# HOLMGREN'S UNIQUENESS THEOREM AND SUPPORT THEOREMS FOR REAL ANALYTIC RADON TRANSFORMS

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**1. Introduction.** If $f$ is a continuous function on $\mathbf{R}^n$ decaying at infinity faster than any negative power of $|x|$, and the integral of $f$ is zero over all hyperplanes not intersecting a given compact convex set $K$, then $f$ must vanish outside $K$; this is the well-known support theorem of Helgason [He1], [He2]. Here we shall study the corresponding problem when $f$ is only assumed to decay fast in a certain open cone $\Gamma$ in $\mathbf{R}^n$. It turns out that $f$ must then vanish in the set

$$\bigcap_{x \in K} (x + \Gamma \cup (-\Gamma)); \tag{1}$$

see Corollary 3. Examples show that the set (1) is the largest set for which this statement is true. More generally we consider a generalized Radon transform

$$R_\rho f(H) = \int_H f(x)\rho(x, H) ds, \tag{2}$$

assuming the weight function $\rho$ is positive and real analytic and also real analytic at infinity in a certain sense (see Corollary 2); here $ds$ is the Euclidean surface measure on the hyperplane $H$, and integration is to be performed with respect to $x$. It will be convenient to formulate our main theorem in a projective setting; the decay condition then means that $f$ is flat on an open subset of a hyperplane in the projective space $\mathbf{P}^n$.

The proof depends on microlocal regularity properties of solutions to the equation $R_\rho f = 0$ together with a recent vanishing theorem [B2] stating that if $f$ is flat along a real analytic hypersurface $S$ and all conormals to $S$ are absent in the analytic wave front set $WF_A(f)$ of $f$, then $f$ must vanish in some neighborhood of $S$. This theorem is closely related to a well-known vanishing theorem of Hörmander, which is a crucial part of the elegant proof of Holmgren's uniqueness theorem given in [Hö1] (see also [Hö2], section 8.6). As in the proof of Holmgren's theorem, in our proof we need to construct a family of "non-characteristic" surfaces, covering the claimed zero-set of $f$. In our problem the characteristic set consists of all conormals to planes intersecting the set $K$. Support theorems for generalized Radon transforms have been proved by similar methods in [BQ1], [BQ2], [B1], [GQ], [Q2], [Q3].

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**2. The support theorem.** To describe the zero set of $f$ in the projective setting we shall need to introduce the set $\mathrm{sh}_K(E)$, the shadow of $E$ with respect to $K$. Denote the $n$-dimensional real projective space by $\mathbf{P}^n$, and if $x$ and $y$ are two distinct points of $\mathbf{P}^n$, let $G(x,y)$ be the geodesic through $x$ and $y$ with the point $x$ removed. If $E$ is a subset of $\mathbf{P}^n \setminus \{x\}$, we define $P(x,E)$ as the union of all $G(x,y)$ (considered as subsets of $\mathbf{P}^n$) for $y \in E$. If $K$ is a non-empty subset of $\mathbf{P}^n$ and $K \cap E = \emptyset$, we define $\mathrm{sh}_K(E)$ as the intersection

$$\bigcap_{x \in K} P(x,E).$$

Equivalently, $\mathrm{sh}_K(E)$ is equal to the complement in $\mathbf{P}^n \setminus K$ of the union of all geodesics intersecting $K$ but not intersecting $E$. At least in simple cases $\mathrm{sh}_K(E) \setminus E$ consists of the two shadows of $E$ obtained by letting $K$ act as a light source and assuming light propagates along geodesics in one or the other direction. If $E$ is an open set or an open subset of a hyperplane disjoint from $K$, then $\mathrm{sh}_K(E)$ is open. Furthermore, it is easy to see that $\mathrm{sh}_K$ is a hull operation, that is, $\mathrm{sh}_K(E) = \mathrm{sh}_K(\mathrm{sh}_K(E))$ for any $E \subset \mathbf{P}^n \setminus K$.

