

Analyse mathématique/*Mathematical Analysis*
 (Équations aux dérivées partielles/*Partial Differential Equations*)

A local vanishing theorem for distributions

Jan BOMAN

Abstract — If the analytic wave front set of the distribution f is disjoint from the conormal of a real analytic hypersurface S and f is flat along S , then f must vanish in a neighborhood of S .

Un théorème local d'annulation pour les distributions

Résumé — Si le front d'onde analytique d'une distribution f ne rencontre pas la conormale d'une hypersurface S et f est plate sur S , alors f s'annule dans un voisinage de S .

Version française abrégée — Soit f une distribution dans un ouvert $X \subset \mathbb{R}^d$, $d \geq 1$, et S une hypersurface analytique réelle dans X . On note par $WF_A(f)$ le front d'onde analytique de f et par $N^*(S)$ la variété conormale de S dans $T^*(X)$. Si

$$(1) \quad N^*(S) \cap WF_A(f) = \emptyset$$

on peut définir la restriction, f_S , de f dans S , qui est une distribution dans S (voir [3], section 8.2). Comme l'opération de dérivation n'augmente pas le front d'onde analytique, (1) reste valable pour toutes les dérivées $\partial^\alpha f$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, donc les distributions $(\partial^\alpha f)_S$ dans S sont bien définies. Sous l'hypothèse (1) disons que $f \in \mathcal{D}'(X)$ est plate sur S si $(\partial^\alpha f)_S = 0$ pour tout $\alpha \in \mathbb{N}^d$.

THÉORÈME. — Supposons que la distribution $f \in \mathcal{D}'(X)$ et l'hypersurface analytique réelle S satisfassent à (1) et que f soit plate sur S . Alors $f=0$ dans un voisinage de S .

Ce résultat joue un rôle essentiel dans un théorème d'unicité de type Holmgren pour la transformation de Radon avec poids analytiques réels [1].

Dans la preuve il suffit de traiter le cas où S est donnée par l'équation $x_d=0$. On utilise des suites de fonctions tronquantes χ_k de type Ehrenpreis-Hörmander, dont les dérivées d'ordres $\leq k$ sont majorées par $C^{k+1} k!$. L'hypothèse (1) implique que les transformées de Fourier de $\chi_k f$ satisfont à des estimations $|\widehat{\chi_k f}(0, \xi_d)| \leq C(A k / |\xi_d|)^k$. Cela nous donne de bonnes estimations pour les dérivées des images directes $(\chi_k f)_* \in C^\infty(\mathbb{R})$ par la projection $\mathbb{R}^d \ni (x', x_d) \mapsto x_d \in \mathbb{R}$. Le fait que f est plate sur S implique que les $(\chi_k f)_*$ sont plates sur $x_d=0$. En appliquant la formule de Taylor on déduit que $\lim_{k \rightarrow \infty} (\chi_k f)_* = 0$ dans un voisinage

fixe de l'origine, et faisant varier la suite χ_k on conclut que $f=0$ au voisinage de S .

Il est facile de construire une fonction $f \in C^\infty(\mathbb{R})$, plate à l'origine, telle que $0 \in \text{supp}(f)$ et $(0, dx) \notin WF_A(f)$; donc l'hypothèse qu'aucune des deux conormales opposées de S soit contenues dans $WF_A(f)$ est essentielle.

Si l'on fait l'hypothèse plus forte que f s'annule dans l'un des côtés de S , et si l'une des conormales n'est pas contenue dans $WF_A(f)$, alors il est bien connu que $f=0$ dans un voisinage de S ([3], Theorem 8.5.6); ce théorème est valable aussi pour les hyperfonctions ([5], Theorem 4.4.1).

Note présentée par Bernard MALGRANGE.

Il existe une hyperfonction f dans \mathbb{R}^2 telle que (1) soit satisfaite pour $S = \{x \in \mathbb{R}^2; x_2 = 0\}$, f soit plate sur S , mais $(0, 0) \in \text{supp } f$. Donc le théorème ci-dessus n'est pas valable pour la classe des hyperfonctions (voir [5], note 3.3).

