Lecture 2: The discrete spectral theorem

Sources: ·) Laugesen-notes, Chapter 4

·) Friedman: Foundations of Modern Analysis (Hilbert spaces)

·) Evans: Partial Differential Equations (Sobolev spaces)

Alternatively: ·) Schmüdgen: Unbounded Self-adjoint Operators on Hilbert space, Chapter 10

Preliminaries I: Hilbert spaces

- $\mathcal{H}$ vector space over $\mathbb{C}$
- $(\cdot,\cdot): \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ scalar product:
  - ·) linear in the first entry: $(\lambda u + \mu v, w) = \lambda (u,w) + \mu (v,w)$
  - ·) symmetric: $(u,v) = \overline{(v,u)}$
  - ·) positive def. $(u,u) \geq 0$ and $(u,u) = 0 \Leftrightarrow u = 0$.
- $\|u\| := \sqrt{(u,u)}$

Definition $(\mathcal{X}, (\cdot,\cdot))$ Hilbert space if it is complete (w.r.t. $\|\cdot\|$), i.e. every Cauchy sequence in $(\mathcal{X}, \|\cdot\|)$ has a limit.

Ex. ·) $\mathbb{R}^d$, $\mathbb{C}^d$ with $\left(\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix} \right) := \sum_{j=1}^{d} x_j \overline{y_j}$

finite-dim.

·) Infinite-dim.: $L^2(\Omega) = \{$equivalence classes of $u : \Omega \to \mathbb{C}$ s.t. $\int_\Omega |u|^2 dx < \infty \}$

with $(u,v)_{L^2} := \int_\Omega u \overline{v} \, dx$

($\Omega \subset \mathbb{R}^d$ domain)

Properties of Hilbert spaces:

- Cauchy-Schwarz inequality: $|(u,v)| \leq \|u\| \cdot \|v\|$

- Orthogonality: $u \perp v \quad :\Leftrightarrow \quad (u,v) = 0$.
- Orthogonal complements:

  For $A \subset \mathcal{H}$, $\quad A^{\perp} := \{ u \in \mathcal{H} : (u,v) = 0 \ \forall v \in A \}$

  always closed linear subspace

- Orthogonal projections P_X on a closed subspace X of $\mathcal{H}$

  ($P_X^2 = P_X$, $\operatorname{ran} P_X = X$, $\ker P_X = X^{\perp}$)

- Fréchet-Riesz theorem: For each $f : \mathcal{H} \to \mathbb{C}$ linear and bdd. ($|f(u)| \leq c\|u\|$) there ex.! $v \in \mathcal{H}$ s.t. $f(u) = (u,v)$ $\forall u \in \mathcal{H}$. Moreover, $\|f\| = \|v\|$

  norm in dual space $\qquad$ norm in $\mathcal{H}$

- Orthonormal systems: $\{e_j\}$ ON-system if $(e_j, e_k) = \delta_{jk}$
- $\{e_j\}$ ON-basis if $\{e_j\}$ ON-system s.t. $\forall u \in \mathcal{H}$ we have a (unique) representation $u = \sum_{j=1}^{\infty} \lambda_j e_j$

(convergence in $\|\cdot\|$), in this case: $\lambda_j = (u, e_j)$.

<u>Ex.</u> $\cdot$) $e_j = \begin{array}{c} \\ \\ \\ \\ \leftarrow j \\ \\ \end{array} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ in $\mathbb{C}^d$ ON-basis

$\cdot$) $u_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$ ON-basis of $L^2(0,\pi)$

$j \in \mathbb{N}$

EF of $-\Delta$ with Dirichlet-BC

$\cdot$) $\mathbb{1}$, normalized versions of $\sin x, \cos x, \sin(2x), \cos(2x), \ldots$

ON-basis in $L^2(0, 2\pi)$

EF of $-\Delta$ with periodic BC

(Expansion of any $u \in L^2(0, 2\pi)$ in this ON-basis is classical Fourier series)

<u>Linear operators</u> $\quad B : \mathcal{X} \to \mathcal{X}$

$\cdot$) linear if $B(\lambda u + \mu v) = \lambda Bu + \mu Bv$

$\cdot$) bdd. if $\exists\, C > 0 : \|Bu\| \leq C\|u\|$

$\cdot$) self-adjoint if $(Bu, v) = (u, Bv)$

·) compact if for each bdd. sequence (u_n) in $\mathcal{X}$, (Bu_n) has a convergent subsequence.

Spectral theorem for self-adjoint, compact operators

Let $B: \mathcal{X} \to \mathcal{X}$ linear, self-adjoint, compact (implying bdd.). Then there ex. an ON-basis $\{u_j\}$ of $\mathcal{X}$ and real numbers γ_j with $\gamma_j \to 0$ s.t.

$$B u_j = \gamma_j u_j \quad \forall j.$$

(u_j EV corr. to E-values γ_j)

Preliminaries II: Sobolevspaces

Def.: Let $u \in L^2(\Omega)$. Then $v \in L^2(\Omega)$ is called weak derivative of u in direction j ($v = \partial_j u = \frac{\partial u}{\partial x_j}$) if

$$\int_\Omega u \, \partial_j \varphi \, dx = - \int_\Omega v \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

compact support

- differentiable fcts. are weakly diffb.
- Ex. $u(x) = |x|$ on $(-1,1)$ is weakly diffb.
with $u'(x) = \operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

but sgn on $(-1,1)$ is not weakly diffb.

