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Siciak's Theorem on Separate Analyticity

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Dedicated to the memory of Mikael Passare

Abstract. We give a simple proof of an important special case of the famous theorem of Jósef Siciak on separate analyticity.

1. Introduction

The well-known theorem of Hartogs states that a function $u(z_1, z_2)$ of two complex variables which is separately analytic must be analytic. By separately analytic we mean here that $z_1 \mapsto u(z_1, z_2)$ is analytic for each fixed z_2 and vice versa. Similar statements are also true if the "fixed" variables are restricted to sets of real dimension 1, or even to arbitrary sets of positive capacity. An important theorem of that kind was proved by Siciak in 1969, [11]. In the same paper Siciak gave a precise description of the maximal domain in $\mathbf{C}^2$ to which the function can be analytically continued. Many sharpenings and extensions of Siciak's theorem have been given later, for instance in [15], [13], [14], and [8], and a couple of years ago Jarnicki and Pflug wrote a whole book on the subject, [4]. Surveys of results related to separate analyticity can be found in [10] and [5]. The purpose of this note is to give a short proof of the most important special case of Siciak's theorem (Theorem $1'$ in Section 3) using only very well-known tools. More exactly, we shall treat the special case when the "fixed" variables range over bounded intervals on the real line. We will treat only the case of functions of two variables; the extension to functions on $\mathbf{C}^n \times \mathbf{C}^m$ is straightforward.

The well-known example $x_1x_2/(x_1^2+x_2^2)$ shows that a separately real analytic function need not be real analytic. Let us say that a function $u(x_1, x_2)$ is *uniformly* separately real analytic in the domain $D \in \mathbf{R}^2$, if the functions $x_1 \mapsto u(x_1, x_2^0)$ and $x_2 \mapsto u(x_1^0, x_2)$ are analytically continuable to complex disks with radius $r(x_1^0, x_2^0)$ around x_1^0 and x_2^0, respectively, where r is some positive continuous function on D. It is a corollary of Siciak's theorem that a uniformly separately real analytic function is real analytic. If the additional assumption is made that the continued

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function u is locally bounded, then this statement is easy to prove and very well known, but in its general form we think the theorem is not as well known as it deserves to be. Here we give a proof of that theorem (Proposition $1'$) using an important lemma of Lelong (Lemma 2), which is a sharpening of the well-known Hartogs lemma.

The notation and terminology used here follows that of Siciak's papers. Let E be an open bounded interval on the real axis $\mathbf{R}$ and let G be a simply connected bounded domain in $\mathbf{C}$ which contains $\overline{E}$, the closure of E. If G is regular with respect to the Dirichlet problem we define the function $h_{G,E}(z)$ on $\overline{G}$ as the solution to the Dirichlet problem in $G \setminus \overline{E}$ with boundary values 0 on $\overline{E}$ and 1 on ∂G, the boundary of G. Note that the domain $G \setminus \overline{E}$ is also regular, since E is an interval. We remark that a bounded domain in $\mathbf{C}$ must be regular if it has finite connectivity and ∂G contains no isolated point.

A first version of this paper was written in 1994 while my student Ozan Öktem was writing the paper [9] and we both were struggling to understand Siciak's proof from his original paper [11]. Having learnt from Christer Kiselman about Lelong's lemma (Lemma 2) and its relevance in this context I wrote a new version containing Theorem $1'$ in 2004. In submitting the paper I made a new revision and added references to the book [4] and a couple of articles that have appeared after 2004.

I am indebted to two referees for a number of very valuable comments leading to a considerably improved article.

2. Separate analyticity with boundedness assumption

We shall begin by making the simplifying assumption that the original function is locally bounded. In Section 3 we shall discuss the case when no boundedness assumptions are made.

