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On stable region-of-interest reconstruction in tomography: examples of non-existence of bounded inverse

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Abstract

In applications of computerized tomography (CT) it is often of interest to reconstruct a function in a proper subset, the region of interest (ROI), of its support from a proper subset of a full CT-scan. In several recent works it has been shown that stable ROI reconstruction is possible in certain limited data situations. In this paper we investigate the limits of what is possible in that direction by proving that reconstruction in an interior ROI is severely unstable for a certain set of data that contains integrals over all lines that intersect the ROI and is large enough for the unknown function to be uniquely determined.

Keywords: region-of-interest, tomography, stable, reconstruction

In this note we study the possibility to reconstruct some part of a compactly supported function f on the plane from incomplete knowledge of its two-dimensional Radon transform Rf,

$$Rf(L) = \int_L f \, ds;$$

here ds denotes arc length measure along the line L. It is well known that exact reconstruction of f requires complete Radon data, that is, knowledge of $Rf(L)$ for all lines L that intersect the support of f. In many applications it is enough to compute the unknown function in some proper subset, the region of interest (ROI), of the support of the function. For a long time it was believed that complete data were needed also in such cases; recall that the standard inversion formula for the Radon transform requires complete data even for computing the value of f at a single point. However, beginning in 2002, a number of exact formulas have been given for computing the restriction of f to a ROI from incomplete knowledge of Rf, and it was shown by numerical experiments that those formulas can be used for stable reconstruction; see for instance [2–8], the survey article [1], and references given there. This makes it natural to ask the following general question. Given a compact set K in the plane and a proper subset $G \subset K$, for which sets Λ of lines is it possible to stably compute f in G from the restriction of Rf to Λ, if $\operatorname{supp} f \subset K$? This problem is too general at this moment; here we will only give a negative result for a specific class of sets G, K, and Λ.

Recall that ROI reconstruction is trivial for the three-dimensional Radon transform integrating over planes in 3-space, since the standard inversion formula for the Radon transform is local in odd dimensions.

Let G be an open subset of K, and let Λ be an open set of lines in the plane. It is well known that a necessary condition for stable reconstruction of f in G from $Rf(L)$ for $L \in \Lambda$ to be possible is that

$$\text{every line that meets } G \text{ is contained in the closure, } \overline{\Lambda}, \text{ of } \Lambda. \tag{1}$$

To see this assume that a line $L \notin \overline{\Lambda}$ with normal θ intersects G and that $x^0 \in L \cap G$. Then there must exist a neighborhood $U \subset G$ of x^0 and a neighborhood Γ of θ such that no line in Λ that intersects U has normal in Γ. Take $\psi \in C_c^\infty(U)$ with $\psi(x^0) \neq 0$ and choose $f_\lambda(x) = \psi(x)\mathrm{e}^{\mathrm{i}\lambda x\cdot\theta}$. Then $\int |f_\lambda(x)|^2 \mathrm{d}x$ is independent of λ, and $Rf_\lambda(L)$ tends to zero as $\lambda \to \infty$ for all $L \in \Lambda$, and the same is true for all derivatives of Rf_λ with respect to the coordinates on the manifold of lines, hence $\|Rf_\lambda\|_{\Lambda,s}$ tends to zero as $\lambda \to \infty$ for any Sobolev norm $\|\cdot\|_{\Lambda,s}$ (see below for definition).

It is well known that condition (1) in general not even suffices for f to be uniquely determined by the data $Rf(L)$ for $L \in \Lambda$. For instance, if G is a disk whose closure is contained in the interior of K and Λ is the set of all lines meeting G, then we have the so-called interior problem, which is not uniquely solvable; see [9], section 4.4.

An obvious necessary condition for stable reconstruction in G from $Rf(L)$ for $L \in \Lambda$ is that

$$\text{a function } f \text{ with } \operatorname{supp} f \subset K \text{ is uniquely determined in } G \text{ by } Rf(L) \text{ for } L \in \Lambda. \tag{2}$$

It is rather well understood for which sets Λ the condition (2) is valid. For instance, (2) holds if Λ is open, connected, contains at least one line not meeting $\operatorname{supp} f$, and every point in G is contained in some line in[1] Λ. An example is the set Λ consisting of all lines with normal in an open set of unit vectors. See also [9, section 2.3].

