Lecture 3: Partial differential operators with discrete spectrum

Sources: ·) Langesen-notes, Chapters 5, 6, 8

·) Evans: Partial Diff. Equations
(for PDE properties)

Repetition Discrete spectral thm

Let $\mathcal{K}, \mathcal{H}$ Hilbert spaces, $\mathcal{K} \subset \mathcal{H}$ continuously, compactly densely embedded,
$a: \mathcal{K} \times \mathcal{K} \to \mathbb{C}$ scalar product, bounded, elliptic.

$\Rightarrow \exists\, u_1, u_2, \ldots \in \mathcal{K}$, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ s.t.

$$a(u_j, v) = \lambda_j (u_j, v)_{\mathcal{H}} \quad \forall v \in \mathcal{K}, \qquad \{u_j\} \text{ ON-basis in } \mathcal{H}.$$

Proof ① $\forall f \in \mathcal{H}\ \exists!\, u \in \mathcal{K}: a(u,v) = (f,v)_{\mathcal{H}}\ \forall v \in \mathcal{K}$ | "unique soln of $-\Delta u = f$, $u \in H_0^1(\Omega)$"

$\tilde{B}: \mathcal{H} \to \mathcal{K}$, $f \mapsto u$ | "inverse of $-\Delta$ on $H_0^1(\Omega)$"

$\tilde{B}$ bdd.

② Let $\mathbb{B}: \mathcal{H} \to \mathcal{H}$, $\mathbb{B}u := \tilde{\mathbb{B}}u$. Then for $\iota: X \to \mathcal{H}$, $u \mapsto u$,

$$\mathbb{B} = \underbrace{\iota}_{\text{comp. } X \to \mathcal{H}} \underbrace{\tilde{\mathbb{B}}}_{\text{bdd}: \mathcal{H} \to X} \quad \text{compact.}$$

$\mathbb{B}$ self-adjoint:

$$(\mathbb{B}f, g)_{\mathcal{H}} = \overline{(g, \underbrace{\mathbb{B}f}_{\in X})_{\mathcal{H}}} \overset{\text{def } \mathbb{B}}{=} \overline{a(\mathbb{B}g, \mathbb{B}f)} = a(\mathbb{B}f, \mathbb{B}g)$$

$$\overset{\text{def } \mathbb{B}}{=} (f, \mathbb{B}g)_{\mathcal{H}}.$$

$\Rightarrow$ $\mathbb{B}$ self-adjoint.

$\Rightarrow$ (spectr. thm.) $\exists\, \gamma_j \in \mathbb{R}$, $\{u_j\}$ ONB of $\mathcal{H}$ s.t.

$$\mathbb{B}u_j = \gamma_j u_j \quad \forall j, \qquad \gamma_j \to 0$$

③ The γ_j are positive:

$$\gamma_j\, a(u_j, v) = a(\mathbb{B}u_j, v) = (u_j, v)_{\mathcal{H}} \quad \forall v \in X. \quad (*)$$

Set $v = u_j \Rightarrow \gamma_j \underbrace{a(u_j, u_j)}_{>0} = \underbrace{(u_j, u_j)_{\mathcal{H}}}_{>0} \Rightarrow \gamma_j > 0.$

④ Let $\lambda_j := \frac{1}{\gamma_j}$. Then (*) gives

$$a(u_j, v) = \lambda_j (u_j, v)_{\mathcal{X}} \qquad \forall\, v \in \mathcal{X}.$$

a-orthonormality of $\frac{1}{\sqrt{\lambda_j}} u_j$ in $\mathcal{X}$:

$$a(u_j, u_k) = \lambda_j (u_j, u_k)_{\mathcal{X}} = \lambda_j \delta_{jk} = \sqrt{\lambda_j}\sqrt{\lambda_k}\, \delta_{jk}.$$

$\square$

<u>Laplacian with Dirichlet boundary conditions</u>

Let $\Omega \subset \mathbb{R}^d$ bounded domain.

