On stable inversion of the attenuated Radon transform with half data

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We shall consider weighted Radon transforms of the form

$$R_\rho f(L) = \int_L f(x)\rho(x, L) ds,$$

where ρ is a given smooth, positive weight function on the set of pairs (x, L) of oriented hyperplanes $L \subset \mathbf{R}^n$ and points $x \in L$, and ds is Euclidean surface measure on the plane L. Identifying in the usual way the set of oriented hyperplanes with $S^{n-1} \times \mathbf{R}$ and the set of pairs (x, L) with $\mathbf{R}^n \times S^{n-1}$, we can write

$$R_\rho f(\omega, p) = \int_{x\cdot\omega=p} f(x)\rho(x, \omega) ds, \quad (\omega, p) \in S^{n-1} \times \mathbf{R}.$$

An important example is the attenuated Radon transform in $\mathbf{R}^2$, which corresponds to

$$\rho(x, \omega) = \exp\big(- \int_0^\infty \mu(x + t\omega^\perp) dt\big) \tag{1}$$

for some given attenuation function $\mu(x)$ of compact support; here $\omega^\perp$ denotes the vector ω rotated 90 degrees counterclockwise.

An explicit inversion formula for the attenuated Radon transform was recently given by Roman G. Novikov [No], and a somewhat less explicit solution to the same problem was given earlier by Arbuzov, Bukhgeim, and Kazantsev [ABK]. Since then several treatments of this important problem have appeared [Na2], [BS], [Bal], [BK]. See also the survey article [F]. In [BS] we proved an inversion formula very similar to Novikov's for the class of all $\rho(x, \omega)$ for which the directional derivative $\langle \omega, \partial_x \rangle \log \rho(x, \omega)$ is locally equal to an affine function of ω, that is

$$\langle \omega^\perp, \partial_x \rangle \log \rho(x, \omega) = a(x) + \omega_1 b_1(x) + \omega_2 b_2(x) \tag{2}$$

for some functions $a(x)$, $b_1(x)$, and $b_2(x)$. The weight functions (1) correspond of course to the case when $b_1(x) = b_2(x) = 0$.

From the point of view of numerical inversion Novikov's formula has the weakness that it requires data over the full circle. On the other hand it has been shown that a solution f of compact support is uniquely determined already by data $R_\rho f(\omega, p)$ for ω in an arbitrarily small open subset of the circle S^1 and all p, [No], [BS]. Knowing that the inverse operator exists but lacking an explicit formula, it is natural to ask about the continuity properties of the inverse operator. If $R_\rho f(\omega, p)$ is measured only for ω in some subset of S^1 whose total length is strictly less than π, then the inverse operator is known to be strongly unstable, because some microlocal singularities are not detected by R_ρ. More precisely, if lines with normal close to ω^0 are missing in the data set and f is a local plane wave of the form $\psi(x)v(x\cdot\omega^0)$, where ψ is a smooth function with compact support, then $R_\rho f$ will be smooth even if v is not smooth. On the other hand, in this note we will show that the inverse is continuous in the usual Sobolev norms (c.f. [Na1, Theorem 5.1]) with data over a little more than half of the circle (Corollary 3). Since our arguments apply with almost no change for arbitrary dimension n, we give our main result (Theorem 2) in that generality.

After an earlier version of this article had been written Hans Rullgård proved that the estimate (5) with $s = 0$ holds for the attenuated Radon transform with data over *exactly* half of the circle S^1 [R2, Theorem 4]. This result of course supersedes ours in dimension 2. However, since our proof is very simple and the methods used have independent interest (though well known in the partial differential equations community), we think it might still be justified to publish this article.

Rullgård has also given an inversion formula for R_ρ with ρ of the type (1), where $\mu(x)$ is constant on the support of f, using only data $Rf(\omega, p)$ for ω in half of the circle [R1].