It seems not to have been known whether the conditions (1) and (2) together are sufficient for stable reconstruction of f in G from $Rf(L)$ for $L \in \Lambda$. However, here we will prove that they are *not*. In other words, the two conditions (1) and (2) together do not imply that f can be stably reconstructed in G from $Rf(L)$ for $L \in \Lambda$.

To discuss stability of reconstruction we need to define the Sobolev norms $\|g\|_{\Lambda,s}$ for a function $g(L)$ defined on Λ. We will represent a line L as a pair $(\omega, p) \in S^1 \times \mathbf{R}$ in the familiar way: $L(\omega, p) = \{x \in \mathbf{R}^2;\, x\cdot\omega = p\}$. Note that $L(\omega, p) = L(-\omega, -p)$. Thus a function g on Λ can be represented as an even function $g(\omega, p)$ on some subset of $S^1 \times \mathbf{R}$, which will sometimes also be denoted by Λ. Writing $\omega = \omega(\alpha) = (\cos\alpha, \sin\alpha)$ we define for integers $s \geqslant 0$ the Sobolev H^s norm of a function g defined on all of $S^1 \times \mathbf{R}$ by

$$\|g\|_s^2 = \sum_{j+k\leqslant s} \int_0^{2\pi} \int_{\mathbf{R}} |\partial_\alpha^j \partial_p^k g(\omega(\alpha), p)|^2 \mathrm{d}p \, \mathrm{d}\alpha. \tag{3}$$

[1] This is an easy consequence of the following *local* injectivity theorem, which is proved, although not stated, in [10]: if $f \in C_c(\mathbf{R}^2)$ vanishes on one side of the line L_0 and $Rf(L) = 0$ for all L in some neighborhood of L_0, then $f = 0$ in some neighborhood of L_0.

If g is only defined on the subset Λ of $S^1 \times \mathbf{R}$ we define the norm $\|g\|_{\Lambda,s}$ by

$$\|g\|_{\Lambda,s} = \inf\{\|\tilde{g}\|_s;\, \tilde{g}|_\Lambda = g\}, \tag{4}$$

where the infimum is taken over all even functions $\tilde{g}(\omega, p)$ on $S^1 \times \mathbf{R}$ for which the restriction to Λ is equal to g and the norm $\|\tilde{g}\|_s$ is finite.

The Sobolev norms for non-integral s will not be needed in this article, except in remark 1 below. In spite of that we give a simple definition here. For arbitrary real s one can define the norm $\|g\|_s$ as follows. Denote by $\hat{g}(n, \xi)$ the Fourier transform with respect to p of the nth Fourier coefficient of the function $\alpha \mapsto g(\omega(\alpha), p)$, that is,

$$\hat{g}(n, \xi) = \int_{\mathbf{R}} \mathrm{e}^{-\mathrm{i}p\xi} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}n\alpha} g(\omega(\alpha), p)\mathrm{d}\alpha \mathrm{d}p.$$

Define the norm $\|g\|_s$ as the square root of

$$\sum_{n=-\infty}^{\infty} \int_{\mathbf{R}} |\hat{g}(n, \xi)|^2 (1 + |n|^2 + |\xi|^2)^s \mathrm{d}\xi, \quad s \in \mathbf{R}. \tag{5}$$

If s is a non-negative integer, the expression (5) is equal to

$$\sum_{j+k\leqslant s} c_{jk}^s \sum_{n=-\infty}^{\infty} |n|^{2j} \int_{\mathbf{R}} |\xi|^{2k} |\hat{g}(n, \xi)|^2 \mathrm{d}\xi = \sum_{j+k\leqslant s} c_{jk}^s \int_0^{2\pi} \int_{\mathbf{R}} |\partial_\alpha^j \partial_p^k g(\omega(\alpha), p)|^2 \mathrm{d}p\, \mathrm{d}\alpha,$$

where $c_{jk}^s = s!/(j!k!)$. This shows that the norm defined by (5) is equivalent to the norm defined by (3), if s is a non-negative integer.

