Lecture 6: Nodal domains

Sources: Å. Pleijel: "Remarks on Courant's nodal line theorem" (1956)

Evans: Partial Differential Equations

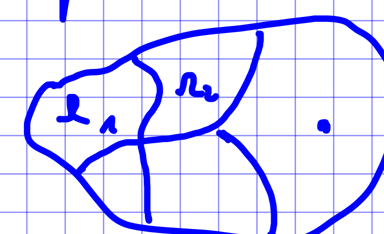
Nodal domains and properties of eigenfunctions

$\Omega \subset \mathbb{R}^d$ bdd. domain, $d = 1, 2, \dots$

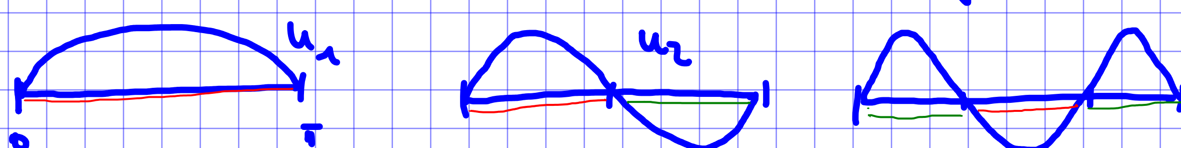
Definition For $u : \Omega \to \mathbb{R}$, the connected components of

$$\Omega \setminus \overline{\{x \in \Omega : u(x) = 0\}}$$

are called nodal domains of u and their number $\nu(u)$ is called nodal count.



Ex. Let λ_j, u_j the EV and EF of $-\Delta$ on $\Omega = (0, \pi)$ with Dirichlet BC. Then $u_j(x) = \sin(jx)$, $j = 1, 2, \dots$



$\nu(u_j) = j \quad \forall j$.

- physical meaning (wave equation, Chladni plate experiment)
- give an impression of what the EF may look like and how much it oscillates

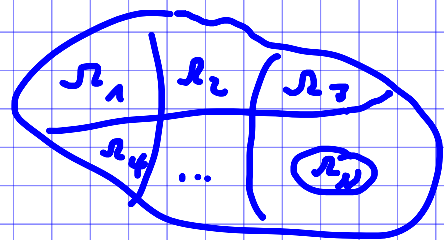
Courant's nodal domain theorem

Let $\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$ the EV of the Dirichlet Laplacian on Ω and $u_1, u_2, u_3, \dots$ any ON-basis of corr. EF (chosen real-valued). Then $\nu(u_j) \le j$ for all $j \in \mathbb{N}$.

(The same holds for the Neumann Laplacian if $\partial\Omega$ is suff. smooth.)

Proof Assume $N := \nu(u_j) > j$ for some j and let $\Omega_1, \dots, \Omega_N$ the nodal domains of u_j. Define

$$v_k := \begin{cases} u_j & \text{on } \Omega_k, \\ 0 & \text{else}, \end{cases} \qquad k = 1, \dots, N.$$



Then $v_k \in H_0^1(\Omega)$, $k = 1, \dots, N$. As $N \geq j$, there ex. $\alpha_k \in \mathbb{R}$ s.t.

$$0 \neq \sum_{k=1}^{j} \alpha_k v_k \perp \operatorname{span}\{u_1, \dots, u_{j-1}\}.$$

Thus (by max-min)

$$\lambda_j \leq \frac{\int_\Omega |\nabla \sum \alpha_k v_k|^2 dx}{\int_\Omega |\sum \alpha_k v_k|^2 dx} = \frac{\sum |\alpha_k|^2 \int_{\Omega_k} |\nabla v_k|^2 dx}{\int_\Omega |\sum \alpha_k v_k|^2 dx} \overset{\text{Green}}{=}$$

$$= \frac{\sum |\alpha_k|^2 \int_{\Omega_k} (-\Delta u_j) u_j \, dx}{\int_\Omega |\sum \alpha_k v_k|^2 dx} = \lambda_j.$$

Equality is only possible if $\sum_{k=1}^{j} \alpha_k v_k$ is an EF corr. to λ_j.
But $\operatorname{supp}\left(\sum_{k=1}^{j} \alpha_k v_k\right) \subset \bigcup_{k=1}^{j} \overline{\Omega_k}$, so the fct. vanishes constantly on Ω_{j+1}. But an eigenfunction cannot be zero on any non-empty open set ("unique continuation principle").