Let $K$ and $E$ be disjoint subsets of $\mathbf{R}^n$, $K$ non-empty. Identifying $\mathbf{R}^n$ with a subset of $\mathbf{P}^n$ we can define $\mathrm{sh}_K(E)$ as before. In some cases $\mathrm{sh}_K(E)$ will contain points at infinity. Specifically, if $K$ and $E$ are disjoint compact convex subsets of $\mathbf{R}^n$ with nonempty interior, then the restriction to $\mathbf{R}^n$ of $\mathrm{sh}_K(E) \setminus E$ will have two or three components, one between $E$ and $K$, one "behind" $E$ as viewed from $K$, and sometimes one on the opposite side of $K$ from that of $E$. In the second case the last two components are unbounded; if we move up to projective space they are of course parts of the same component.

Let $\mathcal{G}_n$ denote the set of hyperplanes in $\mathbf{P}^n$. We now choose, for every $H \in \mathcal{G}_n$, a measure $d\sigma$ on $H$ as follows. Using a choice of coordinates $(x_0; x_1; \dots; x_n)$ on $\mathbf{P}^n$ we obtain a $2-1$ map $S^n \to \mathbf{P}^n$ and similarly a $2-1$ map $S^{n-1} \to H$ for every $H \in \mathcal{G}_n$. We define $d\sigma$ to be the push-forward of the Euclidean surface measure on $S^{n-1}$ under that map. For a continuous function $f$ on $\mathbf{P}^n$ we can now define our Radon transform $R_\rho$ by

$$R_\rho f(H) = \int_H f(x)\rho(x,H)\,d\sigma, \quad H \in \mathcal{G}_n. \tag{3}$$

The weight function $\rho$ is defined on the manifold $Z$ consisting of all pairs $(x,H)$ where $x \in H$ and $H \in \mathcal{G}_n$. It is clear that $Z$ is a real analytic submanifold of the product space $\mathbf{P}^n \times \mathcal{G}_n$. We shall say that the function $f$ is *flat* on $E \subset \mathbf{P}^n$ if $f(x)$ tends to zero faster than any power of the distance to $E$ as $x$ tends to $E$.

If $L$ is a hyperplane in $\mathbf{P}^n$ and $K$ is a subset of $\mathbf{P}^n \setminus L$ we shall say that $K$ is a *convex* subset of $\mathbf{P}^n \setminus L$ if $K$ is the image of a convex subset of $\mathbf{R}^n$ under some affine bijection $\mathbf{R}^n \to \mathbf{P}^n \setminus L$. This property is obviously independent of the choice of the hyperplane $L$.

**Theorem.** *Assume that the weight function $\rho$ is real analytic and positive on $Z$. Let $L$ be a hyperplane in $\mathbf{P}^n$ and let $K$ be a compact convex subset of $\mathbf{P}^n \setminus L$. Let $f$*

*be continuous on* $\mathbf{P}^n \setminus K$*, and assume* $R_\rho f(H) = 0$ *for all* $H \in \mathcal{G}_n$ *not intersecting* $K$*. Let* $E$ *be an open subset of* $L$ *and assume* $f$ *is flat on* $E$*. Then* $f = 0$ *on* $\mathrm{sh}_K(E)$*.*

*Remark.* It is sufficient to assume that $f$ tends to zero fast as $x$ approaches $E$ from *one side* in some neighborhood of an arbitrary point of $E$; see remark after Proposition 2.

**Corollary 1.** *With the same hypotheses as in the theorem, assume* $f$ *is flat on all of* $L$*, i.e.,* $E = L$*. Then* $f = 0$ *in the complement of* $K$*.*

*Proof.* If $E$ is an entire hyperplane not intersecting $K$, then $\mathrm{sh}_K(E)$ is equal to the complement of $K$.

In the remaining corollaries we shall assume given a real analytic Radon transform $R_\rho$ on $\mathbf{R}^n$. Then $\rho(x, H)$ is defined on the manifold $Z_0$ consisting of all pairs $(x, H)$, $x \in \mathbf{R}^n$, $H$ hyperplane in $\mathbf{R}^n$. An affine bijection $\mathbf{R}^n \to \mathbf{P}^n \setminus L$ for some fixed $L \in \mathcal{G}_n$, for instance the map $\alpha : (x_1, \ldots, x_n) \mapsto (1; x_1; \ldots ; x_n)$, induces an identification of $Z_0$ with a dense subset of our manifold $Z \subset \mathbf{P}^n \times \mathcal{G}_n$. This identification depends of course on the choice of imbedding of $\mathbf{R}^n$ into $\mathbf{P}^n$, but the assumptions in the following corollary do not depend on this choice.