If $g(\omega, p)$ is equal to the Radon transform $Rf(\omega, p)$ of some compactly supported function f, then the derivatives with respect to α in the definition of $\|g\|_s$ can be omitted. In fact, if we define $\|g\|_s'$ by

$$\|g\|_s'^2 = \sum_{k\leqslant s} \int_0^{2\pi} \int_{\mathbf{R}} |\partial_p^k g(\omega(\alpha), p)|^2 \mathrm{d}p\, \mathrm{d}\alpha,$$

then it is obvious that $\|g\|_s' \leqslant \|g\|_s$, and the opposite inequality,

$$\|g\|_s \leqslant C_s \|g\|_s', \tag{6}$$

is known to hold for all g of the form $g = Rf$ for some compactly supported function f, [9, theorem 5.2]. However, if we define $\|g\|_{\Lambda,s}'$ by $\|g\|_{\Lambda,s}' = \inf\{\|\tilde{g}\|_s';\, \tilde{g}|_\Lambda = g\}$, then it is not obvious that $\|g\|_{\Lambda,s} \leqslant C_s \|g\|_{\Lambda,s}'$ for arbitrary subsets $\Lambda \subset S^1 \times \mathbf{R}$ and $g = Rf$, $f \in C_c^\infty(\mathbf{R}^2)$, on[2] Λ. Therefore we have to use the definition (3) and (4) for $\|g\|_{\Lambda,s}$ in the proof of the theorem below.

For functions on $\mathbf{R}^2$ we will only use the Sobolev norms with $s = 0$, in which case we will omit the parameter s. Then

$$\|f\|_G^2 = \|f\|_{G,0}^2 = \int_G |f(x)|^2 \mathrm{d}x.$$

In our opinion stable reconstruction in G from data in Λ for functions supported in K should be possible, if an inequality

[2] A proof of $\|g\|_{\Lambda,s} \leqslant C_s \|g\|_{\Lambda,s}'$ might run as follows. By the definition of $\|g\|_{\Lambda,s}'$ there exists an extension $\tilde{g}$ of g such that $\|\tilde{g}\|_s' \leqslant 2\|g\|_{\Lambda,s}'$. Assume we could find another extension, $\tilde{g}_1$, of g, in the range of R, such that $\|\tilde{g}_1\|_s' \leqslant C_1 \|\tilde{g}\|_s'$. By (6) we then know that $\|\tilde{g}_1\|_s \leqslant CC_1 \|\tilde{g}\|_s'$, both C and C_1 depending on s. By the definition of $\|g\|_{\Lambda,s}$ it would follow that $\|g\|_{\Lambda,s} \leqslant \|\tilde{g}_1\|_s \leqslant CC_1 \|\tilde{g}\|_s' \leqslant 2CC_1 \|g\|_s'$. But we do not know how to find such $\tilde{g}_1$ in general.

