Lecture 1: Computable spectra

Sources: •) Laugesen lecture notes "Spectral theory of PDEs" Chapters 2/3

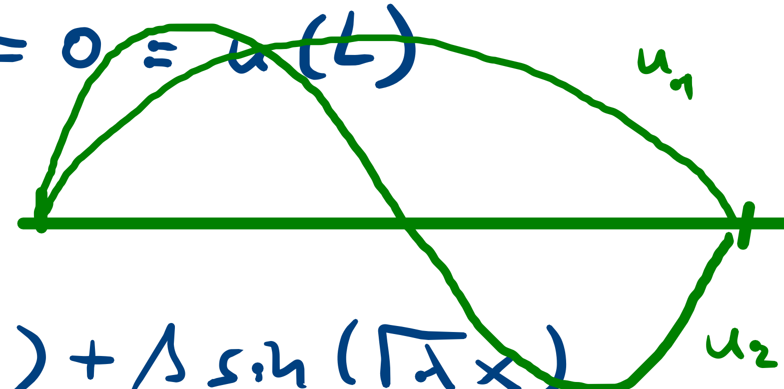
•) Pinchover/Rubinstein "An introduction to Partial Differential Equations" Sec. 9.5

Laplacian $\quad -\Delta u = -\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \dots - \frac{\partial^2 u}{\partial x_d^2}$

$\Omega \subset \mathbb{R}^d$, $d \geq 1$, domain (= open, connected)

1. Laplacian on an interval $\quad \Omega = (0, L)$

(a) Dirichlet-BC $\quad u(0) = 0 = u(L)$



$$-u'' = \lambda u$$

Sol. $\quad u(x) = \alpha \cos(\sqrt{\lambda}\, x) + \beta \sin(\sqrt{\lambda}\, x)$

BC $\quad 0 \overset{!}{=} u(0) = \alpha$, $\quad 0 \overset{!}{=} u(L) = \beta \sin(\sqrt{\lambda}\, L)$

$\Leftrightarrow \sin(\sqrt{\lambda}\, L) = 0 \Leftrightarrow \sqrt{\lambda}\, L \in \mathbb{Z}\pi$

$\Leftrightarrow \lambda_k = \frac{k^2 \pi^2}{L^2}$, $\quad k \in \mathbb{Z}$.

EF $\quad u(x) = \sin\left(\frac{k\pi}{L} x\right)$

(b) Neumann BC $u'(0) = 0 = u'(L)$

$$\mu_k = \frac{(k-1)^2}{L^2}\pi^2 \,, \quad k \geq 1$$

(0 is EV with EF $u(x) \equiv$ const)

$$u_k(x) = \cos\left(\frac{k-1}{L}\pi x\right) \,, \quad k \geq 2$$

Observations:

- ∞ many EV converging to $+\infty$, discrete
- quadratic in k
- EF corr. to λ_k has $k-1$ zeroes inside Ω
- scale quadratically in L, decrease if L increases

(c) Periodic BC

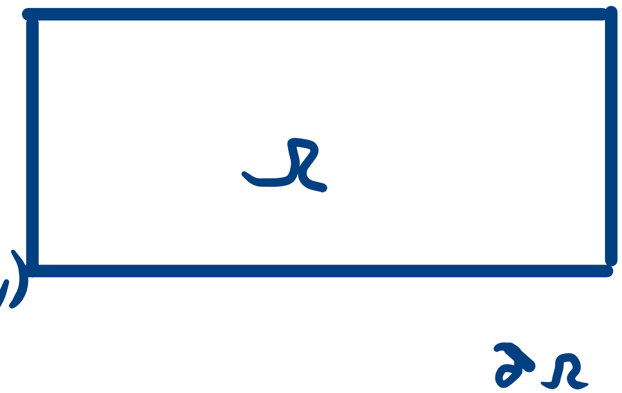
(Identify $(0,L)$ with circle of length L)

$$\begin{cases} u(0) = u(L) \\ u'(0) = u'(L) \end{cases}$$

$\Omega = (0, 2\pi)$, EV $0, 1^2, 1^2, 2^2, 2^2, 3^2 \ldots$

2. Laplacian on a rectangle $\Omega = (0,L)\times(0,M)$

Separate variables: ansatz $u(x,y) = \varphi(x)\psi(y)$



(a) Dirichlet-BC: $u|_{\partial\Omega} = 0$

$$\lambda\, \varphi(x)\psi(y) = -\varphi''(x)\psi(y) - \varphi(x)\psi''(y)$$

$$\Leftrightarrow \quad \underbrace{\lambda}_{\text{const}} = \underbrace{-\frac{\varphi''}{\varphi}}_{\text{dep. on } x} \underbrace{- \frac{\psi''}{\psi}}_{\text{dep. on } y}$$

$\Rightarrow \exists\, \mu, \nu$ const.: $\quad -\varphi'' = \mu\varphi, \qquad -\psi'' = \nu\psi$