$\Rightarrow \lambda_j < \lambda_j$ ↯

□

Remark Have used that eigenfunctions cannot vanish on open, non-empty sets. This is counterpart of uniqueness of solutions to ODEs of second order in higher dimensions.

One ex. is the following:

Lemma Assume $-\Delta u = \lambda u$ on Ω, $\underbrace{u|_{\partial\Omega} = 0}_{\text{i.e. } u \in H_0^1(\Omega)}$, $\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0$.

Then $u \equiv 0$ identically on Ω.

Proof Define $\tilde{u} = \begin{cases} u & \text{on } \Omega, \\ 0 & \text{elsewhere on } \mathbb{R}^d. \end{cases}$

Due to both Dirichlet and Neumann BC, $\tilde{u} \in H^1(\mathbb{R}^d)$ and $-\Delta \tilde{u} = \lambda \tilde{u}$ on $\mathbb{R}^d$. (Does NOT work with just Dirichlet BC.)

Apply Fourier transform: (Reminder: $(\mathcal{F} f)(\xi) = \frac{1}{(2\pi)^d} \int f(x) e^{-i\xi x} dx$)

$$\lambda (\mathcal{F}\tilde{u})(\xi) = (\mathcal{F}(-\Delta \tilde{u}))(\xi) = +|\xi|^2 (\mathcal{F}\tilde{u})(\xi) \quad \forall \xi \in \mathbb{R}^d$$

$\Rightarrow \mathcal{F}\tilde{u} \equiv 0$ on $\mathbb{R}^d$ $\Rightarrow \tilde{u} \equiv 0$ a.e. on $\mathbb{R}^d$ $\Rightarrow u \equiv 0$ on Ω. $\square$

Theorem Let λ_1 the first EV of the Dirichlet Laplacian. Then the corr. EF u_1 is unique up to scalar multiples (i.e. the eigenspace corr. to λ_1 is one-dimensional) and can be chosen positive in Ω. Moreover, all eigenfcts. corr. to some λ_j with $j \geq 2$ must have a sign change in Ω.

Proof. Let u_1 be any EF corr. to λ_1. As $\nu(u_1) = 1$ and u_1 is continuous in Ω, u_1 does not change sign in Ω, w.l.o.g. $u_1 \geq 0$. In particular,

$$-\Delta u_1 = \lambda u_1 \geq 0 \quad \text{in } \overline{\Omega}$$

$\Rightarrow$ u_1 subharmonic. By the minimum principle, (see Evans, Sec. 6.4) u_1 can only vanish at some point in Ω if $u_1 \equiv 0$. Hence $u_1 > 0$ in Ω.

In part., no EF corr. to λ_1 has a zero in Ω.

Assume now the eigenspace corr. to λ_1 had dimension ≥ 2 and u_1, v_1 be two corr. lin. indep. EF. Then for $x_0 \in \Omega$, the fct.

$$v_1(x_0)\, u_1 - u_1(x_0)\, v_1$$

is an EF corr. to λ_1 and vanishes at x_0, a contradiction.

Finally, let u_j EF corr. to λ_j, $j \geq 2$. Then

$$0 = \int_\Omega u_1 u_j \, dx, \qquad \text{and } u_1 > 0 \text{ in } \Omega,$$

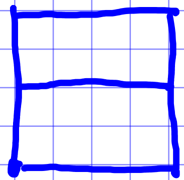
hence u_j must take both negative and pos. values. □

<u>Corollary</u> Let u_j EF corr. to λ_j and Ω_0 a nodal domain of u_j. Then $u_j|_{\Omega_0}$ is an EF of the Dirichlet Laplacian on Ω_0 and the corr. EV is $\lambda_1(\Omega_0) = \lambda_j$.

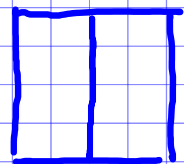
Is it possibly true that $\nu(u_j) = j \;\; \forall j$?

- always true for $j = 1, 2$.
- On the square $(0, \pi)^2$ we $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 5$, and the EF for $\lambda_2 = \lambda_3 = 5$ are

$\alpha \underbrace{\sin(x)\sin(2y)} + \beta \underbrace{\sin(2x)\sin(y)}$



two nodal dom.



two nodal dom.

and it can be shown that all of these linear combinations only has two nodal domains.

