Lecture 4: Variational characterization of EV and consequences

Sources: .) Laugesen-notes, Chapters 9-10

.) Schmüdgen: Unbounded self-adjoint operators on Hilbert space, Chapter 12

Repetition: Settings with purely discrete spectrum:

- Laplacian on Ω bdd. with Dirichlet/Neumann BC
- similar for more general elliptic op. on Ω bdd.
- $-\Delta + V$ on $\mathbb{R}^d$ with $-C \le V(x) \to +\infty$

Motivation: Courant-Fischer thm

Let $A \in \mathbb{C}^{n\times n}$ Hermitian with $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$.

Then

$$\lambda_k = \min_{\substack{U \subseteq \mathbb{C}^n \text{ subsp.} \\ \dim U = k}} \max_{u \in U\setminus\{0\}} \frac{(Au,u)}{\|u\|^2}$$

E.g. $$\lambda_1 = \min_{u \in \mathbb{C}^n \setminus\{0\}} \frac{(Au,u)}{\|u\|^2}$$

- $\mathcal{X}, \mathcal{H}$ Hilbert, $\mathcal{X} \subset \mathcal{H}$ densely, boundedly, compactly embedded
- $a(\cdot,\cdot) : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ scalar prod. with
  - .) bdd, $|a(u,v)| \le C \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}$
  - .) elliptic, $a(u,u) \ge c \|u\|_{\mathcal{X}}^2$ for some $c > 0$.

Know from discr. spectral thm: the weak EV problem

$$a(u,v) = \lambda (u,v)_{\mathcal{H}} \quad \forall v \in \mathcal{X}$$

has positive EV $0 < \lambda_1 \le \lambda_2 \le \dots \to \infty$ and the corr. eigenvectors $u_1, u_2, \dots$ can be chosen to form an ON-basis of $\mathcal{H}$.

$\frac{u_1}{\sqrt{\lambda_1}}, \frac{u_2}{\sqrt{\lambda_2}}, \dots$ ON-basis of $(\mathcal{X}, a(\cdot,\cdot))$.

Theorem Poincaré min-max principle

Under the above assumptions,

$$\lambda_k = \min_{\substack{U \subset \mathcal{X} \text{ subsp.} \\ \dim U = k}} \max_{u \in U \setminus \{0\}} \frac{a(u,u)}{\|u\|_{\mathcal{H}}^2}$$

Rayleigh

Proof Let $\{u_j\}$ be an ON-basis of $(\mathcal{X}, a(\cdot,\cdot))$ s.t.

$$a(u_j, v) = \lambda_j (u_j, v)_{\mathcal{X}} \quad \forall v \in \mathcal{X}.$$

Set $U_0 := \{0\}$, $\underbrace{U_j}_{j\text{-dim.}} := \operatorname{span}\{u_k : k \le j\}$, $j = 1, 2, \ldots$

$U_{j-1}^{\perp}$ orth. comp. of U_j in $(\mathcal{X}, a(\cdot,\cdot))$.

Then $u \in U_{j-1}^{\perp}$ can be written $u = \sum_{k=j}^{\infty} c_k u_k$ with $c_k \in \mathbb{C}$,

where the series converges in $(\mathcal{X}, a(\cdot,\cdot))$ and thus also in $\mathcal{X}$.

some work to do

$$\|u\|_{\mathcal{X}}^2 = (u,u)_{\mathcal{X}} = \sum_{k=j}^{\infty} |c_k|^2 (u_k, u_k)_{\mathcal{X}} = \sum_{k=j}^{\infty} \frac{1}{\lambda_k} |c_k|^2 a(u_k, u_k)$$

$\lambda_k \ge \lambda_j$

$$\le \frac{1}{\lambda_j} \sum_{k=j}^{\infty} |c_k|^2 a(u_k, u_k) = \frac{1}{\lambda_j} a(u,u)$$

$$\Rightarrow \quad \lambda_j \le \frac{a(u,u)}{\|u\|_{\mathcal{X}}^2} \quad \forall u \in U_{j-1}^{\perp} \setminus \{0\},$$ with equality for $u = u_j$

$$\Rightarrow \quad \lambda_j = \min_{u \in U_{j-1}^{\perp} \setminus \{0\}} \frac{a(u,u)}{\|u\|_{\mathcal{X}}^2}. \qquad (A)$$

Analogously, for $u \in U_j$, $\|u\|_{\mathcal{X}}^2 \geq \frac{1}{\lambda_j} a(u,u)$, hence

$$\lambda_j \geq \frac{a(u,u)}{\|u\|_{\mathcal{X}}^2} \quad \forall\, u \in U_j \setminus \{0\},$$

$$\Rightarrow \qquad \lambda_j = \max_{u \in U_j \setminus \{0\}} \frac{a(u,u)}{\|u\|_{\mathcal{X}}^2}. \qquad (B)$$