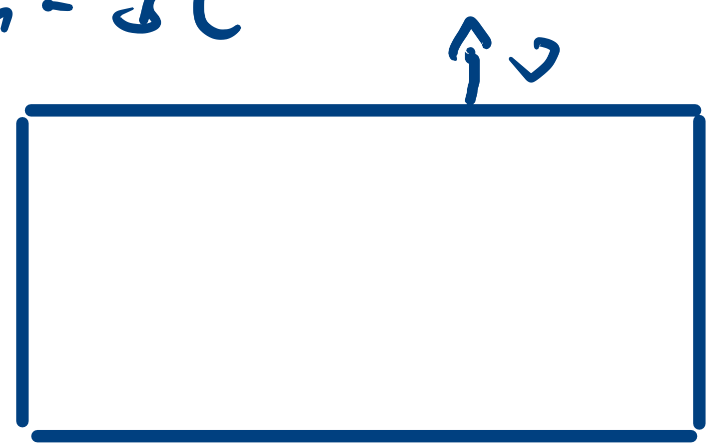
$$\varphi(0) = 0 = \varphi(L) \qquad \psi(0) = 0 = \psi(M)$$

EV: $\lambda = \mu + \nu = \frac{k^2\pi^2}{L^2} + \frac{j^2\pi^2}{M^2}, \quad k, j = 1, 2, \dots$

EF: $u(x,y) = \sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{j\pi}{M}y\right), \quad k, j = 1, 2, \dots$

Similar for Neumann-BC

$\partial_\nu u \big|_{\partial\Omega} = 0$



$$\frac{k^2\pi^2}{L^2} + \frac{l^2\pi^2}{M^2}$$

Observations:

·) $L = M = \pi$:

Dirichlet: $1+1=2$, $1+4=5$, $4+1=5$,

$4+4=8$, $1+9=10$, $9+1=10$, ...

$\exists$ EV of higher multiplicities

lowest EV has multiplicity one

·) EV discrete, converge $+\infty$

Asymptotics?

Proposition (Weyl's law for rectangles)

Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$ Dirichlet Laplacian EV,

$0 = \mu_1 \leq \mu_2 \leq \mu_3 \cdots$ Neumann Laplacian EV

on $(0, L) \times (0, M)$. Then

$$\lambda_j \sim \mu_j \sim \frac{4\pi j}{|\Omega|} \qquad \text{as } j \to +\infty,$$

where $|\Omega|$ is the area of Ω, $|\Omega| = L \cdot M$.

Proof (Dirichlet case)

$$N(\alpha) := \#\{\text{EV below / equal } \alpha\}$$

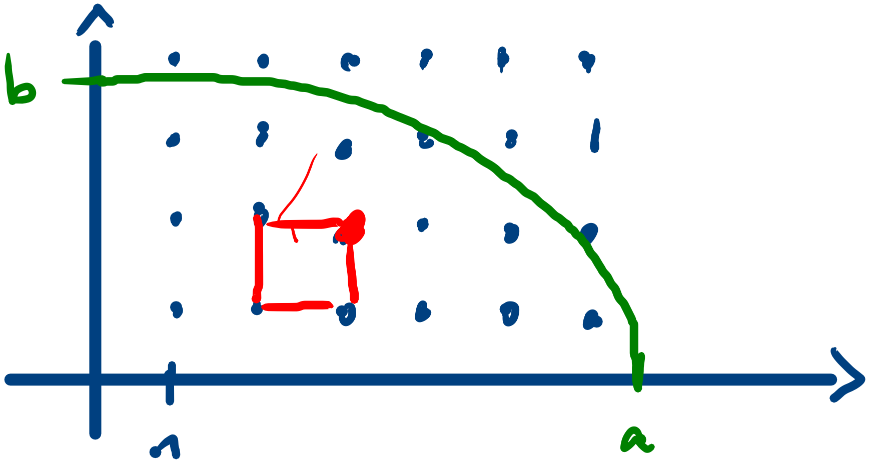
$$= \#\left\{(j,k)\in\mathbb{N}\times\mathbb{N} : \frac{j^2\pi^2}{L^2} + \frac{k^2\pi^2}{M^2} \le \alpha\right\}$$

$$= \#\left\{(j,k) : \frac{j^2}{\alpha\frac{L^2}{\pi^2}} + \frac{k^2}{\alpha\frac{M^2}{\pi^2}} \le 1\right\}$$

$$= \#\{(j,k) : (j,k)\in E\}$$

where E is the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \le 1$

with $a = \sqrt{\alpha}\,\frac{L}{\pi}$, $b = \sqrt{\alpha}\,\frac{M}{\pi}$.