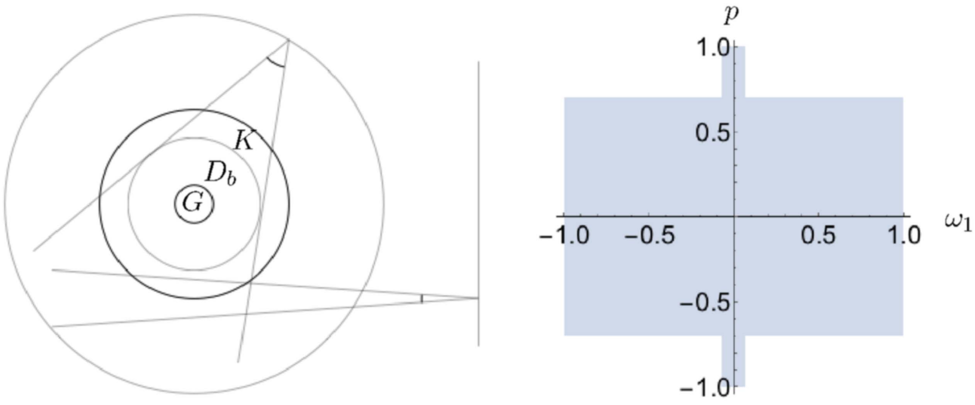

Figure 1. Left: illustration of set of lines $\Lambda(b, \delta)$ with $b = 0.7$ and $\delta = 0.07$. K is the closed unit disk; sources on the big circle send rays in all directions through the disk $D_b = \{x; |x| < b\}$; sources on the vertical line send rays in directions that make angle less than $\arcsin(\delta)$ to the horizontal; the region of interest G is contained in D_b. Right: the gray area is the set of (ω_1, p) coordinates of lines in $\Lambda(b, \delta) = \{L(\omega, p); |p| < b\} \cup \{L(\omega, p); |\omega_1| < \delta\}$ with $b = 0.7$ and $\delta = 0.07$.

$$\|f\|_G \leqslant C\|Rf\|_{\Lambda,1/2}$$

holds for smooth functions that are supported in K (see also remark 1 below).

Let K be the closed unit disk $\{x; |x| \leqslant 1\}$ in the plane. For $0 < b < 1$ and $\delta > 0$ we shall consider the set of lines

$$\Lambda(b, \delta) = \{L(\omega, p); |p| < b \text{ or } |\omega_1| < \delta\}. \tag{7}$$

Thus $\Lambda(b, \delta)$ is the union of the set of lines that intersect the disk $\{x; |x| < b\}$ and the set of lines with unit normal $\omega = (\omega_1, \omega_2)$ satisfying $|\omega_1| < \delta$. The ROI will be an arbitrary open subset of the disk $\{x; |x| < b\}$; see Figure 1. Then it is clear that (1) is satisfied with $G \subset \{x; |x| < b\}$. Moreover, it is well known that a compactly supported function is uniquely determined by $Rf(\omega, p)$ for ω in an open set and all p; hence $\Lambda(b, \delta)$ also satisfies condition (2). However, if δ is sufficiently small, there can be no stable reconstruction, as is seen from the following theorem.

Theorem. *Let $\Lambda(b, \delta)$ be defined by (7) and let* G *be a non-empty open subset of the disk $\{x; |x| < b\}$. If $\delta < (1 - b)/8e$ there exists a sequence f_n of smooth functions supported in the unit disk such that*

$$\lim_{n\to\infty} \frac{\|Rf_n\|_{\Lambda(b,\delta),s}}{\|f_n\|_G} = 0, \quad \text{for every } s \geqslant 0. \tag{8}$$

Remark 1. Of special interest is the case $s = 1/2$. Recall that in the case of full data we have the well known estimates

$$c\|Rf\|_{1/2} \leqslant \|f\| \leqslant C_K\|Rf\|_{1/2}$$

with $c > 0$ for f with support in the compact set K, [9, theorem 5.1]. For $s = 1/2$ formula (8) says that there exists no constant C such that the inequality $\|f\| \leqslant C\|Rf\|_{\Lambda(b,\delta),1/2}$ holds for f

supported in the unit disk. Since arbitrary $s > 0$ is allowed in (8), the theorem also rules out stability in much weaker sense in this situation.

Remark 2. The results of [2, 3] probably imply estimates of the kind $\|f\|_G \leqslant C\|Rf\|_{\Lambda,1/2}$ for the regions of interests G and data sets Λ considered there, although this seems not to have been proved. This makes it interesting to check what can be deduced from those results in the case of our data sets $\Lambda(b, \delta)$. It turns out that if $\delta > \sqrt{1 - b^2}$ the results in [3] show that stable reconstruction from data in $\Lambda(b, \delta)$ should be possible at all points of a strip $|x_1| < a$, if a is small enough and f is supported in the closed unit disk $\bar{D}$. (Here the possibility of 'stable reconstruction' should probably be understood in a practical sense: it is asserted that the given formulas have been used to produce useful reconstructions.) To see this we note that the main result of [3] ('Differentiated backprojection with Hilbert filtering', [1, p 72]) shows that $f(x)$ can be stably computed from data in Λ for all x on the line L_0, if Λ is open and every line that intersects $L_0 \cap \operatorname{supp} f$ is contained in Λ. Let L_0 be the line $x_1 = a$. We claim that every line that intersects $L_0 \cap \bar{D}$ is in $\Lambda(b, \delta)$ if $\delta > \sqrt{1 - b^2}$ and a is small enough. Indeed, if the line $L(\omega, p)$ intersects the line $\{(0, x_2); |x_2| \leqslant 1\}$ but does not meet D_b, then $|\omega_1| \leqslant \sqrt{1 - b^2} < \delta$. By continuity the claim must be true if a is small enough.