Theorem 1. *Let E_1 and E_2 be open bounded intervals on $\mathbf{R}$, and let G_1 and G_2 be simply connected bounded regular domains in $\mathbf{C}$ such that $G_j \supset \overline{E}_j$ for $j = 1, 2$. Let u be defined in the set*

$$X = (E_1 \times G_2) \cup (G_1 \times E_2) \subset \mathbf{C}^2,$$

and assume that u is separately analytic in X; by definition this means that for every $x_2 \in E_2$ the function $z_1 \mapsto u(z_1, x_2)$ is analytic in G_1, and for every $x_1 \in E_1$ the function $z_2 \mapsto u(x_1, z_2)$ is analytic in G_2. Assume furthermore that u is locally bounded on X. Then u can be continued analytically to the set

$$\widetilde{X} = \{z \in G_1 \times G_2;\ h_{G_1,E_1}(z_1) + h_{G_2,E_2}(z_2) < 1\}. \tag{1}$$

Our proof consists of three steps. The first step is to use the above-mentioned fact that a uniformly separately real analytic function is real analytic, in the easy special case when the function is assumed bounded (Proposition 1). It follows that u must be real analytic on $E_1 \times E_2$, which means by definition that u can be continued to an analytic function on some neighborhood of $E_1 \times E_2$ in $\mathbf{C}^2$.

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In the second step (Proposition 2) we construct an analytic extension of u to an open neighborhood Σ in $\mathbf{C}^2$ of the set X. The third step is to prove that any function which is analytic in Σ can be extended to an analytic function on $\widetilde{X}$ (Proposition 3).

Proposition 1. *Let E_1 and E_2 be open bounded intervals on $\mathbf{R}$ with open complex neighborhoods V_1 and V_2, respectively, and assume that $z_1 \mapsto u(z_1, x_2)$ is analytic on V_1 for each $x_2 \in E_2$ and that $z_2 \mapsto u(x_1, z_2)$ is analytic on V_2 for each $x_1 \in E_1$. Assume furthermore that u is bounded on $(E_1 \times V_2) \cup (V_1 \times E_2)$. Then there exists an analytic function $\widetilde{u}$ on some complex neighborhood of $E_1 \times E_2$ in $\mathbf{C}^2$ that agrees with u on $E_1 \times E_2$.*

This theorem is well known; for the proof we refer for instance to [6].

Using the notion of analytic wave front set one can give a short proof of Proposition 1 as follows.[1] Let u be a compactly supported integrable function or distribution in $\mathbf{R}^n$. It is known that $(x^0, \xi^0) \in T^*(\mathbf{R}^n)$, $\xi^0 \neq 0$, belongs to the complement of the analytic wave front set of u, $WF_A(u)$, if and only if the so-called FBI transform of u,

$$F_u(x,\xi) = \int_{\mathbf{R}^n} u(y) e^{-|\xi||y-x|^2} e^{-iy\cdot\xi} dy,$$

decays exponentially as $|\xi|$ tends to infinity for ξ in a conic neighborhood of ξ^0 uniformly for x in some neighborhood of x^0. It follows immediately from the definition that the analytic wave front set is conic in the second variable, i.e., that $(x^0, \xi^0) \in WF_A(u)$ if and only if $(x^0, \lambda\xi^0) \in WF_A(u)$ and $\lambda > 0$. It is a basic fact that a function (distribution) is real analytic in some neighborhood of x^0 if and only if $(x^0, \xi^0) \notin WF_A(u)$ for every $\xi^0 \neq 0$. Assume now that $u \in L^1(\mathbf{R}^2)$ is uniformly separately real analytic in some neighborhood of x^0. Write $F_u(x,\xi)$ as a repeated integral with inner integral

$$\int_{\mathbf{R}} u(y_1, y_2) e^{-|\xi|(y_1-x_1)^2} e^{-iy_1\xi_1} dy_1. \tag{2}$$

By the assumption of real analyticity with respect to x_1 we can use Cauchy's theorem to deform the path of integration a little bit into the complex near $y_1 = x_1^0$ and thereby prove that the integral (2) tends to zero exponentially as $|\xi_1|$ tends to infinity for x_1 close to x_1^0, and hence the same is true of $F_u(x,\xi)$. Similarly, real analyticity with respect to x_2 implies that $F_u(x,\xi)$ is exponentially decreasing as

[1] The analytic wave front set for distributions was introduced by Hörmander 1970 in connection with a new proof of Holmgren's uniqueness theorem for partial differential equations with real analytic coefficients. A parallel theory was developed independently by M. Sato. There the so-called singular support of a hyperfunction was defined in terms of the possibility to represent the (hyper-)function as a sum of boundary values of analytic functions in regions $\{x + iy;\ x \in U \subset \mathbf{R}^n,\ y \in \Gamma_k,\ |y| < \varepsilon\}$, where U is open and Γ_k are certain cones in $\mathbf{R}^n$; see [1], ch. 9. The fact that the concepts were equivalent for distributions was proved a few years later. The third equivalent definition used here was given by Bros and Iagolnizer in 1975; see [1], Theorem 9.6.3.