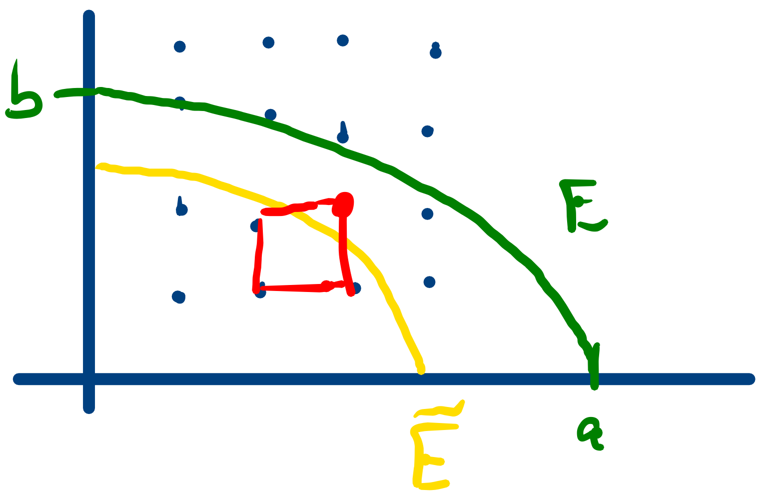


To each (j,k) associate the square

$$S(j,k) := (j-1,j) \times (k-1,k)$$

$$\Rightarrow \quad N(\alpha) = \sum_{\substack{(j,k)\in\mathbb{N}\times\mathbb{N},\\ (j,k)\in E}} |S(j,k)| \leq \text{area of } E \text{ in 1. quadr.}$$

$$= \frac{1}{4}\pi a b = \frac{1}{4}\cancel{\pi}\,\alpha\,\frac{LM}{\pi^{\cancel{2}}} = \frac{|\Omega|}{4\pi}\alpha\,.$$



Let $\tilde{E}$ be E shifted one unit left and one unit down.

Then $$N(\alpha) = \sum_{(j,k)\in E\cap(\mathbb{N}\times\mathbb{N})} |S(j,k)|$$

$$\geq \text{area of } \tilde{E} \text{ in 1. quadrant}$$

$$\geq \frac{1}{4}\pi ab - a - b = \frac{|\Omega|}{4\pi}\alpha - \frac{L+M}{\pi}\sqrt{\alpha}$$

$$= \frac{|\Omega|}{4\pi}\alpha - \frac{Per(\Omega)}{2\pi}\sqrt{\alpha},$$ $Per(\Omega)$ perimeter

$$\Rightarrow \quad \underbrace{N(\alpha)}_{\approx j \text{ maybe } > j} \sim \frac{|\Omega|}{4\pi}\alpha \quad \text{as } \alpha \to +\infty.$$

Set $\alpha = \lambda_j$.

$$j \sim \frac{|\Omega|}{4\pi}\lambda_j$$

$$\Leftrightarrow \quad \lambda_j \sim \frac{4\pi j}{|\Omega|}$$

$\square$

$L = M = \pi$:

$\lambda_1 = 2$

$\lambda_2 = \lambda_3 = 5$

$N(5) = 3$

$N(\lambda_2) = 3$

Remark

$\Omega = (0, L) \times (0, M)$

If L fixed, $M \searrow 0$ then

D $\quad \lambda_1 = \dfrac{\pi^2}{L^2} + \dfrac{\pi^2}{M^2} \longrightarrow +\infty$

N $\quad \mu_2 = \dfrac{\pi^2}{L^2}$ indep. of M

first nontrival EV

Neumann:

$$\frac{(j-1)^2\pi^2}{L^2} + \frac{(k-1)^2\pi^2}{M^2}$$

3. Laplacian on a disk (sketch)

$$\Omega = \{ x^2 + y^2 < R^2 \}$$

Idea: Write EQ in polar coordinate

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\vartheta\vartheta} = -\lambda u, \quad 0 < r < R, \ 0 \leq \vartheta \leq 2\pi$$

$$u(R, \vartheta) = 0, \qquad 0 \leq \vartheta \leq 2\pi.$$

Separate variables, ansatz $u(r, \theta) = \psi(r)\phi(\vartheta)$

$$-\phi''(\vartheta) = \mu\,\phi(\vartheta)$$

EV $n^2 : n \in \mathbb{N}$

$$\left.\begin{aligned} \phi(0) &= \phi(2\pi) \\ \phi'(0) &= \phi'(2\pi) \end{aligned}\right\} \text{periodic BC}$$

$$\psi''(r) + \frac{1}{r}\psi'(r) + \left(\lambda - \frac{\mu}{r^2}\right)\psi(r) = 0, \quad 0<r<R$$

$$\lim_{r\to 0} \psi(r) \text{ finite} \qquad \psi(R) = 0$$

Substitute $s = \sqrt{\lambda}\, r$ to get

$$(s\psi')' + \left(s - \frac{n^2}{s}\right)\psi = 0, \quad 0<s<\sqrt{\lambda}R.$$

(Bessel eq.)

EV: $\left(\frac{j_{n,m}}{R}\right)^2, \quad n \geq 0, \quad m \geq 1$

where $j_{n,m}$ = m-th positive root of the Bessel fct. J_n of order n.