Lecture 5: Weyl asymptotics, Faber-Krahn inequality

Sources: .) Laugesen-notes, Chapter 11

.) Henrot: Extremum problems for eigenvalues of elliptic operators, Chapters 3-4

Repetition: λ_1 - first Dirichlet Laplacian EV on Ω bdd

μ_2 - first nontrivial Neumann Laplacian EV on Ω bdd

$$\lambda_1 = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}, \qquad \mu_2 = \min_{\substack{u \in H^1(\Omega), \\ \int_\Omega u\, dx = 0}} \frac{\int |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}.$$

Theorem (Weyl's law)

Let $\Omega \subset \mathbb{R}^d$ bdd. domain, $d \geq 1$, with piecewise smooth bdr., λ_j, ρ_j, μ_j Dirichlet / Robin / Neumann Laplacian EV on Ω. Then

$$\lambda_j \sim \rho_j \sim \mu_j \sim 4\pi^2 \left(\frac{j}{\omega_d |\Omega|} \right)^{2/d} \quad \text{as } j \to \infty$$

where $|\Omega|$ d-dim. volume of Ω, ω_d volume of unit ball in $\mathbb{R}^d$.

E.g.

$$\lambda_j \sim \varrho_j \sim \mu_j \sim \begin{cases} \dfrac{\pi^2 j^2}{|\Omega|^2} & (d=1) \\ \dfrac{4\pi j}{|\Omega|} & (d=2) \\ \left(\dfrac{6\pi^2 j}{|\Omega|}\right)^{\frac{2}{3}} & (d=3) \end{cases}$$

<u>Rem.</u> ·) $d=1$ clear from explicit computations (Dirichlet and Neumann) and $\mu_j \leq \varrho_j \leq \lambda_j \ \forall j$.

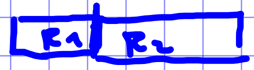
·) $d=2$ already seen in Lecture 1

<u>Proof for $d=2$</u>, general Ω

Step 1: For rectangles we know $\lambda_j \sim \varrho_j \sim \mu_j \sim \dfrac{4\pi j}{|\Omega|}$.

Step 2: Unions of rectangles. Let $R_1, \dots, R_n$ disjoint open rectangles, and let

$$\tilde{\Omega} := \bigcup_{m=1}^{n} R_m, \qquad \Omega := \left(\bigcup_{m=1}^{n} \overline{R_m}\right)^{\circ}$$



$\tilde{\Omega}$

not connected

Ω

Let $\tilde{\lambda}_j, \tilde{\mu}_j$ Dirichlet / Neumann EV on $\tilde{\Omega}$. Then

$$\tilde{\mu}_j \overset{\le}{\nearrow} \mu_j \le \rho_j \le \lambda_j \overset{\le}{\nearrow} \tilde{\lambda}_j \qquad \forall j$$

restricted reverse domain monotonicity

dom. mon. for Dirichlet

Hence suffices to prove Weyl's law for D and N EV on $\tilde{\Omega}$.

Define the counting fcts.

$$N_N(\lambda; R_m) := \#\{j \ge 1 : \mu_j(R_m) \le \lambda\},$$

$$N_D(\lambda; R_m) := \#\{j \ge 1 : \lambda_j(R_m) \le \lambda\}.$$

Know

$$N_N(\lambda; R_m) \sim N_D(\lambda; R_m) \sim \frac{|R_m|}{4\pi}\lambda, \qquad \lambda \to \infty.$$

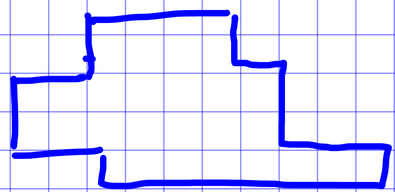
Hence

$$N_N(\lambda;\tilde{\Omega}) = \sum_{m=1}^{n} N_N(\lambda; R_m) \sim \sum_{m=1}^{n} \frac{|R_m|}{4\pi}\lambda = \frac{|\Omega|}{4\pi}\lambda,$$

and

$$N_D(\lambda;\tilde{\Omega}) \sim \frac{|\Omega|}{4\pi}\lambda.$$

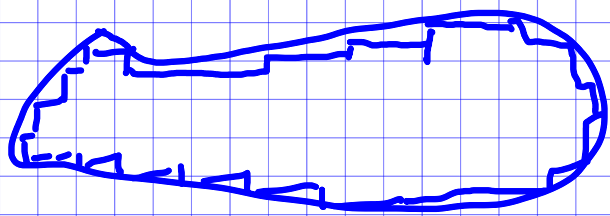
By plugging in $\lambda = \lambda_j$ or $\lambda = \mu_j$, resp., and inverting we get the result for Ω = union of rectangles.



Step 3: general domains

Approximate Ω by unions of, use continuity of EV

(Details to be filled.)



$\square$

Symmetrization and the Faber-Krahn inequality

$\lambda_1(\Omega)$ – first EV of Dirichlet Laplacian on Ω

= lowest frequency of vibrating membrane of shape Ω (wave equation)

= fundamental tone of a drum of shape Ω.

·) Know: $0 \xleftarrow{|\Omega| \to \infty} \lambda_1(\Omega) \xrightarrow{|\Omega| \to 0} \infty$,

actually $\lambda_j(t\Omega) = \frac{1}{t^2} \lambda_j(\Omega)$.

