \varphi_k(x).$$

c_k, d_k determined uniquely by initial cond.,

$$c_k = \frac{1}{2} (f, \varphi_k)_{L^2} - \frac{i}{2\sqrt{\lambda_k}} (g, \varphi_k)_{L^2},$$

$$d_k = \frac{1}{2} (f, \varphi_k)_{L^2} + \frac{i}{2\sqrt{\lambda_k}} (g, \varphi_k)_{L^2}$$

$\rightsquigarrow$ frequency decomposition, λ_1 fundamental tone, $\lambda_2, \lambda_3, \dots$ overtones.

<u>Question</u> (Kac 1966): "Can one hear the shape of a drum?", i.e. does the sequence of eigenvalues of the Dirichlet Laplacian on Ω determine Ω uniquely (up to congruence)?

Theorem Let $\Omega \subset \mathbb{R}^d$ bdd. domain. Then the sequence $\{\lambda_k\}$ of Dirichlet-Laplacian EV determines $|\Omega|$ uniquely.

Proof By Weyl's law, $\lambda_j \sim 4\pi^2 \left(\frac{j}{\omega_d |\Omega|}\right)^{\frac{2}{d}}$, ω_d volume of unit ball in $\mathbb{R}^d$. Hence $\lambda_j^{d/2} \sim (2\pi)^d \frac{j}{\omega_d |\Omega|}$ or

$$\lim_{j\to\infty} (2\pi)^d \frac{j}{\omega_d \lambda_j^{d/2}} = |\Omega|. \qquad \square$$

By expanding Weyl asymptotics further one can see that also the surface measure (perimeter in 2D) of Ω is uniquely determined by the EV.

One cannot hear the shape of a drum

- First counter example: Gordon, Webb, Wolpert, 1992
- We discuss a construction due to Sunada, Bérard and others.

We follow Arendt / ter Elst / Kennedy:

For $j=1,2$, let $\mathcal{K}_j, \mathcal{H}_j$ Hilbert spaces, $\mathcal{K}_j \subset \mathcal{H}_j$ densely, boundedly, compactly embedded, $a_j : \mathcal{K}_j \times \mathcal{K}_j \to \mathbb{C}$ scalar product, bounded, elliptic.

Lemma Let $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ bounded lin., $\underbrace{\text{unitary}}_{\phi^* = \phi^{-1}}$ operator.

The following are equivalent:

(i) $\phi(\mathcal{K}_1) = \mathcal{K}_2$ and $a_2(\phi u, \phi v) = a_1(u,v)$ $\forall u, v \in \mathcal{K}_1$.

(ii) $a_1(u,v) = \lambda (u,v)_{\mathcal{H}_1}$ $\forall v \in \mathcal{K}_1$ if and only if $\phi u \in \mathcal{K}_2$ and

$a_2(\phi u, w) = \lambda (\phi u, w)_{\mathcal{H}_2}$ $\forall w \in \mathcal{K}_2$.

Proof: (i) $\Rightarrow$ (ii): Assume $a_1(u,v) = \lambda (u,v)_{\mathcal{H}_1}$ $\forall v \in \mathcal{K}_1$. By (i), $\phi(u) \in \mathcal{K}_2$ and for all $w \in \mathcal{K}_2$:

$$a_2(\phi u, \underbrace{w}_{=\phi\phi^{-1}w}) = a_1(u, \phi^{-1} w) = \lambda (u, \underbrace{\phi^{-1}}_{=\phi^*} w)_{\mathcal{H}_1} = \lambda (\phi u, w).$$

Other implication in (ii) works analogously.