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$|\xi_2| \to \infty$ for x_2 close to x_2^0. Hence $(x^0, \xi^0) \notin WF_A(u)$ for every $\xi^0 \neq 0$, so u is real analytic in a neighborhood of x^0.

The second step consists in using the assumption of separate analyticity to extend the region of joint analyticity in one direction at the time.

Proposition 2. *Let G be a simply connected bounded domain in $\mathbf{C}$, U an open disk with $\overline{U} \subset G$, and F a compact interval in $\mathbf{R} \subset \mathbf{C}$. Let u be an analytic function in a complex neighborhood of $U \times F$. Assume that the function $U \ni z_1 \mapsto u(z_1, x_2)$ can be extended to an analytic function in G for every $x_2 \in F$ and that the extended function $u(z_1, x_2)$ is locally bounded in $G \times F$. Then there exists an analytic function $\widetilde{u}$ on some complex neighborhood of $G \times F$ that agrees with u on $U \times F$.*

As a preparation for the proof of Proposition 2 we shall first consider the case when G is the open disk $U_R = \{\zeta \in \mathbf{C}; |\zeta| < R\}$ and U is a smaller disk containing the origin. This lemma is part of the standard proof of Hartogs' theorem on separate analyticity (see, e.g., [2], Lemma 2.2.11), but we include it here for the sake of completeness and in order to facilitate the discussion in Section 3.

Lemma 1. *Let $U_\varepsilon = \{\zeta \in \mathbf{C}; |\zeta| < \varepsilon\}$ and let F be a compact interval in $\mathbf{R} \subset \mathbf{C}$. Let u be analytic in some complex neighborhood of $U_\varepsilon \times F$ and assume that $U_\varepsilon \ni z_1 \mapsto u(z_1, x_2)$ can be extended to an analytic function in U_R for every $x_2 \in F$. Assume moreover that the function $u(z_1, x_2)$ is bounded for $(z_1, x_2) \in U_R \times F$. Then there exists an analytic function $\widetilde{u}$ on some complex neighborhood of $U_R \times F$ that agrees with u on $U_\varepsilon \times F$.*

Proof. By the first assumption u can be expanded in a Taylor series with respect to z_1

$$u(z_1, z_2) = \sum_{k=0}^{\infty} a_k(z_2) z_1^k, \tag{3}$$

where $a_k(z_2)$ are analytic in some neighborhood of F. Denoting by $V_\delta(F)$ the complex δ-neighborhood of $F \subset \mathbf{C}$ we choose $\delta > 0$ and $\varepsilon > 0$ so that u is analytic and bounded in $U_\varepsilon \times V_\delta(F)$. By Cauchy's inequality we then obtain

$$|a_k(z_2)| \leq C_1 \varepsilon^{-k}, \quad z_2 \in V_\delta(F), \quad k = 0, 1, \dots. \tag{4}$$

By the second assumption we also have the estimates

$$|a_k(x_2)| \leq C_0 R^{-k}, \quad x_2 \in F, \quad k = 0, 1, \dots. \tag{5}$$

Let $g(w)$ be the solution to the Dirichlet problem in $V_\delta(F) \setminus F$ with boundary values 0 on F and 1 on the boundary of $V_\delta(F)$. The function $\log |a_k(w)|$ is subharmonic in $V_\delta(F)$ and $\leq A_0$ in F and $\leq A_1$ in $V_\delta(F)$, where $A_0 = \log(C_0 R^{-k})$ and $A_1 = \log(C_1 \varepsilon^{-k})$. Hence $\log |a_k(w)| \leq A_0 + (A_1 - A_0) g(w)$ in $V_\delta(F)$, or

$$|a_k(w)| \leq C R_0(w)^{-k},$$

where $R_0(w) = R^{1-g(w)} \varepsilon^{g(w)}$ and $C = \max(C_0, C_1)$. But $R_0(w)$ is continuous and $R_0(w) = R$ for $w \in F$, hence for any given $r < R$ there exists a neighborhood W_r

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of F such that $R_0(w) > r$ for $w \in W_r$. This proves that the series (3) converges in $U_r \times W_r$ for every $r < R$, which completes the proof of the lemma. $\square$