(Thumb rule: "corners" are weakly diffb., jumps are not)

Def. Sobolev space of order $k \in \mathbb{N}$:

$$H^k(\Omega) := \{ u \in L^2(\Omega) : \text{all weak derivatives up to order } k \text{ ex. in } L^2(\Omega) \}$$

Ex. $u(x) = |x|$ on $(-1,1)$ belongs to $H^1(-1,1)$ but not $H^2(-1,1)$.

- With $(u,v)_{H^k} := \sum_{|\alpha| \le k} (D^\alpha u, D^\alpha v)_{L^2}$

(Notation: $\alpha \in \mathbb{N}_0^d$, $D^\alpha = \prod_{j=1}^d \frac{\partial^{\alpha_j}}{\partial x^{\alpha_j}}$, $|\alpha| = \sum_{j=1}^d \alpha_j$)

E.g. For $\alpha = (2,0,3)$: $D^\alpha u = \frac{\partial^5 u}{\partial x_1^2 \, \partial x_3^3}$

- $H_0^1(\Omega) := \overline{C_0^\infty(\Omega)}$ (closure w.r.t. $\|\cdot\|_{H^1}$)
- $\exists\; T: H^1(\Omega) \to L^2(\partial\Omega)$ bdd., lin. s.t. $Tu = u|_{\partial\Omega}$ for u smooth on $\overline{\Omega}$ ("T trace op.") if Ω smooth enough
- $H_0^1(\Omega) = \{ u \in H^1(\Omega) : \underbrace{Tu}_{u|_{\partial\Omega}} = 0 \}$
- $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$, in part. Hilbert space with $\|u\|_{H^1} = \sqrt{(u,u)_{H^1}}$
- $H^k(\Omega)$ and $H_0^1(\Omega)$ are <u>dense</u> in $L^2(\Omega)$ (i.e. each $u \in L^2(\Omega)$ can be approximated by $(u_n) \subset H_0^1(\Omega)$ w.r.t. $\|\cdot\|_{L^2}$)

Discrete spectral theorem

Preview: $-\Delta u = \lambda u$ with $u|_{\partial\Omega} = 0$ (Dirichlet)

can be multiplied by $v \in H_0^1(\Omega)$:

$$\int_\Omega \nabla u \cdot \nabla v\,dx = -\int_\Omega \Delta u\, v\,dx = \lambda \int_\Omega u\, v\,dx \qquad \forall\, v \in H_0^1(\Omega)$$

(weak formulation of EV problem)

Abstract setting: Let $\mathcal{K}, \mathcal{H}$ Hilbert spaces

- ·) $\mathcal{K}$ continuously embedded into $\mathcal{H}$, i.e. $\mathcal{K} \subset \mathcal{H}$,
  $\|u\|_{\mathcal{H}} \leq \|u\|_{\mathcal{K}} \quad \forall\, u \in \mathcal{K}$
- ·) embedding is compact: each sequence $(u_n) \subset \mathcal{K}$ bdd. w.r.t. $\|\cdot\|_{\mathcal{K}}$ has a convergent subsequence w.r.t. $\|\cdot\|_{\mathcal{H}}$
- ·) $a : \mathcal{K} \times \mathcal{K} \to \mathbb{C}$ scalar product with

.) a bounded : $|a(u,v)| \le C \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}$

.) a elliptic : $a(u,u) \geq C \|u\|_{\mathcal{X}}^2$

In part., $\|\cdot\|_{\mathcal{X}}$ is equivalent to $\sqrt{a(\cdot,\cdot)}$

[Think of $\mathcal{H} = L^2(\Omega)$, $\mathcal{X} = H_0^1(\Omega)$, $a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx$]

Spectral theorem Under the above assumptions,

$\exists\, u_1, u_2, \ldots, \in \mathcal{X}$ and $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty$ s.t.

.) $a(u_j, v) = \lambda_j (u_j, v)_{\mathcal{H}} \quad \forall v \in \mathcal{X}$

.) $\{u_j\}$ ON-basis in $\mathcal{H}$

.) $\left\{ \frac{u_j}{\sqrt{\lambda_j}} \right\}$ ON-basis of $(\mathcal{X}, a(\cdot,\cdot))$.

<u>Proof</u> ① For each $f \in \mathcal{X}$ $\exists!\, u \in \mathscr{X}$ s.t.

$$a(u,v) = (f,v)_{\mathcal{X}} \quad \forall\, v \in \mathscr{X}.$$

"unique solvability of $-\Delta u = f$"

Indeed, consider $F : \mathscr{X} \to \mathbb{C}$, $F(v) := (v,f)_{\mathcal{X}}$.

Then F is linear,

$$|F(v)| \leq \|v\|_{\mathcal{X}} \|f\|_{\mathcal{X}} \overset{\text{cont. emb.}}{\leq} C \|v\|_{\mathscr{X}} \|f\|_{\mathcal{X}}$$

$$\overset{a \text{ elliptic}}{\leq} \tilde{C} \sqrt{a(v,v)}\, \|f\|_{\mathcal{X}}$$

$\Rightarrow F : (\mathscr{X}, \sqrt{a(\cdot,\cdot)}) \to \mathbb{C}$ bdd.

Fréchet - Riesz : $\exists!\, u \in \mathscr{X} :\ a(u,v) = F(v) = (f,v)_{\mathcal{X}}$ $\forall\, v \in \mathscr{X}$.

In part., $\tilde{B} : \mathcal{X} \to \mathscr{X}$, $f \mapsto u$, is well-defined, linear, and

$$a(\tilde{B} f, \tilde{B} f) = a(u,u) = F(u) \leq \tilde{C} \sqrt{a(u,u)}\, \|f\|_{\mathcal{X}}$$

$\Rightarrow$ $\tilde{B} : \mathcal{X} \to \mathscr{X}$ bdd.