Let s be an arbitrary real number. If $f(x)$ is a function or distribution on $\mathbf{R}^n$, its Sobolev norm $\|\cdot\|_s$ is defined by

$$\|f\|_s = \Big(\int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2(1+|\xi|^2)^s d\xi\Big)^{1/2};$$

here the Fourier transform $\widehat{f}$ is defined by $\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-ix\cdot\xi}dx$ for $\xi \in \mathbf{R}^n$. If $\phi(\omega, p)$ is a function on $S^{n-1}\times\mathbf{R}$ the same notation will have the meaning

$$\|\phi\|_s = \Big(\int_{S^{n-1}}\int_{\mathbf{R}} |\widehat{\phi}(\omega,\tau)|^2(1+|\tau|^2)^s d\tau d\omega\Big)^{1/2},$$

where $d\omega$ is Euclidean area measure on the sphere S^{n-1}. In this case $\widehat{\phi}(\omega, \tau)$ is the 1-dimensional Fourier transform of the function $p \mapsto \phi(\omega, p)$ for fixed ω.

If K is a compact set in $\mathbf{R}^n$ we shall denote by $C_0^\infty(K)$ and $H_0^s(K)$ the sets of functions in $C^\infty(\mathbf{R}^n)$ and $H^s(\mathbf{R}^n)$, respectively, which are supported in K.

The main step in our proof is Theorem 1, which states that, for a weighted Radon transform with full data, injectivity implies continuity of the inverse. Our main theorem is a very easy consequence of this fact. It is enough to prove Theorem 1 for even ρ, $\rho(x,-\omega) = \rho(x,\omega)$, since if $\rho_1(x,\omega) = \rho(x,-\omega) + \rho(x,\omega)$, then $\|R_{\rho_1} f\|_t \leq 2\|R_\rho f\|_t$ for any t.

It is well known that R_ρ is not always injective on functions with compact support, even if ρ is smooth and positive [Bo2].

Theorem 1. Let $\rho(x,\omega)$, $x \in \Omega \subset \mathbf{R}^n$, $\omega \in S^{n-1}$, be a positive C^∞ weight function, even in ω, and assume that the operator $f \mapsto R_\rho f$ is injective on functions with compact support in Ω. Let $K \subset \Omega$ be compact and $s \in \mathbf{R}$. Then there exists a constant C depending on K, ρ, s, and n such that

$$\|f\|_s \leq C\|R_\rho f\|_{s+(n-1)/2}, \quad f \in H_0^s(K). \tag{3}$$

Corollary 1. If $\rho(x,\omega)$ is positive, real analytic, and even, then (3) holds for some C.

Proof. By Theorem 2.1 in [BQ] (see also [Bo1]) R_ρ is injective on functions with compact support if ρ is real analytic and positive, so the assertion follows from Theorem 1.

Remark. It is sufficient to assume that $\rho(x,\omega)$ is real analytic for ω in some open set $\Gamma \in S^{n-1}$ and all $x \in \Omega$ and positive on all of $\Omega \times S^{n-1}$, since the injectivity theorem in [BQ] implies that a compactly supported function (distribution) f must vanish if $R_\rho f(\omega, p) = 0$ for all $(\omega, p) \in \Gamma \times \mathbf{R}$ and ρ is real analytic and positive in $\Gamma \times \mathbf{R}$.

Corollary 2. The set of ρ for which (3) holds is an open and dense set in the space of all positive, even functions in $C^\infty(\Omega \times S^{n-1})$.

Proof. It is well known that the set of operators with bounded inverse is an open set with the operator norm topology. The mapping $\rho \mapsto A_\rho$ is continuous from $C^\infty(\Omega \times S^{n-1})$ to the space of operators from $H_0^s(K)$ to $H^{s+(n-1)/2}(S^{n-1} \times \mathbf{R})$, so the set of ρ in question must also be an open set. And since it contains the set of real analytic functions it must be a dense set.