1. INTRODUCTION. — Let f be a distribution in an open set $X \subset \mathbb{R}^d$ and let S be a real analytic hypersurface in X . Denote the conormal bundle of S in $T^*(X)$ by $N^*(S)$. If the analytic wave front set of f , denoted $\text{WF}_A(f)$, is disjoint from $N^*(S)$, that is

$$(1) \quad N^*(S) \cap \text{WF}_A(f) = \emptyset,$$

then the restriction of f to S , denoted f_S , is well-defined as a distribution on S (see section 4 below). Since differentiation does not increase the analytic wave front set, the same is true for all derivatives of f . Let us say that a distribution satisfying (1) is *flat* along S , if $(\partial^\alpha f)_S = 0$ for all α ; here we have written $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$, $\partial_\nu = \partial/\partial x_\nu$ for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$.

THEOREM. — Assume that the distribution $f \in \mathcal{D}'(X)$ and the real analytic hypersurface $S \subset X$ satisfy (1) and that f is flat along S . Then $f = 0$ in some neighborhood of S .

This theorem plays a crucial role in a uniqueness theorem of Holmgren type for Radon transforms with real analytic weights [1].

The theorem answers a question posed in the literature on hyperfunction theory; see e. g. [4] (Remark 4.11) and [7] (Remarques 2.4). In fact, if f is only assumed to be a hyperfunction, the corresponding statement is known to be false, due to a counterexample by M. Sato; more precisely, there exists a hyperfunction f on \mathbb{R}^2 such that f satisfies (1) for $S = \{x \in \mathbb{R}^2; x_2 = 0\}$, f is flat on S , and $(0, 0) \in \text{supp } f$ ([5], note 3.3). Results for hyperfunctions related to the theorem above are given in [6].

In the proof of the theorem we shall use special sequences χ_k of cut-off functions of Ehrenpreis-Hörmander type, whose derivatives of order at most k satisfy good estimates. We use Hörmander's definition of the analytic wave front set ([3], section 8.4). The statement is local, so it is sufficient to consider a neighborhood of an arbitrary point in S , which we may assume to be the origin. We can also make a real analytic change of coordinates so that S is contained in the plane $x_d = 0$. The assumption (1) implies that the Fourier transforms $\widehat{\chi_k f}$ decay rapidly in a conic neighborhood of $\pm dx_d$; see (2). This together with the vanishing of all derivatives of f along S implies good estimates for the functions $w_k(x_d) = \int \chi_k(x', x_d) f(x', x_d) dx'$. Letting k tend to infinity here we shall conclude $f = 0$.

2. THE CUT-OFF FUNCTIONS. — Take a fixed function $\phi \in C^\infty(\mathbb{R}^d)$, non-negative, even, with integral equal to 1 and with support in an ε -neighborhood U_ε of the origin and form for $k = 1, 2, \dots$ the function $\theta_k = (k^d \phi(kx))^{*k}$ as the convolution of the function $k^d \phi(kx)$ with itself to the power k . Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ and choose $\chi_k = \theta_k * \varphi$. Then the support of χ_k is contained in an ε -neighborhood of the support of φ , and we have the estimates $|\partial^{\alpha+\beta} \chi_k| = |\partial^\alpha \varphi * \partial^\beta \theta_k| \leq C_\alpha (Bk)^{|\beta|}$ for $|\beta| \leq k$; to prove the last inequality one uses an expression for $\partial^\beta \theta_k$ where each factor in θ_k is differentiated at most once. One can take $B = \sup \int |\partial_\nu \phi| dx$. Since $|\hat{\theta}_k| \leq 1$ and

$$\hat{\theta}_k(\xi) = \hat{\phi}(\xi/k)^k = (1 - c|\xi|^2/k^2 + \dots)^k \rightarrow 1$$

uniformly on compact sets, we have

$$|\partial^\alpha \chi_k - \partial^\alpha \varphi| = |\theta_k * \partial^\alpha \varphi - \partial^\alpha \varphi| \leq \int |(\widehat{\theta}_k(\xi) - 1) \widehat{\partial^\alpha \varphi}(\xi)| d\xi \rightarrow 0 \text{ as } k \rightarrow \infty,$$

hence $\chi_k \rightarrow \varphi$ in $C_0^\infty(\mathbb{R}^d)$.