Let now $V \subset \mathcal{X}$ arb. subsp. with $\dim V = j$. Then $\exists\, v \in (V \cap U_{j-1}^{\perp}) \setminus \{0\}$ and

$$\lambda_j \overset{(A)}{=} \min_{u \in U_{j-1}^{\perp} \setminus \{0\}} \frac{a(u,u)}{\|u\|_{\mathcal{X}}^2} \leq \frac{a(v,v)}{\|v\|_{\mathcal{X}}^2} \leq \max_{u \in V \setminus \{0\}} \frac{a(u,u)}{\|u\|_{\mathcal{X}}^2}.$$

As for $V = U_j$ we have equality (B), we get the claim. $\square$

Analogously one can show the max-min principle

$$\lambda_j = \max_{\substack{U \subset \mathcal{X} \text{ subsp.} \\ \dim U = j-1}} \min_{u \in U^{\perp} \setminus \{0\}} \frac{a(u,u)}{\|u\|_{\mathcal{X}}^2}.$$

Corollary:
$$\lambda_1 = \min_{u \in \mathcal{X} \setminus \{0\}} \frac{a(u,u)}{\|u\|_{\mathcal{X}}^2},$$

Remark: Eigenvectors can be characterised as minimizers, e.g.

$$\frac{a(u,u)}{\|u\|_{\mathcal{X}}^2} = \min_{v \in \mathcal{X} \setminus \{0\}} \frac{a(v,v)}{\|v\|_{\mathcal{X}}^2} = \lambda_1 \iff u \text{ Eigenvector corr. to } \lambda_1.$$

Proof: "$\Leftarrow$": clear

"$\Rightarrow$": We have $a(u,u) = \lambda_1 \|u\|_{\mathcal{X}}^2$ and want to show

$$a(u,v) = \lambda_1 (u,v) \qquad \forall v \in \mathcal{X}.$$

As u is a minimizer, for any $v \in \mathcal{X}$,

$$0 = \frac{d}{dt} \frac{a(u+tv, u+tv)}{(u+tv, u+tv)} \Bigg|_{t=0} =$$

$$= \frac{1}{\|u\|_{\mathcal{X}}^4} \Big(2\,\mathrm{Re}\, a(v,u)\, \|u\|_{\mathcal{X}}^2 - \underbrace{a(u,u)}_{= \lambda_1 \|u\|_{\mathcal{X}}^2}\, 2\,\mathrm{Re}\, (v,u)_{\mathcal{X}} \Big)$$

$$= \frac{2}{\|u\|_{\mathcal{X}}^2} \left(\operatorname{Re} a(v,u) - \lambda_1 \operatorname{Re} (v,u)_{\mathcal{X}} \right).$$

Replacing v by iv we get the same for Im, hence

$$a(u,v) = \lambda_1 (u,v)_{\mathcal{X}} \quad \forall\, v \in \mathcal{X}. \qquad \square$$

Application to PDEs

Dirichlet Laplacian on Ω bdd

$$a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx, \qquad \mathcal{X} = H_0^1(\Omega)$$

$$\lambda_j = \min_{\substack{U \subset H_0^1(\Omega) \\ \dim U = j}} \; \max_{u \in U \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

"quadratic form"

"form domain"

Neumann Laplacian on Ω bdd, $a(u,v) = \int_\Omega u \bar{v}\, dx + \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx$,

$\mathcal{X} = H^1(\Omega)$

EV $\tilde{\mu}_j = \mu_j + 1$ corr. to $a(u,v) = \mu (u,v)$

$$\mu_j + 1 = \tilde{\mu}_j = \min \; \max \frac{\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

$$\boxed{\mu_j = \min_{\substack{U \subset H^1(\Omega),\\ \dim U = j}} \ \max_{u \in U \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}.}$$

Robin Laplacian on Ω bdd. $\left(BC \ \frac{\partial u}{\partial \nu} + \sigma u = 0 \text{ on } \partial\Omega, \ \sigma > 0 \text{ const.} \right)$

$$\boxed{\varrho_j = \min_{\substack{U \subset H^1(\Omega)\\ \dim U = j}} \ \max_{u \in U \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx + \sigma \int_{\partial\Omega} |u|^2 dS}{\int_\Omega |u|^2 dx}}$$

Recall: Neumann and Robin cases require some smoothness of $\partial\Omega$, e.g. Lipschitz.