If Γ is a subset of S^{n-1} and ϕ is a function on $S^{n-1} \times \mathbf{R}$ we shall use the notation

$$\|\phi\|_s^\Gamma = \Big(\int_\Gamma \int_{\mathbf{R}} |\widehat{\phi}(\omega,\tau)|^2 (1+|\tau|^2)^s d\tau d\omega\Big)^{1/2}$$

for the Sobolev norm of ϕ restricted to $\omega \in \Gamma$.

Theorem 2. Let Γ be an open subset of S^{n-1} such that $\Gamma \cup (-\Gamma) = S^{n-1}$, and let $\rho(x,\omega)$ be a positive C^∞ weight function defined on $\Omega \times \Gamma$. Let Γ_0 be an open subset of Γ such that the closure $\overline{\Gamma}_0$ is contained in Γ and $\overline{\Gamma}_0 \cap (-\overline{\Gamma}_0) = \emptyset$, and assume that the operator taking $f \in C_0^\infty(\Omega)$ into the restriction of $R_\rho f(\omega,p)$ to $\Gamma_0 \times \mathbf{R}$ is injective. Let $K \subset \Omega$ be compact and let $s \in \mathbf{R}$. Then

$$\|f\|_s \le C \|R_\rho f\|^\Gamma_{s+(n-1)/2}, \quad f \in H_0^s(K). \tag{4}$$

Proof. Since $\Gamma \cup (-\Gamma) = S^{n-1}$ and $\overline{\Gamma}_0 \cap (-\overline{\Gamma}_0) = \emptyset$ we can find a non-negative function $\psi \in C_0^\infty(\Gamma)$ such that $\psi = 1$ on Γ_0 and $\psi(\omega) + \psi(-\omega) = 1$ on S^{n-1}. Define an even, symmetric weight function ρ_0 for $(x,\omega) \in \Omega \times S^{n-1}$ by

$$\rho_0(x,\omega) = \psi(\omega)\rho(x,\omega) + \psi(-\omega)\rho(x,-\omega).$$

Since ρ is positive and either $\psi(\omega)$ or $\psi(-\omega)$ is positive for every ω, it is clear that $\rho_0(x,\omega) > 0$ for all $(x,\omega) \in \Omega \times S^{n-1}$. Since $\|\cdot\|_t^\Gamma \le \|\cdot\|_t^{S^{n-1}}$ for every t we may assume that $\Gamma \ne S^{n-1}$. Since $\rho = \rho_0$ for $\omega \in \Gamma_0$ the assumption implies that R_{ρ_0} is injective. By Theorem 1 we can now conclude that (3) holds with ρ_0 instead of ρ, i.e.

$$\|f\|_s \le C\|R_{\rho_0} f\|_{s+(n-1)/2}, \quad f \in H_0^s(K).$$

Since

$$R_{\rho_0} f(\omega,p) = \psi(\omega) R_\rho f(\omega,p) + \psi(-\omega) R_\rho f(-\omega,-p)$$

and ψ is supported in Γ, the estimate (4) follows.

Corollary 3. Let Ω be an open subset of $\mathbf{R}^2$ and assume that $\rho(x,\omega)$ is a positive C^∞ weight function on $\Omega \times S^1$ of the type (1), or more generally that ρ satisfies (2). Let Γ be an open subset of S^1 such that $\Gamma \cup (-\Gamma) = S^1$, for instance an interval of length $> \pi$. Let $K \subset \Omega$ be compact and let $s \in \mathbf{R}$. Then

$$\|f\|_s \le C\|R_\rho f\|^\Gamma_{s+1/2}, \quad f \in H_0^s(K). \tag{5}$$

Proof. By [BS] it is known that R_ρ is injective when ω is restricted to an arbitrary open interval, if ρ is a weight function satisfying (2). The assertion therefore follows from Theorem 2.