3. ESTIMATES FOR $\widehat{\chi_k f}$. — By the definition of the analytic wave front set the assumption (1) implies that there exists a conic open neighborhood Γ of $\pm dx_d = (0, \dots, 0, \pm 1)$ and distributions $f_k, k = 1, 2, \dots$, with compact support, equal to f in a fixed open neighborhood V of the origin, and constants C and A_0 , which are independent of k , such that

$$|\widehat{f}_k(\xi)| \leq C(A_0 k / |\xi|)^k \text{ for all } \xi \in \Gamma \text{ and all } k \geq 1,$$

and, if f has order m , $|\widehat{f}_k(\xi)| \leq C(1 + |\xi|)^m$ for all k . More precisely, Lemma 8.4.4 in [3] shows that these statements are true for any V and Γ for which $\bar{V} \times \bar{\Gamma} \cap \text{WF}_A(f) = \emptyset$; here $\bar{}$ denotes closure. We need to check that $\widehat{\chi_k f}$ satisfies similar estimates, where the constant A_0 is independent of the function φ . In fact, the arguments in Lemma 8.4.4 in [3] give the following estimate.

Choose a number $b, 0 < b < 1/2$, and a conic neighborhood Γ_1 of $\pm dx_d$ such that $\xi \in \Gamma_1$ and $|\eta| < b|\xi|$ implies $\xi + \eta \in \Gamma$. Let K be a compact subset of V , choose $\varepsilon, 0 < \varepsilon < 1$, so small that $K + \bar{U}_\varepsilon \subset V$, let $\varphi \in C_0^\infty(K)$, and let $\chi_k = \theta_k * \varphi$ be the sequence of functions constructed above.

LEMMA 1. — *With the choices made we have the estimate*

$$(2) \quad |\widehat{\chi_k f}(\xi)| \leq C \left(\frac{A k}{1 + |\xi|} \right)^k \text{ for all } \xi \in \Gamma_1 \text{ and all } k \geq 1,$$

where $A = 2 \max \{1, 2A_0, B/b\}$.

4. ESTIMATES FOR $\int \chi_k f dx'$. — Recall that if $\chi \in C_0^\infty(\mathbb{R}^d)$ and $f \in \mathcal{D}'(\mathbb{R}^d)$, the integral

$\int \chi f dx' = w$ is defined as the push-forward $\pi_*(\chi f)$ with respect to the projection $\pi: (x', x_d) \mapsto x_d$ from \mathbb{R}^d to \mathbb{R} . This means that $\langle \pi_*(\chi f), \varphi \rangle = \langle \chi f, \varphi \circ \pi \rangle$ for $\varphi \in C_0^\infty(\mathbb{R})$; here $\langle \cdot, \cdot \rangle$ is the pairing between distributions and test functions. In general $\pi_*(\chi f)$ is a distribution on \mathbb{R} . Let now V be an open set and assume that

$$(3) \quad V \times \{ \pm dx_d \} \cap \text{WF}(f) = \emptyset.$$

Here $\text{WF}(f)$ is the (C^∞) wave front set of f . Then $w \in C^\infty$, if $\text{supp } \chi \subset V$. The restrictions $f_t = f|_{V_t}$ can then be defined as distributions on $V_t = \{x \in V; x_d = t\}$, and writing $\chi_t(x') = \chi(x', t)$ we have $w(t) = \langle f_t, \chi_t \rangle$, i. e.,

$$(4) \quad \pi_*(\chi f)(t) = \langle f_t, \chi_t \rangle.$$

Indeed, this formula is trivially true for $f \in C_0^\infty(V)$, and the definition of the restriction f_t is done by continuous extension from $C_0^\infty(V)$ to the space of all distributions satisfying (3), the latter space being equipped with a suitably chosen topology ([3], ch. 8.2). Since $\text{WF}(\partial^\alpha f) \subset \text{WF}(f)$ for any α , the same is true with f replaced by $\partial_d^\alpha f$. Assume that the distribution f satisfies (3). Repeated application of the formula

$$\partial \pi_*(\chi f) = \pi_*(\chi \partial_d f) + \pi_*(f \partial_d \chi)$$

together with (4) now proves (the well known fact) that $\pi_*(\chi f)$ is flat at $x_d=0$ for all $\chi \in C_0^\infty(V)$ if and only if the restriction $\partial_d^\alpha f|_{\{x_d=0\}}$ vanishes for all α .