**Corollary 2.** *(Helgason's theorem for real analytic densities,* [B1]*.) Let* $R_\rho$ *be a real analytic Radon transform in* $\mathbf{R}^n$*, and assume* $\rho(x, H)$ *can be extended to a real analytic and positive function on* $Z$*. Let* $K$ *be a compact convex subset of* $\mathbf{R}^n$*. Let* $f$ *be continuous on* $\mathbf{R}^n \setminus K$ *and decaying faster than any negative power of* $|x|$ *as* $|x|$ *tends to infinity, and assume* $R_\rho f(H) = 0$ *for all* $H$ *not intersecting* $K$*. Then* $f = 0$ *outside* $K$*.*

*Proof.* The imbedding $\alpha$ introduced above transforms the measure $ds$ on $H \subset \mathbf{R}^n$ into a measure $\alpha_*(ds) = b(x, H) d\sigma$ on $H \subset \mathbf{P}^n$, where the density $b(x, H)$ factors into a product of one function depending only on $x$ and one depending only on $H$, $b(x, H) = b_0(x) b_1(H)$ (see Lemma 1 in [B1]). Using this fact we showed in [B1] that the assumptions in Corollary 2 allow us to transform the problem to one in $\mathbf{P}^n$, where $f$ is flat on an entire hyperplane in $\mathbf{P}^n$ (for details we refer to [B1]). The conclusion now follows from Corollary 1.

**Corollary 3.** *Let* $R_\rho$ *and* $K$ *be as in Corollary 2. Let* $\Gamma$ *be an open conic subset of* $\mathbf{R}^n$*, let* $f$ *be continuous on* $\mathbf{R}^n \setminus K$ *and decaying enough at infinity to be integrable on hyperplanes, for instance* $f(x) = \mathcal{O}(|x|^{-n})$ *as* $|x| \to \infty$*. Assume moreover that* $f$ *decays faster than any negative power of* $|x|$ *as* $x$ *tends to infinity in* $\Gamma$*. Assume finally* $R_\rho f(H) = 0$ *for all* $H$ *not intersecting* $K$*. Then* $f = 0$ *in the set (1).*

*Proof.* Imbedding $\mathbf{R}^n$ in $\mathbf{P}^n$ as in Corollary 2 we obtain the situation in the theorem, or in the remark after it, with $L = H_\infty$, the hyperplane at infinity, and $E = H_\infty \cap \overline{\Gamma}$, where $\overline{\Gamma}$ is the closure of $\Gamma$ in $\mathbf{P}^n$. In this case $\mathrm{sh}_K(E)$ is the set (1).

**Corollary 4.** *Let* $R_\rho$ *and* $K$ *be as in Corollary 2. Let* $f$ *be continuous and decaying enough at infinity to be integrable on hyperplanes, for instance* $f(x) = \mathcal{O}(|x|^{-n})$ *as* $|x| \to \infty$*, and assume* $R_\rho f(H) = 0$ *for all* $H$ *not intersecting* $K$*. Let* $L$ *be a hyperplane in* $\mathbf{R}^n$ *not intersecting* $K$*, let* $E$ *be an open subset of* $L$*, and assume* $f$ *is flat on* $E$*. Then* $f = 0$ *in* $\mathrm{sh}_K(E)$*.*

*Proof.* Imbedding $\mathbf{R}^n$ in $\mathbf{P}^n$ as in Corollary 2 we obtain the statement immediately from the theorem.

*Remark.* The function $f$ in the theorem and the corollaries may be any distribution on $\mathbf{P}^n$, provided the flatness condition used here is replaced by an appropriate flatness condition valid for distributions (cf. Proposition 2).

**3. The microlocal regularity theorem.** Our Radon transform can be written

$$R_\rho f(H) = \int K(x,H) f(x) dx,$$

where the distribution $K(x,H)$ on $\mathbf{P}^n \times \mathcal{G}_n$ is a measure supported on the hypersurface $Z$, in fact a smooth positive density on that surface. The wave front set of $K$ is therefore contained in the conormal manifold $N^*(Z) \subset T^*(\mathbf{P}^n \times \mathcal{G}_n)$ to $Z$. The corresponding operator is an especially simple kind of Fourier integral operator. Using the natural identification $T^*(\mathbf{P}^n \times \mathcal{G}_n) \simeq T^*(\mathbf{P}^n) \times T^*(\mathcal{G}_n)$ we can consider the wave front set of $K$, $WF(K)$, as a subset of $T^*(\mathbf{P}^n) \times T^*(\mathcal{G}_n)$. It is a trivial fact that (cf. [Hö2], ch. 8.2)

$$WF(R_\rho f) \subset \{(H,\eta);\ (x,\xi) \in WF(f),\ (x,\xi;H,-\eta) \in WF(K)\}. \tag{4}$$