Proof of Proposition 2. Since G is simply connected it is easy to see that one can construct a locally finite covering $G = \cup_{k=0}^{\infty} G_k$ of G by open disks $G_k \subset G$ with $\overline{G}_k \subset G$ such that $G_0 = U$ and for every k the union $\cup_{j=0}^{k} G_j$ is simply connected and contains the center of G_{k+1}. Set $H_k = \cup_{j=0}^{k} G_j$ for all k. We claim that

$$\text{for every } k \text{ there exists a complex neighborhood } V_k \text{ of } F \text{ and an analytic function } \widetilde{u}_k \text{ in } H_k \times V_k \text{ that agrees with } u \text{ on } U \times F. \qquad (P_k)$$

To prove this we use induction over k. For $k = 0$ there is nothing to prove. Assume that the statement (P_k) is true. Let ζ_0 be the center of G_{k+1}. Since $\zeta_0 \in G_k$ we can choose $\varepsilon > 0$ so that $U_\varepsilon(\zeta_0) = \{\zeta \in \mathbf{C};\ |\zeta - \zeta_0| < \varepsilon\}$ is contained in $G_k \cap G_{k+1}$. By the induction assumption $\widetilde{u}_k$ is then analytic in $U_\varepsilon(\zeta_0) \times V_k$. Applying Lemma 1 with $U_R = U_R(\zeta_0)$ and $U_\varepsilon = U_\varepsilon(\zeta_0)$ and R chosen so that $\overline{G}_{k+1} \subset U_R(\zeta_0) \subset G$ we can find a complex neighborhood V_{k+1} of F and a function $\widetilde{u}_{k+1}$ that is analytic in $G_{k+1} \times V_{k+1}$ and agrees with $\widetilde{u}_k$ on $U_\varepsilon(\zeta_0) \times F$. Shrinking V_{k+1}, if necessary, we may assume that $V_{k+1} \subset V_k$. Extending $\widetilde{u}_{k+1}$ suitably we therefore get an analytic function on $H_{k+1} \times V_{k+1}$, that we also denote by $\widetilde{u}_{k+1}$. Since H_{k+1} is simply connected, analytic continuation from G_0 to G_{k+1} along a different chain of disks would give the same values in $G_{k+1} \times V_{k+1}$. This proves (P_{k+1}) and hence shows that the statement (P_k) is true for all k.

To finish the proof of the proposition we observe that the union W of all $H_k \times V_k$ is a complex neighborhood of $G \times F$ and that all the $\widetilde{u}_k$ agree on their common domains, which shows that they define an analytic function on W. This completes the proof. $\square$

Proposition 3. *Let G_1 and G_2 be simply connected bounded regular domains in $\mathbf{C}$ and let E_1 and E_2 be open bounded intervals on the real axis such that $\overline{E}_j \subset G_j$ for $j = 1, 2$. Let X and $\widetilde{X}$ be defined as in Theorem 1, and let u be analytic in some open neighborhood Σ of X. Then there exists an analytic function $\widetilde{u}$ on $\widetilde{X}$ that agrees with u on X.*

Proof. Set $h(z) = h_{G_1,E_1}(z_1) + h_{G_2,E_2}(z_2)$, and for $\varepsilon > 0$ and $0 < t < 1$ define the region $\widetilde{X}_\varepsilon(t)$ by

$$\widetilde{X}_\varepsilon(t) = \{z \in G_1 \times G_2;\ h(z) < \min(1 - \varepsilon, t + \varepsilon|z|^2\}.$$

Choose $M > 1$ so that $|z|^2 \le M$ in $G_1 \times G_2$. We shall prove that, for every sufficiently small $\varepsilon > 0$, there exists an analytic function $\widetilde{u}_\varepsilon$ on

$$\widetilde{X}_\varepsilon(1 - 2\varepsilon M) \tag{6}$$

that agrees with u on X. Since the union of all the regions (6) is equal to $\widetilde{X}$ and all the functions $\widetilde{u}_\varepsilon$ agree on their common domains of definition, this proves the assertion of the proposition. We first claim that

$$\widetilde{X}_\varepsilon(t) \subset \Sigma$$

if ε and t are sufficiently small. To prove this we observe that the continuous function $h(z)$ is positive on the compact set $(\overline{G}_1 \times \overline{G}_2) \setminus \Sigma$, hence $h(z) \geq \delta$ on that set for some $\delta > 0$. It follows that $\widetilde{X}_\varepsilon(t) \subset \Sigma$ if ε and t are so small that $t + \varepsilon M < \delta$. Fix an arbitrary $\varepsilon > 0$ with $2\varepsilon M < \delta < 1$ and set

$$t_0 = \sup\{t < 1 - 2\varepsilon M; \text{there exists a function } \widetilde{u}_{\varepsilon,t} \text{ that is analytic in } \widetilde{X}_\varepsilon(t) \\ \text{and is equal to } u \text{ on } X\}.$$

