NOTES

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Plane Intersections of Rotational Ellipsoids

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The theorem in this note was discovered by the first author in the course of optical experiments. We remind the readers that a *prolate rotational ellipsoid* is the surface obtained by rotating an ellipse around its major axis.

Theorem 1. *Let C be the curve of intersection between a prolate rotational ellipsoid in 3-space and an arbitrary plane. This curve looks circular when viewed from any of the focal points of the ellipsoid. In other words, the cone generated by all straight lines through C and one of the focal points is a circular cone.*

This result could have been proved by the Greeks more than two thousand years ago, so we certainly did not expect it to be a new theorem. However, since we found it (1996) we have asked a large number of experts, we have searched the literature, and we have posted queries on the internet, all without finding anyone who has heard of the theorem. We thus feel comfortable in concluding that the theorem is, if not unknown, at least not well known. If a reader of this note knows a reference to the theorem or to related facts, we would appreciate being informed.

Before we give a mathematical proof of the theorem, we will explain in principle how the theorem was discovered experimentally. An extremely short light pulse is emitted in a wide range of directions from a point A. The duration of the light pulse is about 1 picosecond $= 10^{-12}$ seconds, which produces a light pulse of length about 0.3 mm. As the light hits an arbitrary fixed plane π, it is scattered in all directions. At a point B, different from A, a "photograph" is taken with extremely short exposure time. The outcome of the experiment is that a bright circle is seen on the photographic plate. In other words, a bright curve—which is actually an ellipse—is seen on the plane π, and this curve looks circular as viewed from B. (The technique used to carry out such experiments is described in [**2**].)

How can we interpret this experiment as supporting our theorem? Well, let us assume that photons are emitted precisely at time $t = 0$ from the point A and that the "photograph" is taken at time $t = t_1 > 0$. Then photons originating at A can contribute to the picture only if their travel time is exactly t_1, and hence the distance that they have travelled is exactly $s = ct_1$, where c is the speed of light. This means that for any photon that has made one bounce at some point P on the plane π on its way from A to B, the sum of the distances AP and PB must be equal to s. What is the locus of the set of all such points P? If for a moment we forget about the plane π, the locus of all the points P in space such that the sum of the distances AP and PB is equal to s is a rotational ellipsoid E with axis of symmetry through A and B and focal points at A and B. Accordingly the set of bounce points P on the plane π is the curve in which π intersects E. This is the bright curve that is viewed by our camera located at the point B. It is an elementary fact that this curve is an ellipse, and as already explained, the assertion of the theorem is that this curve looks circular when viewed from B.

Proof of the theorem The equation of an ellipse of eccentricity e $(0 \leq e < 1)$ with one of its foci at the origin can be written in polar coordinates r and θ as

$$r = d/(1 - e \sin\theta),$$

where $d > 0$. Writing $r = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ and $\sin\theta = x_3/r$, we obtain $r - ex_3 = d$, or

$$|x| - ex_3 - d = 0, \tag{1}$$

which is the equation for the ellipsoid obtained by placing our ellipse with one focus at the origin and with its major axis along the x_3-axis, and then rotating it about that axis. The equation of an arbitrary plane in $\mathbf{R}^3$ can be written

$$\xi \cdot x - p = 0, \tag{2}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ is a unit vector and $p \geq 0$. Here $x = (x_1, x_2, x_3)$, and "$\cdot$" signifies the standard scalar product. We assume that the plane intersects the ellipsoid in more than one point, hence in a curve C that must be an ellipse. The condition for this to occur is that $p < p_0$, where p_0 is a number depending on ξ and d that we will not need to compute. We are looking for the surface Σ that is generated by all rays that pass through the origin and C. This surface is *conic*, which means by definition that whenever a point x lies on Σ the entire ray $\{tx : t > 0\}$ is contained in Σ.

Our proof depends on two easy mathematical facts. The first asserts that a surface $\{x : H(x) = 0\}$ is conic if the function $H : \mathbf{R}^3 \to \mathbf{R}$ is positively homogeneous in the sense that

$$H(tx) = tH(x) \qquad (x \in \mathbf{R}^3, t > 0). \tag{3}$$

The second fact is the following: if two surfaces are given by the equations $F(x) = 0$ and $G(x) = 0$, respectively, and λ and μ are arbitrary real numbers that are not both zero, then $\lambda F(x) + \mu G(x) = 0$ is the equation of a surface containing the curve of intersection (if nonempty) of the two original surfaces. (Indeed, if $F(x) = 0$ and $G(x) = 0$, then $\lambda F(x) + \mu G(x) = 0$.)