Using the facts that $\rho > 0$ and that $Z$ has a certain geometric property ($N^*(Z)$ is the graph of an injective map $T^*(\mathbf{P}^n) \mapsto T^*(\mathcal{G}_n)$; see [GS]), one can also prove the opposite inclusion; in other words we have equality in (4). Since the manifold $Z$ and the function $\rho$ are real analytic these facts also hold with $WF$ replaced by $WF_A$ (see [B1], section 4 and references given there). The manifold $N^*(Z)$ consists of the set of pairs $(x,\xi;H,\eta)$ such that $x \in H$, $\xi$ is conormal to $H$, $\eta$ is conormal at $H$ to the hypersurface $\gamma_x = \{L; x \in L\}$ in $\mathcal{G}_n$, and $\xi$, $\eta$ are coupled by a certain linear relation; to see this one can locally represent $Z$ as $F(x,H) = 0$ for some smooth function $F$ with non-vanishing gradient and note that $N^*(Z)$ then becomes $\{(x, \lambda F'_x, H, \lambda F'_H);\ F(x,H) = 0, \lambda \in \mathbf{R}\}$. These facts imply the following regularity theorem.

**Proposition 1.** *Let $f$ be continuous on $\mathbf{P}^n$, or, more generally, let $f \in \mathcal{D}'(\mathbf{P}^n)$, the space of distributions on $\mathbf{P}^n$. Assume $\rho(x,H)$ is real analytic and positive on $Z$ and that $R_\rho f(H) = 0$ for all $H$ in some neighbourhood of $H_0 \in \mathcal{G}_n$. Then*

$$N^*(H_0) \cap WF_A(f) = \emptyset.$$

**4. Vanishing theorems for microanalytic distributions.** Assume for a moment that we knew about our function $f$ that $WF_A(f) \cap N^*(S) = \emptyset$ for some *closed* real analytic surface $S$, for instance a sphere in $\mathbf{R}^n$ with radius 1, and that $f$ is flat along $S$. Then it would follow almost immediately from the definition of the analytic wave front set that the one-variable function $u(t)$ defined as the integral of $f$ over a family of concentric spheres $S_t$ with radius $t$ must be real analytic and flat at $t = 1$, hence vanish identically near that point. Since the same would be true with $f$ replaced by $fh$ for an arbitrary real analytic function $h$, it would follow that $f = 0$ in a neighborhood of $S$. In [B1], where we assumed $f$ tends to zero fast in all directions at infinity, we had that situation. In this paper, however, our function is only assumed to be flat along some open piece of a hyperplane in $\mathbf{P}^n$; it is clear that the argument just outlined does not apply then. However, the vanishing theorem that we need is actually true [B2]. Note that the restriction to a smooth submanifold $S \subset \mathbf{R}^n$ of a distribution $f$ defined in some neighborhood of $S$ is well defined if $N^*(S) \cap WF(f)$ is empty.

**Proposition 2,** [B2]**.** *Let $f$ be a distribution defined in some neighborhood of the real analytic hypersurface $S \subset \mathbf{R}^n$, and assume*

$$N^*(S) \cap WF_A(f) = \emptyset. \tag{5}$$

*Assume the restrictions to $S$ of $f$ and all its derivatives vanish, or if $f$ is continuous, assume equivalently that $f(x)$ tends to zero faster than any power of the distance from $x$ to $S$ as $x$ tends to $S$. Then $f = 0$ in some neighbourhood of $S$.*

*Remarks.* If $f$ is continuous, it is sufficient to assume $f(x)$ tends to zero fast as $x$ approaches $S$ from one side. If $f$ is assumed to *vanish* in a one-sided neighborhood of the surface $S$, it is sufficient to assume $S$ is $C^1$ and piecewise $C^2$, because then we can find a ball inside the zero-set of $f$, tangent to $S$ at an arbitrary point of $S$; this observation will be useful in the proof below. In this case it is also sufficient, by Hörmander's vanishing theorem referred to above, to assume *one* of the two conormals $(x, \pm\xi)$ absent in $WF_A(f)$. For related and sharper theorems, first proved by Kashiwara, see [Hö2], ch. 9.6. Those theorems are valid for hyperfunctions; however, the assertion of Proposition 2 is *not* valid for hyperfunctions (see [K], note 3.3).