Schrödinger with potential $-c \leq V(x) \to +\infty$,

$$\boxed{E_j = \min_{\substack{U \subset H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, |V| dx)\\ \dim U = j}} \ \max_{u \in U \setminus \{0\}} \frac{\int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} V |u|^2 dx}{\int_{\mathbb{R}^d} |u|^2 dx}}$$

First consequences

Theorem (Neumann - Robin - Dirichlet comparison)

Let $\Omega \subset \mathbb{R}^d$ bounded Lipschitz domain. Then

$$\mu_j \le \rho_j \le \lambda_j \qquad \forall\, j \in \mathbb{N}.$$

Proof $\cdot)$ $\mu_j \le \rho_j$: same form domain, Robin quadratic form larger

$\cdot)$ $\rho_j \le \lambda_j$: $\underbrace{\text{form domain for Dirichlet}}_{H_0^1(\Omega)} \subset \underbrace{\text{form dom. for Robin}}_{H^1(\Omega)}$

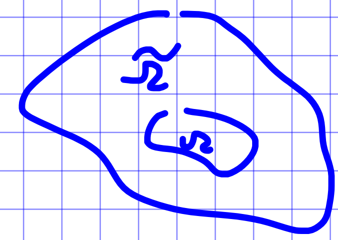
and on the smaller form dom. the Rayleigh quotients coincide. $\square$

Remark: All inequalities in the previous thm. strict.

Theorem (Domain monotonicity for Dirichlet EV)

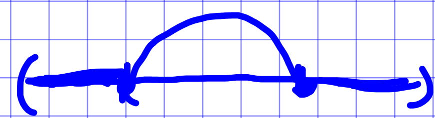
Let $\Omega, \tilde\Omega$ bdd. domains in $\mathbb{R}^d$ with corr. Dirichlet Lapl. EV by $\lambda_j(\Omega)$, $\lambda_j(\tilde\Omega)$. If $\tilde\Omega \supset \Omega$ then

$$\lambda_j(\tilde\Omega) \le \lambda_j(\Omega) \quad \forall j \in \mathbb{N}.$$



Proof For $u \in H_0^1(\Omega)$ the fct. $\tilde u := \begin{cases} u & \text{on } \Omega \\ 0 & \text{otherwise} \end{cases}$

belongs to $H_0^1(\tilde\Omega)$ and



$$\frac{\int_{\tilde\Omega} |\nabla \tilde u|^2\,dx}{\int_{\tilde\Omega} |\tilde u|^2\,dx} = \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega |u|^2\,dx}.$$

Therefore $H_0^1(\Omega)$ "$\subset$" $H_0^1(\tilde\Omega)$ $\Rightarrow \lambda_j(\tilde\Omega) \le \lambda_j(\Omega)$. $\square$

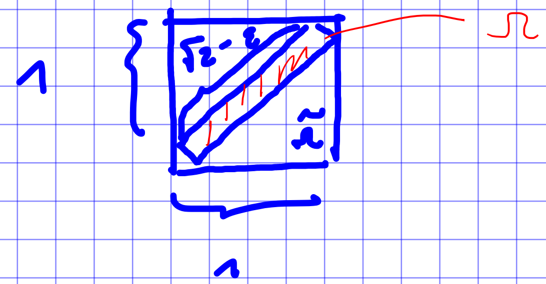
Remarks •) Compare with $d=1$: On $(0,L)$ $\lambda_j = \frac{j^2\pi^2}{L^2}$

$\leadsto$ larger intervals have smaller EV.

•) The inequality in the thm is strict if $\tilde\Omega \setminus \Omega$ contains an open ball.

Example (No domain monotonicity for Neumann EV)

$$\mu_2(\tilde{\Omega}) = \pi^2$$



$$\mu_2(\Omega) = \frac{\pi^2}{(\sqrt{2}-\varepsilon)^2}$$

For small ε: $\mu_2(\Omega) < \mu_2(\tilde{\Omega})$.

On the other hand, any square Ω' of side length $a < 1$ fits into $\tilde{\Omega}$ and has $\mu_2(\Omega') = \frac{\pi^2}{a^2} > \pi^2 = \mu_2(\tilde{\Omega})$.

Theorem (Restricted reverse domain monotonicity for Neumann)

Let $\Omega, \tilde{\Omega}$ bdd. Lipschitz domains in $\mathbb{R}^d$ with Neumann Laplacian EV $\mu_j(\Omega), \mu_j(\tilde{\Omega})$. Moreover, let $\Omega \subset \tilde{\Omega}$ and $\underbrace{|\tilde{\Omega} \setminus \Omega|}_{\text{Lebesgue meas.}} = 0$

Then $\mu_j(\tilde{\Omega}) \geq \mu_j(\Omega)$ $\forall j$.

E.g.