$\Rightarrow \quad \nu(u_3) = 2.$

<u>Definition</u> An EV λ_j of the Dirichlet Laplacian is called <u>Courant-sharp</u> if $\nu(u_j) = j$ for the corr. EF u_j.

<u>Pleijel's theorem</u>

Let $\Omega \subset \mathbb{R}^d$ bounded domain, λ_j, u_j corr. Dirichlet Laplacian EV, EF.

Then

$$\limsup_{j \to \infty} \frac{\nu(u_j)}{j} \leq \gamma(d) := \frac{2^{d-2} d^2 \Gamma\left(\frac{d}{2}\right)^2}{j^d_{\frac{d}{2}-1,1}} < 1.$$

In particular, only finitely many EV are Courant-sharp.

<u>Proof</u> (case $d = 2$). Want to show

$$\limsup_{j \to \infty} \frac{\nu(u_j)}{j} \leq \frac{4}{j_{0,1}^2}.$$

For an EF u_j corr. to λ_j let $\Omega_1, \dots, \Omega_N$ the corr. nodal domains

$N = \nu(u_j) \leq j$. Then $u_j \big|_{\Omega_k}$ is an EF of the Dirichlet

Laplacian on Ω_k corr. $\lambda_1(\Omega_k) = \lambda_j$, and by Faber-Krahn,

$$\lambda_j = \lambda_1(\Omega_k) \geq \frac{\pi j_{0,1}^2}{|\Omega_k|},$$

or

$$\frac{1}{\lambda_j} \leq \frac{|\Omega_k|}{\pi j_{0,1}^2}, \qquad k = 1, \dots, N = \nu(u_j).$$

Summing over k:

$$\frac{\nu(u_j)}{\lambda_j} \leq \frac{|\Omega|}{\pi j_{0,1}^2}.$$

By Weyl's law, $\lambda_j \sim \frac{4\pi j}{|\Omega|}$ as $j \to \infty$, hence

$$\limsup_{j\to\infty} \frac{|\Omega|}{4\pi} \frac{\nu(u_j)}{j} = \limsup_{j\to\infty} \frac{\nu(u_j)}{\lambda_j} \leq \frac{|\Omega|}{\pi j_{0,1}^2}.$$

Finally, as $j_{0,1} \approx 2.4$, $\frac{4}{j_{0,1}^2} < 1$. □

<u>Example</u> $\Omega = (0,\pi)^2$. Let

$$N(\alpha) := \#\{j : \lambda_j \leq \alpha\} \quad \text{EV counting fct.}$$

Know (proof of Weyl's law for rectangles):

$$N(\alpha) \geq \frac{|\Omega|}{4\pi}\alpha - \frac{\mathrm{Per}(\Omega)}{2\pi}\sqrt{\alpha} = \frac{\pi}{4}\alpha - 2\sqrt{\alpha}.$$

For an EV λ_j that is Courant-sharp we may assume $j = N(\lambda_j)$ and hence

$$j \geq \frac{\pi}{4}\lambda_j - 2\sqrt{\lambda_j}.$$

From the proof of Pleijel's thm, $\frac{\nu(u_j)}{\lambda_j} \le \frac{|\Omega|}{\pi j_{0,1}^2} = \frac{\pi}{j_{0,1}^2}$,

hence

$$0.54 \approx \frac{\pi}{j_{0,1}^2} \ge \frac{j}{\lambda_j} \ge \frac{\pi}{4} - \frac{2}{\sqrt{\lambda_j}},$$

$$\Longleftrightarrow \quad \frac{2}{\sqrt{\lambda_j}} \ge \frac{\pi}{4} - \frac{\pi}{j_{0,1}^2} \quad \Longleftrightarrow \quad \lambda_j \le \left(\frac{2}{\frac{\pi}{4} - \frac{\pi}{j_{0,1}^2}} \right)^2 \approx 69.5.$$

$\rightarrow$ only EV $\lambda_j < 70$ can be Courant-sharp.

One can see in this case that only $\lambda_1 = 2$, $\lambda_2 = 5$, and $\lambda_4 = 8$ are Courant-sharp.