Here is an outline of the proof of Theorem 1. For a function $h \in C_0^\infty(\Omega)$, positive on some neighborhood Ω_0 of K, we form the operator

$$Qf = hR^* R_\rho f,$$

which is a pseudodifferential operator of order $1-n$. Since $h(x) > 0$ and $\rho(x, \omega) > 0$ for all $x \in \Omega_0$ and all ω, Q is elliptic in Ω_0. Therefore there exists a pseudodifferential operator P of order $n-1$ such that $PQf = f + Wf$ for f supported in K and W is a pseudodifferential operator of order -1. For any pseudodifferential operator T in Ω of order $t \in \mathbf{R}$, a compact set $K \subset \Omega$, and $h \in C_0^\infty(\Omega)$ there exists a constant C such that $\|hTu\|_s \le C\|u\|_{s+t}$ for all $u \in H_0^s(K)$. Since P has order $n-1$ and W has order -1 it follows from $\|f\|_s \le \|PQf\|_s + \|Wf\|_s$ that

$$\|f\|_s \le C(\|Qf\|_{s+n-1} + \|f\|_{s-1}), \quad f \in H_0^s(K), \tag{6}$$

for some constant C. $R_\rho f$ has compact support and is an even function with respect to (ω, p), that is, $R_\rho f(\omega, p) = R_\rho f(-\omega, -p)$, and R^* is known to be injective on the set of such functions. Moreover, R_ρ is injective by assumption. Hence Q is injective on $C_0^\infty(K)$ (and therefore automatically on $H_0^s(K)$, because the fact that Q is elliptic implies that any solution to $Qu = 0$ must be in C^∞). Now comes the key step. The inequality (6) together with the fact that Q is injective is known to imply that (6) holds without the second term on the right hand side (Lemma 1), i.e.,

$$\|f\|_s \le C\|Qf\|_{s+n-1}, \quad f \in H_0^s(K), \tag{7}$$

with possibly a new C. Finally, the operator R^* obeys the estimate (Lemma 2)

$$\|hR^*\phi\|_{t+(n-1)/2} \le C\|\phi\|_t, \quad \phi \in H_0^t(S^{n-1} \times \mathbf{R}),$$

for any $t \in \mathbf{R}$, hence (7) implies (3).

All the ingredients in the proof outlined above have been standard in the literature on partial differential equations since the 1960:ies. However, since some of our readers may not be familiar with pseudodifferential operators and with Lemma 1, we shall briefly recall how to compute the principal symbol of Q, how to choose the operator P, and give a proof of Lemma 1.

The operator $Q = hR^* R_\rho$ can be written

$$\begin{aligned} Qf(x) &= h(x) \int_{S^{n-1}} R_\rho f(\omega, x \cdot \omega) d\omega \\ &= h(x) \int_{S^{n-1}} \big(\int_{y \cdot \omega = x \cdot \omega} f(y)\rho(y, \omega) ds \big) d\omega = \int_{\mathbf{R}^n} K(x, y) f(y) dy, \end{aligned} \tag{8}$$

where

$$K(x,y) = \frac{h(x)}{|x-y|} \int_{\omega \in S^{n-1} \cap (x-y)^{\perp}} \rho(y,\omega) d\omega_{n-2}, \quad x \neq y.$$

Here $S^{n-1} \cap (x-y)^{\perp}$ denotes the set of $\omega \in S^{n-1}$ that are perpendicular to $x-y$, which is an $n-2$-dimensional submanifold of the sphere, and $d\omega_{n-2}$ denotes the $n-2$-dimensional area measure on that surface. In dimension 2 this set consists of just two opposite points, and $d\omega_0$ is defined as the unit mass at each of those points, so the integral with respect to ω disappears and the expression for K becomes

$$K(x,y) = \frac{h(x)}{|x-y|} \big(\rho(y,\eta) + \rho(y,-\eta)\big) = \frac{2h(x)}{|x-y|} \rho(y,\eta), \quad x \neq y,$$