Recall:
$$\begin{cases} -\Delta u = \lambda u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \text{Dirichlet}$$

eq. to
$$\begin{cases} \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx = \lambda \int_\Omega u \bar{v}\, dx & \forall\, v \in H_0^1(\Omega) \\ u \in H_0^1(\Omega) \end{cases}$$

(weak EF belong to $C^\infty(\Omega)$ — "elliptic regularity")
see Evans, Chap. 6.3

In the discrete spectr. thm, choose

- $\mathcal{H} = L^2(\Omega)$, — $\mathcal{K} = H_0^1(\Omega)$

$\left(\|u\|_{L^2}^2 = \int_\Omega |u|^2 dx, \qquad \|u\|_{H^1}^2 = \int_\Omega |u|^2 dx + \int |\nabla u|^2 dx \right)$

- $a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx$

Check assumptions:

.) $\mathcal{K} \subset \mathcal{H}$, for $u \in \mathcal{K} = H_0^1(\Omega)$:

$$\|u\|_{\mathcal{K}}^2 = \int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx \geq \|u\|_{\mathcal{H}}^2 \quad \text{cont. embedd.}$$

.) $\mathcal{K} \subset \mathcal{H}$ dense

.) embedding compact?

Rellich - Kondrachov thm (e.g. Evans, Chap. 5.7)

.) $H^1(\Omega)$ is compactly emb. into $L^2(\Omega)$ if $\partial\Omega$ is C^1-smooth

.) In part. $H_0^1(\Omega)$ is comp. embd. into $L^2(\Omega)$ on any bdd. Ω.

.) a is a scalar prod. on $\mathcal{X} = H_0^1(\Omega)$: symmetry, sesquilinearity clear.

$$a(u,u) = \int_\Omega \nabla u \cdot \overline{\nabla u}\, dx = \int_\Omega |\nabla u|^2 dx \geq 0$$

Equality implies $|\nabla u|^2 = 0$ a.e. on Ω $\Rightarrow$ $u =$ const.,
$\rightarrow$ ($u \in H_0^1(\Omega)$) $u \equiv 0$ a.e.

.) a bdd.:

$$|a(u,v)| = \left| \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx \right| \overset{CS}{\leq} \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2} \left(\int_\Omega |\nabla v|^2 dx \right)^{1/2}$$

$$\leq \|u\|_{H^1} \|v\|_{H^1} .$$

.) a elliptic:

Poincaré inequality For Ω bdd. $\exists c > 0$:

$$\int_\Omega |\nabla u|^2 dx \geq c \int_\Omega |u|^2 dx \quad \forall u \in H_0^1(\Omega)$$

(not true $\forall u \in H^1(\Omega)$)

(**Proof** (roughly, u real-valued)

$$\|u\|_{L^2}^2 = \int_\Omega |u|^2 dx = \int_\Omega 1 \cdot u^2 \, dx \overset{\text{by}}{\underset{\text{parts}}{=}} - \int_\Omega x_j \underbrace{\partial_j (u^2)}_{= 2\partial_j u \cdot u} dx$$

$$= -2 \int_\Omega x_j \partial_j u \cdot u \, dx \overset{CS}{\leq} 2 \max_{x_j \in \bar\Omega} |x_j| \, \|u\|_{L^2} \|\partial_j u\|_{L^2}$$

$$\leq 2 \max_{x \in \bar\Omega} |x| \, \|u\|_{L^2} \|\nabla u\|_{L^2} . \qquad \square \;)$$

$$\Rightarrow a(u,u) = \int_\Omega |\nabla u|^2 dx = \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \int_\Omega |\nabla u|^2 dx$$

$$\geq \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{c}{2} \int_\Omega |u|^2 dx$$

$$\geq \min\left\{\frac{1}{2}, \frac{c}{2}\right\} \|u\|_{H^1}^2 .$$

a elliptic.

$\Rightarrow$ $\exists\, u_1, u_2, \ldots \in H_0^1(\Omega)$, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ s.t.

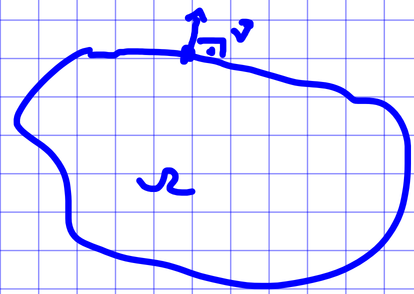
.) u_j are EF of $-\Delta$ with Dirichlet-BC corr. to EV λ_j;

.) $\{u_j\}$ ON-basis of $L^2(\Omega)$

.) $\{u_j / \sqrt{\lambda_j}\}$ ON-basis of $H_0^1(\Omega)$ eq. with $(u,v)_a := \int_\Omega \nabla u \cdot \overline{\nabla v} \, dx$.