To prepare for the proof of the theorem we fix $b \in (0, 1)$ and let $h(t)$ be any smooth function on $\mathbf{R}$ that satisfies the conditions

$$\operatorname{supp} h \subset \{t;\, b \leqslant |t| \leqslant 1\},\ h(t) \geqslant 0,\ h(t) \text{ even, and } \int_b^1 h(t)\mathrm{d}t = 1. \tag{9}$$

Define

$$q(x) = -\frac{1}{\pi}\int_{\min(|x|,1)}^1 \frac{h'(t)}{\sqrt{t^2 - |x|^2}}\mathrm{d}t, \qquad x \in \mathbf{R}^2. \tag{10}$$

Then, by [9, section 4.4], $Rq(\omega, p) = h(p)$ for all p and all ω, so in particular

$$Rq(\omega, p) = 0 \quad \text{for } |p| < b \text{ and all } \omega. \tag{11}$$

Set

$$f_n(x) = \partial_{x_1}^n q(x). \tag{12}$$

Then (11) holds with f_n instead of q, that is,

$$Rf_n(\omega, p) = 0 \quad \text{for } |p| < b \text{ and all } \omega. \tag{13}$$

This follows from the following simple lemma together with a suitable rotation of the coordinate system.

Lemma 1. *Assume that ψ is smooth and compactly supported and that*

$$\int \psi(x_1, x_2)\mathrm{d}x_1 = 0 \quad \text{for } |x_2| < b. \tag{14}$$

Then for all a_1, a_2

$$\int (a_1\,\partial_{x_1} + a_2\,\partial_{x_2})\psi(x_1, x_2)\mathrm{d}x_1 = 0 \quad \text{for } |x_2| < b.$$

Proof. Clearly $\int \partial_{x_1}\psi(x_1, x_2)\mathrm{d}x_1 = 0$ holds for all $\psi \in C_c^\infty(\mathbf{R}^2)$, and taking derivative with respect to x_2 in (14) we obtain $\int \partial_{x_2}\psi(x_1, x_2)\mathrm{d}x_1 = 0$.

The structure of the proof of the theorem is as follows. We will prove that

$$\int_G |f_n(x)|^2 \mathrm{d}x \geqslant \frac{b^2\,\mathrm{area}(G)}{\pi^2}(n!)^2, \quad \text{for } n \text{ even}, \tag{15}$$

if f_n is defined by (10), (12), and h satisfies (9). On the other hand we shall see that Rf_n satisfies (17) below. Later on we will show that one can choose $h = h_n$ depending on n and s and satisfying (9) such that

$$\sup_{k\leqslant n+s} \|h_n^{(k)}\| \leqslant C_{b,s}\left(\frac{8e}{1-b}\right)^n n!, \quad k \leqslant n + s. \tag{16}$$

Combining this with (17) and (15) with f_n replaced by f_{2n} will prove (8).

Lemma 2. *Let the functions f_n be defined by (12) and (10), where h is a smooth function satisfying (9). Then for* $0 < \delta \leqslant 1$

$$\|Rf_n\|_{\Lambda(b,\delta),s} \leqslant C_s\, n^s\delta^{n-s} \sup_{\nu\leqslant n+s} \|h^{(\nu)}\|, \quad n \geqslant s \geqslant 0. \tag{17}$$

Proof. We shall use the following definition of the Fourier transform in $\mathbf{R}^d$ for $d = 1, 2$:

$$\hat{f}(\xi) = \int_{\mathbf{R}^d} f(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\mathrm{d}x, \quad \xi \in \mathbf{R}^d.$$