Assuming that $t_0 < 1 - 2\varepsilon M$ we shall obtain a contradiction. It is clear that all the functions $\widetilde{u}_{\varepsilon,t}$ with $t < t_0$ define a function $\widetilde{u}_{\varepsilon,t_0}$ that is analytic on $\widetilde{X}_\varepsilon(t_0)$. On the other hand, by the definition of t_0 there must exist $z^0 \in \partial\widetilde{X}_\varepsilon(t_0)$ such that $\widetilde{u}_{\varepsilon,t_0}$ cannot be continued to any neighborhood of z^0. We claim that

$$\partial\widetilde{X}_\varepsilon(t) \subset \{z \in \mathbf{C}^2;\, h(z) = t + \varepsilon|z|^2\}, \quad \text{if } t < 1 - 2\varepsilon M. \tag{7}$$

Indeed, if $h(z) < t+\varepsilon|z|^2$ and $t < 1-2\varepsilon M$, then $h(z) < 1-2\varepsilon M+\varepsilon M = 1-\varepsilon M < 1-\varepsilon$, so z cannot belong to the boundary of $\widetilde{X}_\varepsilon(t)$, which proves (7).

The functions h_{G_j,E_j} are harmonic in $G_j \setminus \overline{E}_j$, hence $h(z) - \varepsilon|z|^2$ is *strictly plurisuperharmonic*, so (7) implies that the domain $\widetilde{X}_\varepsilon(t_0)$ is *strictly pseudoconcave.* This implies that u must be continuable to an analytic function in some neighborhood of z^0. This is a contradiction and hence completes the proof of the proposition. □

Proof of Theorem 1. By Proposition 1 there exists an analytic function $\widetilde{u}_0$ in some open neighborhood W_0 of $E_1 \times E_2$ that agrees with u on $E_1 \times E_2$. We may assume that W_0 is connected, and then it is clear that $\widetilde{u}_0$ agrees with the given function u on $X \cap W_0$. Applying Proposition 2 to $G = G_1$, an arbitrary closed subinterval $F \subset E_2$, and an open disk $U \subset G_1$ such that $U \times F \subset W_0$ we can then find $\widetilde{u}_1$ that is analytic in some complex neighborhood of $G_1 \times F$ and agrees with u on $U \times F$, hence agrees with u on $G_1 \times F$. Varying $F \subset E_2$ we get a function $\widetilde{u}_1$ that is analytic in some complex neighborhood W_1 of $G_1 \times E_2$ and agrees with u on $G_1 \times E_2$. Similarly we can find $\widetilde{u}_2$ that is analytic in some complex neighborhood W_2 of $E_1 \times G_2$ and agrees with u on $E_1 \times G_2$. Since $\widetilde{u}_1$ and $\widetilde{u}_2$ agree on an open set, it is clear that they together define an analytic function $\widetilde{u}$ in a complex neighborhood Σ of X. An application of Proposition 3 now completes the proof of the theorem. □

3. The general case

We shall now discuss the situation when no boundedness assumption is made in Theorem 1. We shall use the convention that a primed theorem, proposition etc. is the analogue without boundedness assumption of the unprimed theorem (proposition etc.) with the same number.

Theorem 1′. *The statement of Theorem* 1 *is true without the assumption that u is locally bounded.*

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The proof of this theorem consists of three steps, analogous to those of the proof of Theorem 1. Only the first two steps need to be modified. The following lemma of Lelong is essential for both those steps ([7], Théorème 10; see also [11], Theorem 2.1). It is an important extension of the well-known Hartogs lemma.