(iii) $\Rightarrow$ (i): Let $u \in \mathcal{X}_1$. Then $u = \sum_{k=1}^{\infty} c_k u_k$ (convergent in $\mathcal{X}_1$)

for weak eigenvectors u_k of a_1, certain $c_k \in \mathbb{C}$. Hence by continuity

of ϕ,

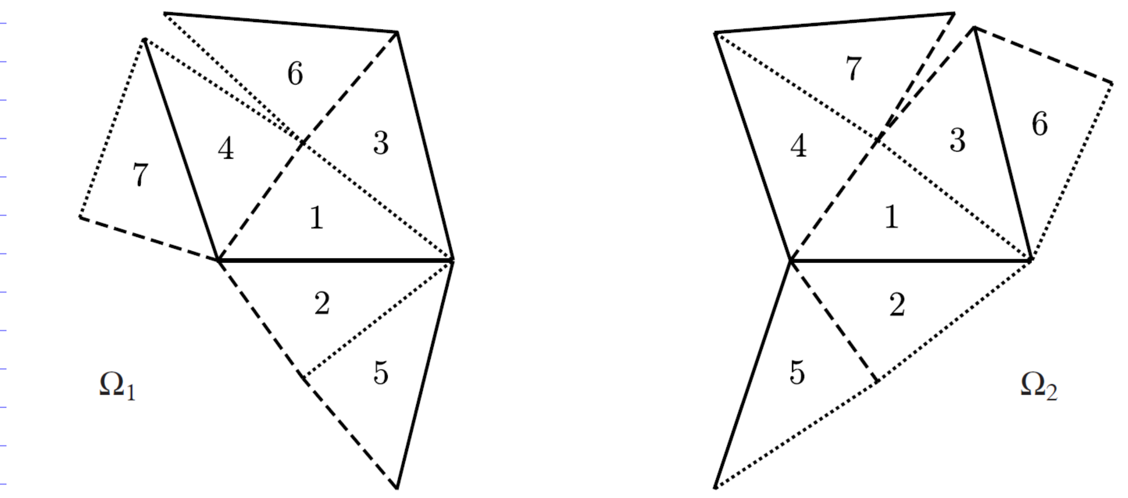
$$\phi u = \sum_{k=1}^{\infty} c_k \underbrace{\phi u_k}_{\in \mathcal{X}_2} \quad \in \mathcal{X}_2$$

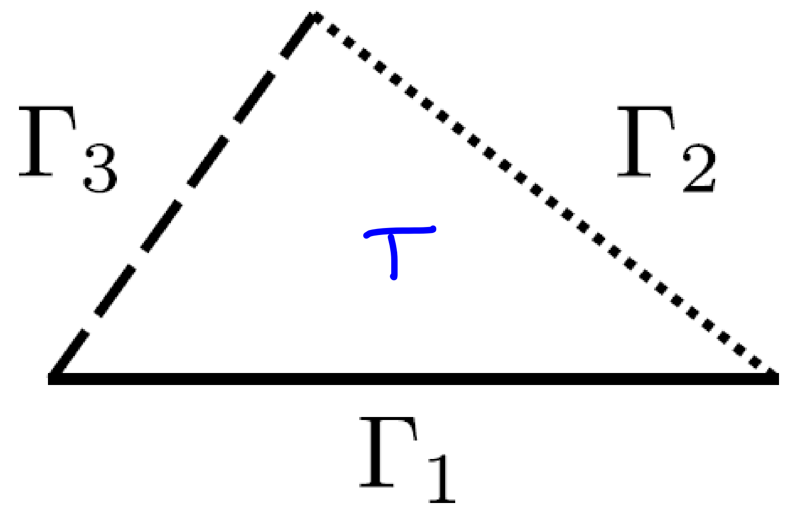
and for all $v \in \mathcal{X}_1$:

$$a_2(\phi u, \phi v) = \sum c_k \, a_2(\phi u_k, \phi v) = \sum c_k \lambda_k (\phi u_k, \phi v)_{\mathcal{X}_2}$$

$$= \sum c_k \lambda_k (u, v)_{\mathcal{X}_1} = \sum c_k \, a_1(u_k, v) = a_1(u, v). \quad \square$$

Short: a_1, a_2 have the same EV (isospectral) if and only if they are intertwined by a unitary operator.