The one-dimensional Fourier transform of $g(\omega, p)$ with respect to p will be denoted $\hat{g}(\omega, \tau)$. Set $g_n = Rf_n$. By the choice of f_n and the fact that $\widehat{Rf}(\omega, \tau) = \hat{f}(\tau\omega)$ we have

$$\hat{g}_n(\omega, \tau) = \hat{f}_n(\tau\omega) = (\mathrm{i}\tau\omega_1)^n\hat{q}(\tau\omega) = (\mathrm{i}\tau\omega_1)^n\hat{h}(\tau),$$

hence

$$g_n(\omega, p) = \omega_1^n\, \partial_p^n h(p).$$

Let ψ be a smooth, even function on $\mathbf{R}$ such that $\psi(t) = 0$ for $|t| > 2$ and $\psi(t) = 1$ for $|t| \leqslant 1$ and choose

$$\tilde{g}_n(\omega, p) = \psi(\omega_1/\delta)g_n(\omega, p) = \psi(\omega_1/\delta)\omega_1^n\, \partial_p^n h(p).$$

Then $\tilde{g}_n$ is a smooth, even, and compactly supported function on $S^1 \times \mathbf{R}$ whose restriction to $\{(\omega, p);\ |\omega_1| < \delta,\ p \in \mathbf{R}\}$ is equal to g_n, so $\|g_n\|_{\Lambda(b,\delta),s} \leqslant \|\tilde{g}_n\|_s$ by the definition of the norm $\|\cdot\|_{\Lambda(b,\delta),s}$. To estimate the norm $\|\tilde{g}_n\|_s$ we shall use ω_1 as a coordinate on the circle S^1 in a neighborhood of the support of $\tilde{g}_n$. Since Sobolev norms computed with different coordinates are equivalent as norms, it is then clear that

$$\|\tilde{g}_n\|_s^2 \leqslant C_s \sum_{j+k\leqslant s} \int_{-\delta}^{\delta} |\partial_{w_1}^j(\psi(\omega_1/\delta)\omega_1^n)|^2 \mathrm{d}\omega_1 \int_{\mathbf{R}} |\partial_p^{k+n}h(p)|^2\mathrm{d}p. \tag{18}$$

With $\psi_1(t) = t^n\psi(t)$ we have $t^n\psi(t/\delta) = \delta^n\psi_1(t/\delta)$ and hence

$$\partial_t^j(t^n\psi(t/\delta)) = \delta^{n-j}\psi_1^{(j)}(t/\delta). \tag{19}$$

Using Leibnitz' formula it is easy to see that

$$\sup_{j\leqslant s} |\psi_1^{(j)}| = \sup_{t\in\mathbf{R}, j\leqslant s} |\partial_t^j (t^n \psi(t)| \leqslant n^s M_s \tag{20}$$

for some constants M_s that depend only on s (and on the choice of the function ψ). The estimate (17) with a new C_s now follows immediately from (18)–(20).

To choose the function $h = h_n$, depending on n, so that it satisfies the estimate (16), we shall use the well known sequences of cut-off functions that were used by Hörmander to define the analytic wave front set; see e.g. lemma 2.2 in [11].

Lemma 3. *There exists a sequence* $\varphi_n \in C_c^\infty(\mathbf{R})$ *with* $\operatorname{supp} \varphi_n \subset [0, d]$, $\varphi_n \geqslant 0$, $\int \varphi_n(t)\mathrm{d}t = 1$, *such that*

$$\left(\int |\partial^k \varphi_n(t)|^2 \mathrm{d}t\right)^{1/2} \leqslant \frac{4}{\sqrt{d}}\left(\frac{4n}{d}\right)^k, \quad k \leqslant n. \tag{21}$$

Proof. We first construct ϕ_n with $\operatorname{supp} \phi_n \subset [-1, 1]$, $\phi_n \geqslant 0$, $\int \phi_n(t)\mathrm{d}t = 1$, such that

$$\left(\int |\partial^k \phi_n(t)|^2 \mathrm{d}t\right)^{1/2} \leqslant \sqrt{8}\,(2n)^k, \quad k \leqslant n. \tag{22}$$