Lemma 2. *Let F be a compact interval on $\mathbf{R}$ and let $G \subset \mathbf{C}$ be an open set containing F. Let $\varphi_k(z)$, $k = 1, 2, \ldots$, be a sequence of subharmonic functions in G satisfying*

$$\varphi_k(z) \le B, \quad z \in G, \quad k = 1, 2, \ldots, \tag{8}$$

and

$$\varlimsup_{k\to\infty} \varphi_k(x) \le A, \quad x \in F.$$

Then for every $\eta > 0$ there exists a complex neighborhood U of F and a number k_0 such that

$$\varphi_k(z) < A + \eta, \quad z \in U, \quad k \ge k_0. \tag{9}$$

The neighborhood U depends only on the numbers η, B, A, and on the sets F and G (not on the sequence φ_k).

We shall sketch a proof of this lemma using facts from [3]. Let us first make a couple of remarks. If we knew that the function $\varphi(z) = \varlimsup \varphi_k(z)$ were subharmonic, then φ would be majorized in G by the function $h(z) = A + (B - A)h_{G,F}$, the solution to the Dirichlet problem in $G \setminus F$ with boundary values A on F and B on ∂G. Then (9) could be proved just as the classical Hartogs lemma (use the mean value property of φ_k and Fatou's lemma to find k_0 independent of z such that $\varphi_k(z) < \varphi(z) + \varepsilon$ for $k > k_0$). But the limes superior of a sequence of subharmonic functions is not always subharmonic. (What is true is that it must be subharmonic if it is upper semicontinuous; more generally, the upper semicontinuous regularization $\overline{\varphi}$ of $\varlimsup \varphi_k$ is subharmonic, but we do not know that $\overline{\varphi} \le A$ on F.) Lelong proves Lemma 2 by establishing the majorization just mentioned for a class of functions which includes upper limits of sequences of subharmonic functions.

Sketch of proof of Lemma 2. As was indicated above it is sufficient to prove the estimate

$$\varlimsup_{k\to\infty} \varphi_k(z) \le h(z) = A + (B - A)h_{G,F} \tag{10}$$

for $z \in G$. It is clearly sufficient to prove (10) for $z \in G \setminus F$. Assume (10) is false at some point $z \in G \setminus F$. Then there exists a number c such that

$$\varphi_k(z_0) > c > h(z_0) \tag{11}$$

for infinitely many k. Since φ_k is bounded from above, we can take a subsequence $\widetilde{\varphi}_\nu = \varphi_{k_\nu}$ satisfying (11) and converging in $\mathcal{D}'(G)$ to some subharmonic function ψ (Theorem 3.2.12 in [3]). Then $\varlimsup \widetilde{\varphi}_\nu \le \psi$ (Theorem 3.2.13 in [3]), and according to Theorem 3.4.14 in [3] the set

$$M = \{z \in G; \varlimsup \widetilde{\varphi}_\nu(z) < \psi(z)\}$$

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is polar (a polar set is by definition any set on which a subharmonic function can be equal to $-\infty$ without being identically $-\infty$). But $\overline{\lim}\,\widetilde{\varphi}_\nu \le \overline{\lim}\,\varphi_k \le A$ on F by assumption, hence ψ must be $\le A$ on $F \setminus M$, and it is clear that $\psi \le B$ on all of G. Since M is polar, and ψ is subharmonic in G this implies in fact that $\psi \le h$ in G (M is so small that the boundary values on M do not influence the solution to the Dirichlet problem). On the other hand

$$\psi(z_0) \ge \overline{\lim}\,\widetilde{\varphi}_\nu(z_0) \ge c > h(z_0).$$

Thus we have obtained a contradiction and (10) is proved. $\square$

Using Lemma 2 it is easy to prove the analogues of Lemma 1 and Proposition 2 without boundedness assumptions:

Lemma 1′. *Let $U_\varepsilon = \{\zeta \in \mathbf{C};\, |\zeta| < \varepsilon\}$ and let F be a compact interval in $\mathbf{R} \subset \mathbf{C}$. Let u be analytic in some complex neighborhood of $U_\varepsilon \times F$ and assume that $U_\varepsilon \ni z_1 \mapsto u(z_1, x_2)$ can be extended to an analytic function in U_R for every $x_2 \in F$. Then there exists an analytic function $\widetilde{u}$ on some complex neighborhood of $U_R \times F$ that agrees with u on $U_\varepsilon \times F$.*

Proof. Let φ_k be the subharmonic function

$$\varphi_k(w) = \frac{1}{k}\log|a_k(w)|,$$

where $a_k(\cdot)$ is defined by (3). By the first assumption the sequence $\varphi_k(w)$ is uniformly bounded from above in $V_\delta(F)$ for some $\delta > 0$. By the second assumption

$$\overline{\lim_{k\to\infty}}\,\varphi_k(x_2) \le \log(1/R), \quad \text{for all} \quad x_2 \in F.$$