where η is any of the unit vectors perpendicular to $x-y$. With $k(x,z) = K(x,x-z)$ the expression (8) can be written

$$Qf(x) = \int k(x,x-y) f(y) dy = (2\pi)^{-n} \int e^{ix\cdot\xi} \widehat{k}(x,\xi) \widehat{f}(\xi) d\xi,$$

where $\widehat{k}(x,\xi)$ denotes the Fourier transform of $z \mapsto k(x,z)$ for fixed x. By definition this means that $\widehat{k}(x,\xi)$ is the symbol of the operator Q. Taylor expanding $\rho(y,\omega)$ around $y = x$ we can write $Q = Q_0 + Q_1$, $Q_\nu f(x) = \int k_\nu(x,x-y) f(y) dy$, $\nu = 0,1$, where

$$\begin{aligned}
k_0(x,z) &= \frac{h(x)}{|z|} \int_{\omega \in S^{n-1} \cap z^{\perp}} \rho(x,\omega) d\omega_{n-2}, \quad n \geq 3, \\
k_0(x,z) &= \frac{2h(x)}{|z|} \rho(x, z^{\perp}/|z|), \quad n = 2, \\
k_1(x,z) &= \frac{1}{|z|} \sum_{j=1}^{n} z_j c_j(x,z,z/|z|),
\end{aligned}$$

for some $c_j \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n \times S^{n-1})$. Note that $z \mapsto k_0(x,z)$ is homogeneous of degree -1, hence $k_0(x,z)$ is unbounded as a constant times $|z|^{-1}$ for $z \approx 0$, whereas $k_1(x,z)$ is bounded. This implies that Q_1 is an operator of lower order than Q_0, in fact the order of Q_0 is $1-n$ and that of Q_1 is $-n$. The symbol $q_0(x,\xi)$ of Q_0, the principal symbol of Q, is equal to the Fourier transform of $z \mapsto k_0(x,z)$, which we denote by $\widehat{k_0}(x,\xi)$. This Fourier transform can actually be computed (Lemma 3) and is equal to

(9) $$q_0(x,\xi) = 2(2\pi)^{n-1} h(x) |\xi|^{1-n} \rho(x, \xi/|\xi|), \quad \xi \in \mathbf{R}^n \setminus \{0\}, \ n \geq 2.$$

We shall not need this expression, apart from the fact that we can see from (9) that $q_0(x,\xi) \neq 0$ for all $x \in \Omega_0$ and all $\xi \in \mathbf{R}^n \setminus \{0\}$, which means that Q is elliptic.

The fact that R^* is injective on functions $\phi(\omega,p)$ of compact support follows from an old results of Ludwig [L]. This can also be seen from the formula

$$\widehat{R^*\phi}(\xi) = 2|\xi|^{-n+1}\widehat{\phi}(\xi/|\xi|, |\xi|),$$

valid for ϕ which are even in (ω,p). In fact Ludwig proved that R^* is injective on the set of continuous function $\phi(\omega,p)$ tending to zero at infinity.

To get rid of the remainder term in (6) we shall use the following well known lemma (see e.g. [T, Proposition 3.1, p. 108]).

Lemma 1. Let K be a compact subset of $\mathbf{R}^n$, and let T be an injective bounded linear operator $H_0^s(K) \to H^q(\mathbf{R}^n)$ such that the inequality