Take an even function $\theta(t)$ in $C^\infty(\mathbf{R})$ with $\operatorname{supp}\theta \subset [-2/3, 2/3]$, $0 \leqslant \theta(t) \leqslant 1$, $\theta(t)$ decreasing for $t \geqslant 0$, and $\int \theta(t)\mathrm{d}t = 1$. Then $\int |\theta'(t)|\mathrm{d}t = 2\,\theta(0) \leqslant 2$. Choose

$$\psi_n(t) = n\theta(nt) * n\theta(nt) * \ldots * n\theta(nt) \quad (n \text{ factors}).$$

Then $\operatorname{supp} \psi_n \subset [-2/3, 2/3]$, $\psi_n \geqslant 0$, and $\int \psi_n(t)\mathrm{d}t = 1$. Moreover, if $k \leqslant n$

$$\partial^k \psi_n(t) = \underbrace{n^2\theta'(nt) * \ldots * n^2\theta'(nt)}_{k \text{ factors}} * \ldots * n\theta(nt),$$

which shows that

$$\int |\partial^k \psi_n(t)|\mathrm{d}t \leqslant (2n)^k, \quad k \leqslant n.$$

Set

$$\phi_n(t) = \psi_n(t) * 2\theta(2t).$$

Then $\operatorname{supp} \phi_n \subset [-1, 1]$, $\phi_n \geqslant 0$, and $\int \phi_n(t)\mathrm{d}t = 1$, and

$$\sup |\partial^k \phi_n(t)| \leqslant 2 \sup \theta(t) \int |\partial^k \psi_n(t)|\mathrm{d}t \leqslant 2(2n)^k, \quad k \leqslant n.$$

Hence

$$\int |\partial^k \phi_n(t)|^2 \mathrm{d}t \leqslant 2 \cdot 4(2n)^{2k}, \quad k \leqslant n,$$

which implies (22). Finally we choose φ_n as a suitable translate of $t \mapsto (2/d)\,\phi_n(2t/d)$, which immediately gives (21).

Denote by $\varphi_{d,n}(t)$ the function constructed in lemma 3. Choose

$$h_n(t) = \varphi_{1-b,n+s}(|t| - b).$$

Then h_n is even, and since $\varphi_{1-b,n+s}$ is supported in $[0, 1-b]$, we see that h_n is supported in $\{t;\, b \leqslant |t| \leqslant 1\}$. From (21) with $d = 1 - b$ we now obtain

$$\|h_n^{(k)}\| = \sqrt{2}\,\|\varphi_n^{(k)}\| \leqslant \frac{6}{\sqrt{d}}\left(\frac{4(n+s)}{d}\right)^k, \quad k \leqslant n+s.$$

By elementary inequalities

$$(n+s)^{n+s} \leqslant e^{n+s}(n+s)! \leqslant e^{n+s} n!\,(n+s)^s \leqslant e^s e^n n!\; C_s\, 2^n \leqslant C_s'\,(2e)^n n!.$$

Here we have used the fact that $(n+s)^s \leqslant C_s\, 2^n$ for some constant C_s and we have set $C_s' = e^s C_s$. Recalling that $d = 1 - b$ we can now conclude that

$$\begin{aligned}\|h_n^{(k)}\| &\leqslant \frac{6}{\sqrt{d}}\left(\frac{4}{d}\right)^{n+s}(n+s)^{n+s} \leqslant \frac{6}{\sqrt{1-b}}\left(\frac{4}{1-b}\right)^s\left(\frac{4}{1-b}\right)^n C_s'\,(2e)^n n! \\ &= C''_{b,s}\left(\frac{8e}{1-b}\right)^n n!, \quad k \leqslant n+s,\end{aligned} \tag{23}$$

which proves (16).

It remains to prove (15). We begin with a lemma.

Lemma 4. *The function* $(1-t^2)^{-3/2}$ *can be represented*

$$\frac{1}{(1-t^2)^{3/2}} = 1 + \sum_1^\infty b_\nu t^{2\nu}, \quad |t| < 1,$$

where $b_\nu \geqslant 1$.