According to Lemma 2 there must then exist for any $r < R$ a number k_0 such that

$$\varphi_k(x_2) < \log(1/r), \quad \text{if} \quad k > k_0,\ x_2 \in F,$$

or equivalently

$$|a_k(x_2)| \le r^{-k}, \quad \text{if} \quad k > k_0,\ x_2 \in F.$$

Thus we have estimates corresponding to (4) and (5), and the proof can now be finished exactly in the same way as the proof of Lemma 1. $\square$

We can now prove Proposition 2 without boundedness assumption:

Proposition 2′. *Let G be a simply connected bounded domain in $\mathbf{C}$, U an open disk with $\overline{U} \subset G$, and F a compact interval in $\mathbf{R} \subset \mathbf{C}$. Let u be an analytic function in a complex neighborhood of $U \times F$. Assume that the function $U \ni z_1 \mapsto u(z_1, x_2)$ can be extended to an analytic function in G for every $x_2 \in F$. Then there exists an analytic function $\widetilde{u}$ on some complex neighborhood of $G \times F$ that agrees with u on $U \times F$.*

Proof. This statement is proved using Lemma 1′ in exactly the same way as Proposition 2 was proved using Lemma 1. $\square$

We are now ready to prove the theorem on separate real analyticity without boundedness assumptions.

Proposition 1′. *Let E_1 and E_2 be bounded open intervals on $\mathbf{R}$ with complex neighborhoods V_1 and V_2, respectively, and assume that $z_1 \mapsto u(z_1, x_2)$ is analytic on V_1 for each $x_2 \in E_2$ and that $z_2 \mapsto u(x_1, z_2)$ is analytic on V_2 for each $x_1 \in E_1$. Then there exists an analytic function $\widetilde{u}$ on some complex neighborhood of $E_1 \times E_2$ in $\mathbf{C}^2$ that agrees with u on $E_1 \times E_2$.*

Proof. It is enough to prove the assertion for arbitrary closed subintervals $F_1 \subset E_1$ and $F_2 \subset E_2$. Shrinking V_1 and V_2, if necessary, we may also assume that V_1 and V_2 are simply connected and that $z_1 \mapsto u(z_1, x_2)$ is bounded on V_1 for each $x_2 \in F_2$ and that $z_2 \mapsto u(x_1, z_2)$ is bounded on V_2 for each $x_1 \in F_1$. For any natural number N define the set

$$K_N = \{x_1 \in F_1;\ |u(x_1, z_2)| \leq N \text{ for all } z_2 \in V_2\}.$$

We claim that K_N is closed for each N. In fact, let $x_1^\nu \in K_N$ for $\nu = 1, 2, \dots$ and $\lim_{\nu\to\infty} x_1^\nu = x_1^0$. We have to prove that $x_1^0 \in K_N$. Since the family of analytic functions $w_\nu(z_2) = u(x_1^\nu, z_2)$ is uniformly bounded, there exists a subsequence of x_1^ν such that $w_\nu(z_2)$ converges to an analytic function $w(z_2)$ on V_2 with $|w(z_2)| \leq N$. Since $E_1 \ni x_1 \mapsto u(x_1, x_2)$ must be continuous for each $x_2 \in E_2$, we must have $w(x_2) = u(x_1^0, x_2)$ for each $x_2 \in F_2 \subset E_2$. But this implies that $w(z_2) = u(x_1^0, z_2)$ for all $z_2 \in V_2$, and hence proves our claim that K_N is closed. Since $V_2 \ni z_2 \mapsto u(x_1, z_2)$ is bounded for each $x_1 \in F_1$, the union of all K_N must be equal to all of F_1. By Baire's theorem K_N must have an interior point for some N, in other words, we can choose N_1, $x_1^0 \in F_1$ and $\delta_1 > 0$ such that $\{x_1;\ |x_1 - x_1^0| < \delta_1\} \subset F_1$ and

$$|u(x_1, z_2)| \leq N_1 \quad \text{whenever } |x_1 - x_1^0| < \delta_1 \text{ and } z_2 \in V_2. \tag{12}$$