(Pictures from Arendt/Elst/Kennedy).

Let T be a scalene acute triangle, Ω_1, Ω_2 constructed from T by gluing together 7 congruent copies of T in the displayed ways.

Let $T_1, T_2, \ldots, T_7$ the seven disjoint triangles congruent to T forming Ω_1 and $\tau_k : T \to T_k$, $k = 1, \ldots, 7$, the corresponding congruence transformations.

Define $\Phi_1 : L^2(\Omega_1) \to L^2(T)^7$, $\Phi_1(w) := \left(w|_{T_1} \circ \tau_1, \ldots, w|_{T_7} \circ \tau_7 \right)$.

Then Φ_1 is unitary and

$$\mathcal{X}_1 := \Phi_1\left(H_0^1(\Omega_1)\right) = \Big\{ (u_1, \dots, u_7) \in H^1(T)^7 :$$

$$u_1 = u_2 \text{ and } u_4 = u_7 \text{ and } u_3 = u_5 = u_6 = 0 \text{ on } \Gamma_1$$

$$u_1 = u_3 \text{ and } u_2 = u_5 \text{ and } u_4 = u_6 = u_7 = 0 \text{ on } \Gamma_2$$

$$u_1 = u_4 \text{ and } u_3 = u_6 \text{ and } u_2 = u_5 = u_7 = 0 \text{ on } \Gamma_3 \Big\}$$

Define $\tilde{a}_1 : \mathcal{X}_1 \times \mathcal{X}_1 \to \mathbb{C}$,

$$\tilde{a}_1(u,v) = \sum_{k=1}^{7} \int_T \nabla u_k \cdot \overline{\nabla v_k}\, dx .$$

This is a scalar product satisfying the conditions of the discrete spectral thm. Denote by $a_1 : H_0^1(\Omega_1) \times H_0^1(\Omega_1) \to \mathbb{C}$,

$$a_1(u,v) = \int_{\Omega_1} \nabla u \cdot \overline{\nabla v}\, dx .$$

<u>Lemma</u> $\tilde{a}_1(\Phi_1 u, \Phi_1 v) = a_1(u,v) \quad \forall u,v \in H_0^1(\Omega_1)$.

<u>Proof</u>:
$$\tilde{a}_1(\Phi_1 u, \Phi_1 v) = \sum_{k=1}^{7} \int_T \nabla(u \circ \tau_k) \cdot \overline{\nabla (v \circ \tau_k)}\, dx$$

$$= \sum_{k=1}^{7} \int_T (\nabla u) \circ \tau_k \cdot (\overline{\nabla v}) \circ \tau_k \, dx$$

$$= \sum_{k=1}^{7} \int_{T_k} \nabla u \cdot \overline{\nabla v}\, dx = \int_{\Omega_1} \nabla u \cdot \overline{\nabla v}\, dx$$

$$= a_1(u,v). \qquad \square$$

From the two lemmas: a_1 and $\tilde{a}_1$ have same EV.

In the same way for Ω_2: Define $\Phi_2 : L^2(\Omega_2) \to L^2(T)^7$ analogously. Then

$$X_2 := \Phi_2(H_0^1(\Omega_2)) = \{ (u_1, \dots, u_7) \in H^1(T)^7 :$$

$$u_1 = u_2 \text{ and } u_3 = u_6 \text{ and } u_4 = u_5 = u_7 = 0 \text{ on } \Gamma_1,$$

$$u_1 = u_3 \text{ and } u_4 = u_7 \text{ and } u_2 = u_5 = u_6 = 0 \text{ on } \Gamma_2,$$

$$u_1 = u_4 \text{ and } u_2 = u_5 \text{ and } u_3 = u_6 = u_7 = 0 \text{ on } \Gamma_3 \},$$

a_2 quadr. form of the Dirichlet Laplacian on Ω_2, $\tilde{a}_2 : X_2 \times X_2 \to \mathbb{C}$,

$$\tilde{a}_2(u,v) := \sum_{k=1}^{7} \int_T \nabla u_k \cdot \overline{\nabla v_k}\, dx.$$

As above, $\tilde{a}_2(\Phi_2 u, \Phi_2 v) = a_2(u,v)$ $\forall\, u,v \in H_0^1(\Omega_2)$. $\Rightarrow a_2, \tilde{a}_2$ isospectral.