Proof. It is well known that b_ν can be written

$$b_\nu = (-1)^\nu \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\ldots\left(-\frac{3}{2}-(\nu-1)\right)}{\nu!} = \frac{\frac{3}{2}}{1}\cdot\frac{\frac{3}{2}+1}{2}\cdot\ldots\cdot\frac{\frac{3}{2}+\nu-1}{\nu} \geqslant 1,$$

which proves the assertion.

Corollary. *If n is even, then*

$$\partial_t^n (1-t^2)^{-3/2} \geqslant n! \quad \text{for } |t| < 1. \tag{24}$$

In order to estimate $f_n(x) = \partial_{x_1}^n q(x)$ from below we do integration by parts in (10) and use the fact that $h(u) = 0$ for $0 \leqslant u < b$, which leads to

$$q(x) = -\frac{1}{\pi}\int_b^1 \frac{u}{(u^2 - |x|^2)^{3/2}} h(u)\,\mathrm{d}u, \quad |x| < b. \tag{25}$$

We next claim that

$$\partial_{x_1}^n (u^2 - |x|^2)^{-3/2} \geqslant n! \tag{26}$$

if n is even and $|x| < b \leqslant u$. To see this write

$$\frac{1}{(u^2 - x_1^2 - x_2^2)^{3/2}} = \frac{1}{(u^2 - x_2^2)^{3/2}}\left(1 - \frac{x_1^2}{u^2 - x_2^2}\right)^{-3/2}.$$

If $|x| = \sqrt{x_1^2 + x_2^2} < b \leqslant u$, then

$$|x_1| < \sqrt{b^2 - x_2^2} \leqslant \sqrt{u^2 - x_2^2},$$

which shows that we can apply (24) with $t = x_1 \Big/ \sqrt{u^2 - x_2^2}$. Since $\partial/\partial_{x_1} = (\mathrm{d}t/\mathrm{d}x_1)\, \partial/\partial t$ and $\mathrm{d}t/\mathrm{d}x_1 = 1/\sqrt{u^2 - x_2^2} \geqslant 1$, this proves (26). Using the fact that $h(u) \geqslant 0$ and $\int_0^1 h(u)\mathrm{d}u = 1$ we can now conclude from (25) and (26) that

$$|f_n(x)| = |\partial_{x_1}^n q(x)| \geqslant \frac{1}{\pi} \cdot b \cdot n! \int_b^1 h(u)\mathrm{d}u = \frac{b\, n!}{\pi}, \qquad |x| < b,$$

which immediately gives (15).

Let us finally sum up how the pieces are put together.

End of proof of theorem. From (5) we see that the norm $\|g\|_s$ is increasing in s, and therefore the same must be true of the norm $\|g\|_{\Lambda,s}$. It is therefore sufficient to prove the theorem for arbitrary non-negative integers s. Inserting the estimate (16) into (17) we get

$$\|Rf_n\|_{\Lambda(b,\delta),s} \leqslant C_{b,s,\delta} n^s \delta^n \left(\frac{8e}{1 - b}\right)^n n!.$$

On the other hand, the estimate (15) means that

$$\|f_n\|_G \geqslant c_{b,G}\, n! \quad \text{for even } n$$

for some $c_{b,G} > 0$. Combination of those two estimates with f_n replaced by f_{2n} proves (8), if $\delta 8e/(1 - b) < 1$. The proof is complete.

Remark 3. By careful estimates one can strengthen the theorem a little by replacing the condition on δ by $\delta < (1 - b)/2e$. To see this we first observe that we can take the function $\theta(t)$ in the proof of lemma 3 with support in $[-1 + \rho, 1 - \rho]$ for small $\rho > 0$ so that $\sup \theta(t) \leqslant \frac{1}{2} + \rho$ and $\int |\theta'(t)|\mathrm{d}t \leqslant 1 + 2\rho$. This gives the estimate (21) with the right-hand side replaced by $(C_\rho/\sqrt{d})((2 + \rho)n/d)^k$ for arbitrary $\rho > 0$. Second, in (23) we used the estimate $(n + s)^s \leqslant C_s 2^s$; replacing this by $(n + s)^s \leqslant C_{s,\rho}(1 + \rho)^s$ for arbitrarily small $\rho > 0$ we could gain another factor 2.

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