**5. Proof of the theorem.** Let $f$ satisfy the assumptions of the theorem. Since the hyperplane $L$ is disjoint from $K$, we infer immediately from Proposition 1 that $N^*(L) \cap WF_A(f)$ is empty. Working locally near an arbitrary point of $E$ we can then conclude from Proposition 2 that $f$ must vanish in some neighborhood of $E$ in $\mathbf{P}^n$. To prove that $f$ must vanish on $\mathrm{sh}_K(E)$ we are going to reduce our problem to an especially simple situation in $\mathbf{R}^n$ by choosing a convenient hyperplane $M \subset \mathbf{P}^n$ and using $\mathbf{R}^n \cup H_\infty$ as a model for $\mathbf{P}^n$ with $M = H_\infty$. Let $z$ be an arbitrary point of $\mathrm{sh}_K(E) \setminus L$. Since $K$ is a convex subset of $\mathbf{P}^n \setminus L$ and $z \notin K$, we can find a hyperplane $H$ through $z$ that does not meet $K$. Let $\mathcal{J}$ be the set of hyperplanes containing $H \cap L$, that are different from $L$ and do not intersect $K$. Then $\mathcal{J}$ is a one-dimensional manifold with two components; one of those components contains no plane through $z$. Choose the hyperplane $M$ in that component and represent $\mathbf{P}^n$ as indicated above. Then $K$ is a compact convex subset of $\mathbf{R}^n \subset \mathbf{R}^n \cup H_\infty$. Choose a linear form $\phi$ on $\mathbf{R}^n$ which vanishes on $L \cap \mathbf{R}^n$ and is positive on $K$. The planes in $\mathcal{J}$ now take the form $\phi(x) = c$, and our choice of $M$ is easily seen to imply that $z$ lies between $L$ and $K$, i.e., that $0 < \phi(z) < \inf_{x \in K} \phi(x)$. Hence the set $F_z = L \cap P(z, K)$ must be a bounded subset of $L \cap \mathbf{R}^n$. We shall prove that $f = 0$ near $z$ by constructing a family of surfaces filling a region between $E$ and $z$ and whose tangent planes are disjoint from $K$.

From now on all our reasoning will take place in $\mathbf{R}^n$ and we shall denote $L \cap \mathbf{R}^n$ by $L$. Since $F_z$ is a convex compact subset of $E$ we can take a bounded convex set $V$, open in $L$, containing $F_z$, with closure $\overline{V}$ contained in $E$, and with smooth boundary $\partial_L(V)$ as a subset of $L$. Since $z \in \mathrm{sh}_K(E)$, any line through $z$ and $\partial_L(V)$

must be disjoint from $K$, and since $V$ is convex, any hyperplane through $z$, tangent to $\partial_L(V)$, must be disjoint from $K$. Choose a ball $B_\varepsilon(z)$ with center at $z$ and radius $\varepsilon$ so small that all common tangent planes to $\partial_L V$ and $B_\varepsilon(z)$ are also disjoint from $K$, and $\varepsilon$ is smaller than the distance from $z$ to $L$. Let $z_0 \in V$, set $z_t = z_0 + t(z - z_0)$ for $0 < t \le 1$, and let $\Sigma_t$ be the boundary of the convex hull of $B_{\varepsilon t}(z_t) \cup V$ with all points on $L$ removed. Then each $\Sigma_t$ is piecewise smooth, and using the fact that $B_{\varepsilon t}(z_t)$ is contained in the convex hull of $B_\varepsilon(z) \cup V$ it is easy to see that $\Sigma_t$ is non-characteristic in the sense that all tangent planes to $\Sigma_t$ are disjoint from $K$.