Laplacian with Neumann BC (Ω bdd, $\partial\Omega$ C^2-smooth)

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \quad \text{Neumann} \end{cases}$$



- $\mathcal{H} = L^2(\Omega)$
- $\mathcal{K} = H^1(\Omega)$
- $a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\,dx + \int_\Omega u \bar{v}\,dx = (u,v)_{H^1}$

Discrete spec. thm: $\exists\, u_j \in H^1(\Omega)$ ON-basis of $L^2(\Omega)$, $\tilde{\mu}_j > 0$:

$$\int_\Omega \nabla u_j \cdot \overline{\nabla v}\,dx + \int_\Omega u_j \bar{v}\,dx = \tilde{\mu}_j \int_\Omega u_j \bar{v}\,dx \qquad \forall v \in H^1(\Omega)$$

$$\Longleftrightarrow \int_\Omega \nabla u_j \cdot \overline{\nabla v}\,dx = \underbrace{\mu_j}_{:= \tilde{\mu}_j - 1} \int_\Omega u_j \bar{v}\,dx \qquad \forall v \in H^1(\Omega) \quad (**)$$

.) Where is the Neumann-BC?

Green's first identity

$$\int_\Omega (\Delta u)\, \overline{v}\, dx + \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, \overset{\text{trace of } v}{\overline{v}}\, \underset{\text{surface measure}}{dS}$$

$$\forall\, u \in H^2(\Omega),\ v \in H^1(\Omega)$$

If u_j solves $(**)$ then $-\Delta u_j = \mu_j u_j$ (elliptic regularity)

$\Rightarrow$ LHS $= 0$ in Green's identity,

$$\Rightarrow \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu}\, \overline{v}\, dS = 0 \quad \forall\, v \in H^1(\Omega)$$

$\Rightarrow$ ($v \in H^1(\Omega)$ arbitrary) $\dfrac{\partial u_j}{\partial \nu} = 0$ on $\partial\Omega$.

·) $\tilde{\mu}_j = \mu_j + 1 > 0$, what about μ_j?

$$\mu_j = \frac{\int_\Omega |\nabla u_j|^2\, dx}{\int_\Omega |u_j|^2\, dx}$$ from $(**)$ with $v = u_j$.

·) $\mu_1 = 0$ with $u_1 \equiv \dfrac{1}{|\Omega|^{1/2}}$

$1 \overset{!}{=} \int_\Omega |u_1|^2 dx = \int_\Omega \dfrac{1}{|\Omega|^{2/2}} dx$

Remark : - Exact same arguments hold if one replaces $-\Delta$ by a diff. op.

$$-\sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}\left(a_{jk} \frac{\partial}{\partial x_k} u\right)$$

with

·) a_{jk} bdd, smooth fcts. on $\overline{\Omega}$

·) $a_{jk} = \overline{a_{kj}}$ (symmetry)

·) $\displaystyle\sum_{j,k=1}^{d} a_{jk}(x)\, \xi_j \overline{\xi_k} \geq \varepsilon |\xi|^2 , \; x \in \overline{\Omega}, \; \xi \in \mathbb{C}^d$

(elliptic)

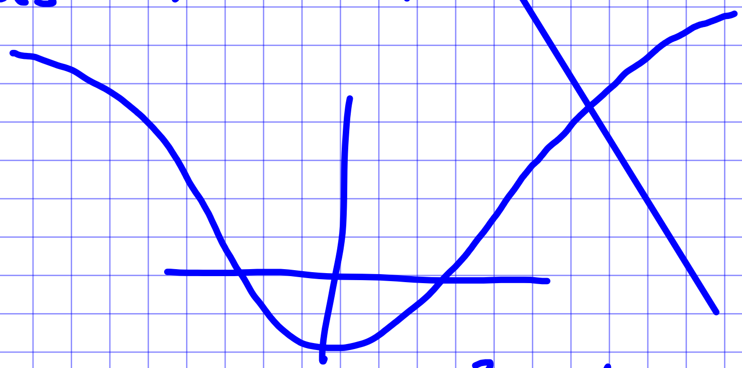
$[\, a_{jk} \equiv \delta_{jk} \,]$

Schrödinger operators with potential growing to $+\infty$

$$(-\Delta + V)u = Eu \quad \text{in } \mathbb{R}^d$$

where $-C \le V(x) \longrightarrow +\infty$ as $|x| \to +\infty$.

for some $C > 0$.