·) For $|\Omega|$ fixed, which shape leads to the lowest (or highest) fundamental tone?

[No maximizer exists ⇝ Exercise]

Schwarz symmetrization (Symmetric rearrangement)

·) For $\Omega \subset \mathbb{R}^d$ bdd. domain let Ω^* denote the open ball of the same volume.

·) For $u : \Omega \to \mathbb{R}$ define the symmetrized fct. $u^* : \Omega^* \to \mathbb{R}$ by

$$u^*(x) := \sup \{ t : x \in \Omega(t)^* \},$$

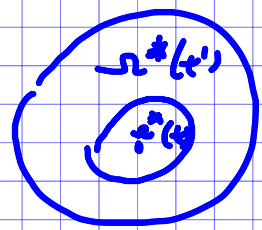
where $\Omega(t) = \{ x \in \Omega : u(x) \geq t \}$ ("superlevel set")

Note: ·) $\Omega(t) \searrow$

·) if u bdd. from below: $\exists\, t_0 : \Omega = \Omega(t_0) \supseteq \Omega(t) \quad \forall t$

·) $\Omega^* = \Omega(t_0)^* \supset \Omega^*(t) \quad \forall t$

·) if u bdd: $\Omega^* \supset \Omega^*(t) \supset \{\}$



$t' < t$

<u>Properties of u^*</u>

·) u^* radially symm. "$u^*(x) = u^*(|x|)$"

·) u^* non-increasing in $|x|$

·) $\inf\limits_{x \in \Omega} u(x) = \inf\limits_{x \in \Omega^*} u^*(x)$ and $\sup\limits_{x \in \Omega} u(x) = \sup\limits_{x \in \Omega^*} u(x)$

·) if u continuous then u^* continuous

·) u, u^* <u>equimeasurable</u> : $|\{x \in \Omega : u(x) \geq t\}| = |\{x \in \Omega^* : u^*(x) \geq t\}|$ $\forall t$

In particular, if u measurable then

$$\int_\Omega \psi(u(x))\,dx = \int_{\Omega^*} \psi(u^*(x))\,dx$$

for each continuous $\psi : \mathbb{R} \to \mathbb{R}$.

<u>Proposition</u>

Let $u \in H_0^1(\Omega)$ be non-negative in Ω. Then $u^* \in H_0^1(\Omega^*)$ and

$$\int_\Omega |\nabla u(x)|^2\,dx \geq \int_{\Omega^*} |\nabla u^*(x)|^2\,dx.$$

(Proof: Bandle : Isoperimetric inequalities and applications, Thm 2.2)

Theorem (Faber-Krahn inequality) ≈ 1925

Let $\Omega \subset \mathbb{R}^d$ bdd. domain and $B \subset \mathbb{R}^d$ any ball with $|\Omega| = |B|$.

Then the lowest EV $\lambda_1(\Omega)$ and $\lambda_1(B)$ of the Dirichlet Laplacians on Ω and B, resp., satisfy

$$\lambda_1(B) \leq \lambda_1(\Omega).$$

"Among all domains of given volume, the ball minimizes the fundamental tone."

Proof: We have $B = \Omega^*$ (up to shift). Let u_1 EF on Ω corr. to $\lambda_1(\Omega)$ and u_1^* its symmetric rearrangement. Then $u_1 \geq 0$ (by choice). Then

$$\lambda_1(B) \leq \frac{\int_{\Omega^*} |\nabla u_1^*(x)|^2 dx}{\int_{\Omega^*} |u_1^*(x)|^2 dx} \leq \frac{\int_{\Omega} |\nabla u_1(x)|^2 dx}{\int_{\Omega} |u_1(x)|^2 dx} = \lambda_1(\Omega).$$

$\square$

Remarks .) Obviously, equality holds if Ω is a ball, but also, e.g., for a ball with a finite number of points removed.
Assuming more regularity of $\partial\Omega$ (e.g. Lipschitz) the ball B the unique minimizer.

.) $\lambda_1(B)$ is explicitly known. (cf. Lecture 1)

E.g. in $d=2$: $\lambda_1(B) = \dfrac{\pi j_{0,1}^2}{|B|}$, $j_{0,1}$ = smallest pos. zero of Bessel fct. J_0.

Corollary On any bdd. domain $\Omega \subset \mathbb{R}^2$, $\lambda_1(\Omega) \geq \dfrac{\pi j_{0,1}^2}{|\Omega|}$.

Corr. result for the Neumann Laplacian:

Szegő-Weinberger inequality ~ 1954

Among all bdd. domains $\Omega \subset \mathbb{R}^d$ with suff. smooth bdr.,

$$\mu_2(\Omega) \leq \mu_2(B)$$

where B is the ball with same volume.

Higher EV?

- $\lambda_2(\Omega)$ minimized by the union of two disjoint balls of half the volume each.
- $\mu_3(\Omega)$ is maximized by the union of two disjoint balls of half of the volume of Ω (Bucur / Henrot, Acta Math. 2019)
- For higher EV shape optimizers open!

For fixed perimeter / surface area

- For λ_1: Balls are minimizers, follows from Faber-Krahn.
- For μ_2: <u>open</u>, Conjecture in $d=2$: for fixed perimeter the maximizers of $\mu_2(\Omega)$ are squares and eq. triangles among convex domains.