$$\|u\|_s \leq C(\|Tu\|_q + \|u\|_t), \quad u \in H_0^s(K), \tag{10}$$

for some constant C and some $t < s$. Then there exists a constant C_1 such that

$$\|u\|_s \leq C_1\|Tu\|_q, \quad u \in H_0^s(K).$$

Proof. The idea is to prove that (10) implies that the range of T is closed in $H^q(\mathbf{R}^n)$. Then the range of T is itself a Banach space, so T is an injective bounded linear operator from one Banach space onto another Banach space, and by Banach's theorem T^{-1} must then be a bounded operator. In proving that the range of T is closed we shall not use the assumption that T is injective, since this is not needed. Take an arbitrary sequence $f_n \in H_0^s(K)$ such that $Tf_n = g_n \to g \in H^q(\mathbf{R}^n)$; we need to prove that f_n converges to some $f \in H_0^s(K)$, from which it follows that $Tf = g$. Assume first that the sequence f_n is bounded, $\|f_n\|_s \leq A$, for some constant A. Since the natural injection $H^s(\mathbf{R}^n) \to H^t(\mathbf{R}^n)$ is compact, there exists a subsequence which converges in $H^t(\mathbf{R}^n)$. Denote the subsequence also by f_n. Applying the inequality (10) to $f_n - f_m$ we see that f_n is also convergent in $H^s(\mathbf{R}^n)$, which proves the claim in this case. Assume finally that $\|f_n\|_s$ is unbounded. Passing to a subsequence we may assume that $\|f_n\|_s \to \infty$. Let V be the nullspace of T. Replacing T by its quotient $\widetilde{T} : H^s/V \to H^q$ does not change the range of T, so we may assume that T is injective. Set $v_n = f_n/\|f_n\|_s$. Since Tf_n is assumed to be convergent, $Tv_n = Tf_n/\|f_n\|_s$ must tend to zero. Arguing as above we find that some subsequence of v_n must converge to some element $v \in H_0^s(K)$ with $\|v\|_s = 1$ and that $Tv = 0$.

This contradicts the assumption that T is injective, hence proves that $\|f_n\|_s$ cannot be unbounded. The proof is complete.

Lemma 2. If $h \in C_0^\infty(\mathbf{R}^n)$ and $t \in \mathbf{R}$, then

$$\|hR^*\phi\|_{t+(n-1)/2} \leq C\|\phi\|_t, \quad \phi \in H^t(S^{n-1} \times \mathbf{R}),$$

for some constant C.

Proof. Let A_h be the operator $u \mapsto hu$ of multiplication with the function h. It is well known that A_h is a bounded operator from $H^q(\mathbf{R}^n)$ into itself for any $q \in \mathbf{R}$. Since h is supported in a fixed compact set, the forward operator $hu \mapsto R(hu)$ is bounded from $H^q(\mathbf{R}^n)$ to $H^{q+(n-1)/2}(S^{n-1} \times \mathbf{R})$ [Na1, Theorem 5.1], hence the operator RA_h is also bounded from $H^q(\mathbf{R}^n)$ to $H^{q+(n-1)/2}(S^{n-1}\times\mathbf{R})$. It follows that the adjoint $(RA_h)^* = A_h^*R^* = A_hR^*$ is bounded from $H^{-q-(n-1)/2}(S^{n-1}\times\mathbf{R})$ to $H^{-q}(\mathbf{R}^n)$. With $q = -t-(n-1)/2$ the claim follows.

For the computation of the principal symbol (9) we shall need the next lemma, which says the following. If we identify homogeneous functions on $\mathbf{R}^n \setminus \{0\}$ with their restrictions to S^{n-1}, then the Fourier transform of an even homogeneous function of order $1-n$ amounts to the Funk transform of the corresponding function on S^{n-1}. This fact is certainly well known, but we have not found a convenient reference so we give the short proof here.