Set $I_{\delta_1} = \{x_1;\ |x_1 - x_1^0| < \delta_1\}$. Applying the same argument to the function u on $(I_{\delta_1} \times V_2) \cup (V_1 \times F_2)$ with the variables interchanged we can find a number $N \geq N_1$, $x_2^0 \in F_2$, and $\delta_2 > 0$ such that $\{x_2;\ |x_2 - x_2^0| < \delta_2\} \subset F_2$ and, in addition to (12),

$$|u(z_1, x_2)| \leq N \quad \text{whenever } |x_2 - x_2^0| < \delta_2 \text{ and } z_1 \in V_1.$$

Set $J_{\delta_2} = \{x_2;\ |x_2 - x_2^0| < \delta_2\}$. Now we can apply Proposition 1 to conclude that u must be real analytic on $I_{\delta_1} \times J_{\delta_2}$. By definition this implies that there exist complex neighborhoods U_1 of I_{δ_1} and U_2 of J_{δ_2} and an analytic function $\widetilde{u}_0$ in $U_1 \times U_2$ that agrees with u on $I_{\delta_1} \times J_{\delta_2}$. Applying Proposition 2′ (Proposition 2 would actually suffice here) with $G = V_1$, $F = F_2$ equal to a closed subinterval of J_{δ_2}, and a disk $U \subset U_1$, we can find an analytic function $\widetilde{u}_1$ in some neighborhood of $V_1 \times F_2$ that agrees with $\widetilde{u}_0$ on $U \times F_2$, hence agrees with u on $(U \cap E_1) \times F_2$. Since $E_1 \ni x_1 \mapsto u(x_1, x_2)$ is real analytic for each x_2, $\widetilde{u}_1$ must agree with u on $E_1 \times F_2$. Thus for an arbitrary closed subinterval $F_1 \subset E_1$ we can now choose a disk $U \subset U_2$ such that $\widetilde{u}_1$ is analytic in a complex neighborhood of $F_1 \times U$. Then we can apply Proposition 2′ with those choices of F_1 and U and $G = V_2$ to conclude that there exists an analytic function $\widetilde{u}_2$ that is analytic in a complex neighborhood of $F_1 \times V_2$ that agrees with $\widetilde{u}_1$ on $F_1 \times U$, hence agrees with u on $F_1 \times E_2$. Since F_1 was an arbitrary closed subinterval of E_1 this gives an analytic function in a complex neighborhood of $E_1 \times E_2$ that is equal to u on $E_1 \times E_2$. The proof is complete. □

Proof of Theorem 1′. We argue in the same way as in the proof of Theorem 1. By Proposition 1′ there exists an analytic function $\widetilde{u}_0$ in some neighborhood W_0 of $E_1 \times E_2$ that agrees with u on $E_1 \times E_2$. Applying Proposition 2′ to $G = G_1$, an arbitrary closed subinterval $F \subset E_2$, and open disks $U \subset G_1$ such that $U \times F \subset W_0$ we find $\widetilde{u}_1$ that is analytic in some complex neighborhood W_1 of $G_1 \times E_2$ and agrees with u on $G_1 \times E_2$. Similarly we can find $\widetilde{u}_2$ that is analytic in a complex neighborhood W_2 of $E_1 \times G_2$ and agrees with u on $E_1 \times G_2$. It is clear that $\widetilde{u}_1$ and $\widetilde{u}_2$ together define an analytic function $\widetilde{u}$ in a complex neighborhood Σ of X. The proof is completed by means of Proposition 3 exactly in the same way as before. □

In [11] Siciak treats also the case when E_1 and E_2 are allowed to be general compact subsets of G_1 and G_2, respectively, not necessarily subsets of the real line. It is assumed that the boundaries of E_1 and E_2 are regular for the Dirichlet problem, which implies in particular that E_1 and E_2 are not too small. An analogous statement in n dimensions where $u(x_1, \ldots, x_n)$ is assumed to be separately analytic in each variable is also proved in [11]. Extension to the case when G_1 and G_2 may be higher-dimensional manifolds is given in [15]. In [13] Siciak gave a new proof of his main result in [11], based on his theory of so-called extremal plurisubharmonic functions. A theorem analogous to Theorem 1 where u is allowed to have singularities on an algebraic curve Γ in $\mathbf{C}^2$ was given in [14]; this proved a conjecture by Öktem, who treated the special case when Γ is a complex line, [9].

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