Set

$$B := \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}, \qquad \Phi : L^2(T)^7 \to L^2(T)^7, \quad \Phi\begin{pmatrix} u_1 \\ \vdots \\ u_7 \end{pmatrix} = B\begin{pmatrix} u_1 \\ \vdots \\ u_7 \end{pmatrix}.$$

Then Φ isomorphism and $\Phi^*\begin{pmatrix} u_1 \\ \vdots \\ u_7 \end{pmatrix} = B^*\begin{pmatrix} u_1 \\ \vdots \\ u_7 \end{pmatrix}$.

Then $\Phi(\mathcal{X}_1) \subset \mathcal{X}_2$, e.g. look at the third triangle in $\mathcal{R}_2$.

If $u = (u_1, \dots, u_7) \in \mathcal{X}_1$, then for $v := \Phi u$ we have

$$v_3 = u_1 - u_4 + u_5 \quad \text{on } T$$

$$v_1 = u_2 + u_3 + u_4 \quad \text{on } T$$

$$v_6 = u_2 - u_7 - u_5 \quad \text{on } T$$

Thus

·) On Γ_1: $v_3 = \underbrace{u_1}_{=u_2} - \underbrace{u_4}_{=u_7} + \underbrace{u_5}_{=0} = u_2 - u_7 - u_5 = v_6$

·) On Γ_2: $v_1 = \underbrace{u_2}_{=u_5} + \underbrace{u_3}_{=u_1} + \underbrace{u_4}_{=0} = u_5 + u_1 - u_4 = v_7$

·) on Γ_3 : $v_3 = \underbrace{u_1 - u_4}_{=0} + \underbrace{u_5}_{=0} = 0.$

Moreover, for all $u \in \mathcal{K}_1$, $v \in \mathcal{K}_2$:

$$\tilde{a}_2(\phi u, v) = \sum_{k=1}^{7} \int_T \nabla (\phi u)_k \cdot \overline{(\nabla v)_k}\, dx = \sum_{k=1}^{7} \int_T \nabla \Big(\sum_{\ell=1}^{7} b_{k\ell} u_\ell \Big) \cdot \overline{\nabla v_k}\, dx$$

$$= \sum_{\ell=1}^{7} \int_T \nabla u_\ell \cdot \nabla \Big(\sum_{k=1}^{7} b_{k\ell} \overline{v_k} \Big)\, dx = \tilde{a}_1(u, \phi^* v).$$

This implies that $\tilde{a}_1, \tilde{a}_2$ are isospectral, e.g. if λ EV of $\tilde{a}_1$ then

$$\lambda (\phi u, v)_{L^2(T)^7} = \lambda (u, \phi^* v)_{L^2(T)^7} = \tilde{a}_1(u, \phi^* v) = \tilde{a}_2(\phi u, v) \quad \forall v \in \mathcal{K}_2$$

$\Rightarrow$ λ EV of $\tilde{a}_2$.

Consequence: a_1 isosp. $\tilde{a}_1$ isosp. $\tilde{a}_2$ isosp. a_2

$\Rightarrow$ Dirichlet Laplacians on Ω_1 and Ω_2 have same EV.
(incl. multiplicities)

Remarks: •) Same reasoning shows that the same pair of domains is also isospectral for the Neumann Laplacians, (replacing each -1 in $\mathbb{B}$ by 1).

•) The question whether one can hear the shape of a convex or smooth drum is still open.
There are some positive results e.g. Zelditch for smooth domains with symmetries.