Now back to the proof. Denote the left-hand sides of (1) and (2) by $F(x)$ and $G(x)$, respectively. With the second fact in mind we try to describe Σ by $H(x) = 0$, where $H(x) = \lambda F(x) + \mu G(x)$ for some real numbers λ and μ. With the first fact in mind we want to choose λ and μ such that $H(x)$ is positively homogeneous. In fact, for the choice $\lambda = p/d$ and $\mu = -1$ the constant terms in (1) and (2) cancel, so we obtain

$$\begin{aligned} H(x) &= (p/d)(|x| - ex_3) - \xi \cdot x \\ &= (p/d)|x| - (ep/d)x_3 - \xi \cdot x. \end{aligned}$$

It is clear that $H(x)$ satisfies (3). If we set $\eta = \xi + (ep/d)\zeta$ with $\zeta = (0, 0, 1)$, the equation $H(x) = 0$ now takes the form

$$\eta \cdot x = (p/d)|x|. \tag{4}$$

This equation is invariant under all rotations around the origin leaving η fixed. In fact the right hand side is invariant under *all* rotations around the origin, and the scalar product $\eta \cdot x$ is invariant under rotations A leaving η fixed, for $\eta \cdot (Ax) = (A\eta) \cdot (Ax) = \eta \cdot x$. This shows that the cone $H(x) = 0$ is circular and that its axis has the direction η, which completes the proof of the theorem. ■

The opening angle of the cone is 2α, where

$$\cos\alpha = \frac{p}{d|\eta|} = \frac{p}{|d\xi + ep\zeta|}.$$

If $p = 0$ the cone coincides with the intersecting plane and the curve of intersection looks locally like a straight line (i.e., like a great circle on the celestial sphere).

It is geometrically obvious that the axis direction of the cone is independent of p (i.e., depends only on the normal direction of the intersecting plane) if the ellipsoid is a sphere ($e = 0$) or if the plane is perpendicular to the axis of the ellipsoid ($\xi = \pm\zeta$). The expression $\eta = \xi + (ep/d)\zeta$ shows that these are the only cases in which the axis direction of the cone is independent of p.

In the proof of the theorem we could in fact have allowed an eccentricity e with $e \geq 1$. In this way we can simultaneously treat the cases of a rotationally symmetric paraboloid and a two-sheeted rotationally symmetric hyperboloid. If $e > 1$, then (1) is the equation of one of the sheets of the hyperboloid, and the equation of the other is $|x| + ex_3 + d = 0$. Thus $\eta = \xi + (ep/d)\zeta$ will have the same value for the two sheets, whereas the factor p/d in (4) will be the same only up to sign. It follows that if the plane intersects both sheets of the hyperboloid, the resulting cone consists of one part of each of the two opposite halves of a circular double cone, so the curve of intersection when viewed from the focus takes the form of two separate circular arcs.

The argument presented here generalizes to establish an n-dimensional analogue of the theorem. If an $(n-1)$-dimensional prolate rotational ellipsoid (an ellipsoid in $\mathbf{R}^n$ with $n-1$ equal principal axes $b_1, \ldots, b_{n-1} = b$ and its nth principal axis a satisfying $a \geq b$) is cut by an arbitrary hyperplane, then the cone generated by the $(n-2)$-dimensional surface of intersection and one of the foci of the ellipsoid is a spherical cone (a cone having $(n-2)$-dimensional spherical symmetry around its axis).

The theorem is related to an important property of Lorentz transformations. To explain this, we use physical arguments to demonstrate that the bright curve observed at B is always circular. If the point of emission A and the point of observation B lie on the same normal L to the plane π, then the situation is rotationally invariant with respect to L, hence the observer will see a circle in this case. Now, for arbitrary positions of A and B there exists a moving coordinate system in which the description of the experiment is rotationally invariant in the way just described. To see this, let L be the normal to π passing through A, let B' be the point on L with the same distance to π as B, let s be the length $|PA| + |PB|$ for a point P of the ellipsoid, let a be the distance from B' to B, and let c be the speed of light. Imagine now that the laboratory moves with constant speed ca/s in the direction of the line $B'B$. An observer originally at B' and moving with the laboratory will then reach the point B at precisely the moment when the "photograph" is taken there. Note that $B'B$ is parallel to π, so π is fixed during this movement. In this laboratory the point of emission and point of observation lie on the same normal to π, making the situation rotationally symmetric as described earlier, hence the observer will see a circle when he passes the point B. The question is now what the stationary observer at B will see. According to the special theory of relativity the stationary observer will also see a circle, but he will see a circle with a different (smaller) radius. In fact, to each Lorentz transformation L of four-dimensional space-time there can be associated a conformal transformation A_L of the two-sphere that describes the correspondence between the celestial spheres viewed from two coordinate systems in uniform relative motion as determined by the transformation L (see, for example, [**1**, secs. 1.2–1.3]). The fact that A_L is conformal implies that a cone of light rays, which looks circular to one observer P, must look circular to another observer in uniform motion relative to P. Applying this argument to the Lorentz transformation

$$(x_1, x_2, x_3, t) = (x, t) \mapsto \frac{1}{\sqrt{1 - v^2/c^2}}(x - vt, t - v \cdot x/c^2),$$

where v is a vector in $\mathbf{R}^3$ with length ca/s and direction $B'B$, leads to an alternative proof of the theorem.

REFERENCES

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