We now apply this to our functions $w_k = \pi_*(\chi_k f)$. Restricting to $\xi'_d=0$ in (2) we obtain $|\hat{w}_k(\xi_d)| \leq C(Ak)^k(1+|\xi_d|)^{-k}$ for all $k \geq 1$, and hence $|\partial^{k-2} w_k| \leq C_1(Ak)^k$. We have seen that the flatness assumption implies $\partial^j w_k(0)=0$ for all j . Taylor's formula with remainder now gives $|w_k(x_d)| \leq C_1(Ak)^k \delta^{k-2}/(k-2)! \leq C_2 k^2 (Ae\delta)^{k-2}$ for $|x_d| < \delta$. Recalling that $\chi_k \rightarrow \varphi$ in C_0^∞ and choosing δ so small that $Ae\delta < 1$ we can conclude that $\int f(x', x_d) \varphi(x', x_d) dx' = 0$ for $|x_d| < \delta$. Since $\varphi \in C_0^\infty(V)$ is arbitrary this proves that $f=0$ in $\{x \in V; |x_d| < \delta\}$. The proof of the theorem is complete.

5. REMARKS. — If only one of the two conormals to the hypersurface S is assumed to be absent in $WF_A(f)$, the conclusion of the theorem is not valid. Indeed, let $f(x) = \exp((-1 - ix|x|)/\sqrt{|x|})$, for $x \in \mathbb{R}$. Then f is C^∞ , flat at the origin, and real analytic outside the origin. Moreover, f is the boundary value of the holomorphic function in the upper half-plane $F(z) = \exp((-1 - i)/\sqrt{z})$ (principal value of the square-root), hence $(0, -dx) \notin WF_A(f)$ ([3], Theorem 8.4.8).

It is interesting to compare the theorem above with Theorem 8.5.6 in [3]; in the latter theorem f is assumed to be identically zero on one side of the surface, *one* of the two conormals is assumed to be absent in $WF_A(f)$, and the conclusion is that f must vanish. This theorem is also valid for hyperfunctions ([5], Theorem 4.4.1).

If the distribution f is a locally integrable function, it is natural to replace the flatness condition in the theorem by a condition of rapid decay as x tends to S . For instance, let S be $U_\varepsilon \cap \{x_d=0\}$ and let f be continuous and satisfy $N^*(S) \cap WF(f) = \emptyset$. Then f satisfies $(\partial^\alpha f)_S = 0$ for all α if and only if $f(x) = \mathcal{O}(x_d^m)$ as $x_d \rightarrow 0$ for every m .

The theorem remains true if S is replaced by a real analytic surface of arbitrary dimension; the proof requires only obvious modifications.

In another direction the theorem can be sharpened as follows. The analytic wave front set in the assumption (1) can be replaced by the wave front set with respect to an arbitrary quasianalytic Denjoy-Carleman class (see [3], ch. 8.4 for definitions). This result will be proved elsewhere.

Note remise le 28 septembre 1992, acceptée le 6 octobre 1992.

REFERENCES

- [1] J. BOMAN, Holmgren's uniqueness theorem and support theorems for real analytic Radon transforms, *Contemp. Math.* (to appear).
- [2] L. HÖRMANDER, Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients, *Comm. Pure Appl. Math.*, 24, 1971, pp. 671-704.
- [3] L. HÖRMANDER, *The analysis of linear partial differential operators*, I, Springer-Verlag, Berlin, Heidelberg, and New York, 1983.
- [4] A. KANEKO, Remarks on hyperfunctions with analytic parameters, *J. Fac. Sci. Univ. Tokyo*, 22, 1975.
- [5] A. KANEKO, *Introduction to hyperfunctions*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1988.
- [6] A. KANEKO, On hyperfunctions with analytic parameters, in *Algebraic analysis*, I, M. KASHIWARA and T. KAWAI Eds., Academic Press, Boston, 1988, pp. 267-276.
- [7] J.-A. MARTI, Analyticité partielle et suites d'unicité, *C. R. Acad. Sci. Paris*, 314, Series I, 1992, pp. 789-792.

Department of Mathematics, Stockholm University, Box 6701, S-11385 Stockholm, Sweden.