Assume now that $z \in \operatorname{supp} f$. Since $\overline{V} \subset E$, there must be a $\bar{t}$, $0 < \bar{t} \le 1$, such that $\Sigma_t$ is disjoint from the support of $f$ for $t < \bar{t}$ and $\Sigma_{\bar{t}}$ meets the support of $f$. By Proposition 1 and the remark following Proposition 2 this is impossible; hence we have obtained a contradiction and can conclude that $z \notin \operatorname{supp} f$. Since $z$ was an arbitrary point of $\mathrm{sh}_K(E)$ we have proved that $f = 0$ in $\mathrm{sh}_K(E)$.

**6. Counterexamples.** To show that the decay assumption in Helgason's original theorem cannot be omitted Helgason gave the examples $\mathfrak{Re}(x_1 + ix_2)^{-m}$, $m = 2, 3, \ldots$. More generally, Quinto characterized the null space of the Radon transform on (weighted) $L^2$-space on the complement of a ball in $\mathbf{R}^n$, [Q1]. Helgason's examples can also be generalized as follows. Let $g(x)$ be a distribution in $\mathbf{R}^n$ which is positively homogeneous of degree $-n-k$ (as a distribution in $\mathbf{R}^n$), $k$ integer $\ge 0$, and $C^\infty$ in $\mathbf{R}^n \setminus \{0\}$. Furthermore we assume $g$ is even if $k$ is even and odd if $k$ is odd. Write $Rf(\omega, p) = Rf(H_{(\omega,p)})$, where $H_{(\omega,p)}$ is the hyperplane $\{x;\, x\cdot\omega = p\}$, $(\omega, p) \in S^{n-1} \times \mathbf{R}$. The $n$-dimensional Fourier transform $\widehat{g}$ of $g$ is $C^\infty$ outside the origin and homogeneous of degree $k$, hence a locally bounded function. The one-dimensional Fourier transform of $Rg(\omega, p)$ with respect to $p$, which we denote by $\widehat{Rg}(\omega, \tau)$, is connected with $g$ by the familiar formula $\widehat{Rg}(\omega, \tau) = \widehat{g}(\tau\omega)$. Since $\widehat{g}$ is even if $k$ is even and odd if $k$ is odd, we then obtain

$$\widehat{Rg}(\omega, \tau) = \widehat{g}(\tau\omega) = \tau^k \widehat{g}(\omega), \quad \tau \in \mathbf{R},$$

and (we denote the Dirac measure at the origin by $\delta$)

$$Rg(\omega, p) = i^k \delta^{(k)}(p) \widehat{g}(\omega),$$

which means in particular that $Rg(H) = 0$ for all hyperplanes $H$ not containing the origin. Thus, to generalize Helgason's example we can take any non-trivial smooth function $g$ in $\mathbf{R}^n \setminus \{0\}$ which has the correct parity, is homogeneous of degree $-n-k$ and *can be extended to a homogeneous distribution in* $\mathbf{R}^n$. It is important to note that such extension is not always possible (see [Hö2], ch. 3). For instance, if $k = 0$ the necessary and sufficient condition is that the integral of $g$ over a sphere centered at the origin is zero. Since the product of a homogeneous distribution and a homogeneous polynomial is homogeneous, it follows that for $k \ge 1$ the product $pg$, for any homogeneous polynomial of degree $k$, must have integral zero over such spheres. This condition is also sufficient. In other words, for $k = 0$ we need to take $g$ homogeneous of degree $-n$, even, and with mean zero.

To prove that the set (1) in Corollary 3 cannot be replaced by any larger set we take $g$ homogeneous of degree $-n$, even, with mean zero, equal to zero in the symmetric cone $\Gamma$, and choose

$$f(x) = \int_K g(x - y)\,dy, \quad x \notin K.$$

Then $f$ vanishes in the set $\cap_{x \in K}(x + \Gamma)$, and in general $f$ does not vanish in any larger set. If the set $K$ has interior points we can of course make $f$ continuous in $\mathbf{R}^n$ by taking a non-trivial smooth function $\psi$, supported in $K$, and choosing $f(x) = \text{p.v.} \int_{\mathbf{R}^n} g(x-y)\psi(y)dy$, $x \in \mathbf{R}^n$; here "p.v." indicates that we must take the principal value of the divergent integral. By transferring these examples to $\mathbf{P}^n$ we see that the set $\text{sh}_K(E)$ cannot be replaced by any larger set in the conclusion of the theorem.

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