Lemma 3. Let $a(\omega)$ be an even function in $L^1(S^{n-1})$ and define the function f_a on $\mathbf{R}^n$, $n \geq 2$, by

$$f_a(x) = |x|^{1-n}a(x/|x|), \quad x \in \mathbf{R}^n \setminus \{0\}.$$

Then the Fourier transform of f_a is given by

$$\widehat{f_a}(\xi) = \frac{\pi}{|\xi|}\int_{S^{n-1}\cap\xi^\perp} a(\omega)d\omega_{n-2}, \quad \xi \in \mathbf{R}^n \setminus \{0\}.$$

Proof. By the definition of the Fourier transform on the class $\mathcal{S}'$ of tempered distributions the following holds for an arbitrary test function $\varphi \in \mathcal{S}$

$$\begin{aligned}\langle \widehat{f_a}, \varphi\rangle = \langle f_a, \widehat{\varphi}\rangle &= \int_{\mathbf{R}^n} |x|^{1-n}a(x/|x|)\widehat{\varphi}(x)dx \\ &= \int_{S^{n-1}} a(\omega)\int_0^\infty \widehat{\varphi}(t\omega)dtd\omega = \frac{1}{2}\int_{S^{n-1}} a(\omega)\int_{-\infty}^\infty \widehat{\varphi}(t\omega)dtd\omega.\end{aligned}$$

In the last equality we have used the fact that $a(\omega)$ is even. Applying the formula $\int_{\mathbf{R}} \widehat{v}(p)dp = 2\pi v(0)$ to the function $v(p) = \int_{x\cdot\omega=p}\varphi(x)ds$ observing that $\widehat{v}(t) = \widehat{\varphi}(t\omega)$ we see that the inner integral in the last expression can be written $2\pi \int_{x\cdot\omega=0} \varphi(x)ds$. Changing the order of integration and using the fact that $dsd\omega = |x|^{-1}d\omega_{n-2}dx$ we therefore obtain

$$\begin{aligned}\langle \widehat{f_a}, \varphi\rangle &= \frac{1}{2}\cdot 2\pi \int_{S^{n-1}} a(\omega)\Big(\int_{x\cdot\omega=0} \varphi(x)ds\Big)d\omega \\ &= \pi \int_{\mathbf{R}^n} \Big(\int_{S^{n-1}\cap x^\perp} a(\omega)d\omega_{n-2}\Big)\varphi(x)\frac{dx}{|x|},\end{aligned}$$

which completes the proof.

By Fourier's inversion formula $\widehat{\widehat{f_a}}(x) = (2\pi)^n f_a(x)$, that is, the Fourier transform of $z \mapsto |z|^{-1}\int_{S^{n-1}\cap z^\perp} a(\omega)d\omega_{n-2}$ is equal to $2\cdot(2\pi)^{n-1}|\xi|^{1-n}a(\xi/|\xi|)$. This proves the expression (9) for the principal symbol $q_0(x,\xi)$.

We now have all tools needed to put together the proof of Theorem 1. Since the proof is essentially a repetition of the outline given above, we shall be brief.

Proof of Theorem 1. As described above take $h \in C_0^\infty(\Omega)$, $h > 0$ on $\Omega_0 \supset K$, and form the pseudodifferential operator $Q = hR^*R_\rho$. We have seen that that the principal symbol q_0 of Q is given by the expression (9), hence $q_0(x,\xi)$ is positive on $\Omega_0 \times (\mathbf{R}^n \setminus 0)$, so Q is elliptic. For P we can now choose an inverse of Q_0 modulo operators of lower order, namely

$$Pu(x) = (2\pi)^{-n}\int_{\mathbf{R}^n} e^{ix\cdot\xi}\frac{1}{q_0(x,\xi)}\widehat{u}(\xi)d\xi.$$

Then the very first step of pseudodifferential operator calculus implies that the principal symbol of PQ_0 is $q_0(x,\xi)^{-1}q_0(x,\xi) = 1$, which is the same as saying that $PQ_0 = I + W_1$ on Ω_0, where W_1 has order -1. Since $Q = Q_0 + Q_1$, where Q_1 has order $-n$, and P has order $n-1$, it follows that $PQ_1 = W_2$ has order -1, so $PQ = PQ_0 + PQ_1 = I + W$, where $W = W_1 + W_2$ has order -1. Thus we get the inequality (6). The rest of the proof follows exactly the outline given above.

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