Use discrete spectral thm. with

- $\mathcal{X} = L^2(\mathbb{R}^d)$,
- $\mathcal{K} = H^1(\mathbb{R}^d)$
  $\cap\, L^2(\mathbb{R}^d, |V|\,dx) \subset L^2(\mathbb{R}^d)$

  $\Leftrightarrow \int_{\mathbb{R}^d} |u^2|\, V\, dx < \infty$

- inner product on $\mathcal{K}$:

$$(u,v)_{\mathcal{K}} = \int_{\mathbb{R}^d} \left(\nabla u \cdot \overline{\nabla v} + (1+|V|)\, u \bar{v}\right) dx$$

·) Embedding of $\mathcal{K}$ into $\mathcal{X}$ compact:

·Let $\{f_k\}$ bdd. in $\mathcal{K}$, say $\|f_k\|_{\mathcal{K}} \le M$.

- $\{f_k\}$ is bdd. in $H^1(B(R))$ for any $R > 0$

  ↑ ball of radius R centered at 0

Rellich: $\{f_k\}$ has subsequence converging in $L^2(B(1))$

$\Rightarrow$ this subseq. has a subs. conv. in $L^2(B(2))$

$\vdots$

diagonal sequence conv. in $L^2(B(R))$ $\forall R \in \mathbb{N}$.

Let $\varepsilon > 0$. As $V(x) \to +\infty$, $\exists R$ s.t. $V(x) \geq \frac{1}{\varepsilon}$ when $|x| \geq R$.

$$\Rightarrow \int_{\mathbb{R}^d \setminus B(R)} |f_{k_\ell}|^2 dx \leq \varepsilon \int_{\mathbb{R}^d \setminus B(R)} |f_{k_\ell}|^2 V dx$$

$$\leq \varepsilon \, \|f_{k_\ell}\|_{\mathcal{X}}^2 \leq \varepsilon M^2 \quad \forall \ell.$$

As $\{f_{k_\ell}\}$ converges in each $L^2(B(R))$ we get

$$\limsup_{\ell, m \to \infty} \|f_{k_\ell} - f_{k_m}\|_{L^2(\mathbb{R}^d)} = \limsup_{\ell, m \to \infty} \|f_{k_\ell} - f_{k_m}\|_{L^2(\mathbb{R}^d \setminus B(R))}$$

$$\leq 2\sqrt{\varepsilon}\, M.$$

$\Rightarrow$ $\{f_{k_\ell}\}$ Cauchy sequence in $L^2(\mathbb{R}^d)$ $\Rightarrow$ has a limit in $L^2(\mathbb{R}^d)$.

- $a(u,v) := \int_{\mathbb{R}^d} (\nabla u \cdot \overline{\nabla v} + V u \bar{v}) dx + (2C+1) \int_{\mathbb{R}^d} u \bar{v}\, dx$

$u, v \in \mathcal{X}$

•) a sesquilinear, symmetric, bdd. clear

•) a elliptic: $\boxed{V + 2C + 1 \geq 1 + |V|}$

(clear where $V \geq 0$, otherwise $|V| = -V \leq C$:

$\Rightarrow V + 2C + 1 \geq V - 2V + 1 = |V| + 1$)

$\Rightarrow a(u,u) \geq \|u\|_{\mathcal{X}}^2$

$\Rightarrow \exists$ ON-basis $\{u_j\}$ of $L^2(\mathbb{R}^d)$ and pos. $\gamma_j =: E_j + 2C + 1$:

$$\int_{\mathbb{R}^d} (\nabla u_j \cdot \overline{\nabla v} + V u_j \bar{v}) dx = E_j \int_{\mathbb{R}^d} u_j \bar{v}\, dx \quad \forall v \in \mathcal{X}.$$

$\Rightarrow \quad -\Delta u_j + V u_j = E_j u_j \quad \forall j.$