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On local injectivity for generalized Radon transforms

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Dedicated to the memory of Leon Ehrenpreis

1 Introduction

I met Leon Ehrenpreis already in 1961, when he presented his celebrated ”Fundamental Principle“ at a summer school at Stanford University. Much later I was fortunate to meet Leon again at many Radon transform meetings and during several visits to Temple University. Leon’s enthusiasm and generosity in sharing mathematical ideas, his broad outlook, and his original way of looking at problems was a great source of inspiration for many of us.

It was shown by R. Strichartz in [15] that the classical Radon transform is *locally injective* in the following sense, here for simplicity formulated for the case of $\mathbf{R}^2$. If the continuous function $f(x,y)$ is supported in $y \geq x^2$ and

$$\int f(x, \xi x + \eta)dx = 0 \tag{1}$$

for all (ξ, η) in a neighborhood of $(0,0)$, then f must vanish in some neighborhood of the origin. It is known [2] that the corresponding statement is not always true if a smooth positive weight function $m(\xi, \eta, x)$ is introduced, so that (1) is replaced by

$$\int f(x, \xi x + \eta) m(\xi, \eta, x) dx = 0\,. \tag{2}$$

On the other hand, if $m(\xi, \eta, x)$ is positive and real analytic, then local injectivity again holds [5]. Thus the set of weight functions $m(\xi, \eta, x)$ for which local injectivity holds is dense in the space of smooth weight functions. In [4] we recently extended the construction in [2] by showing that the set of smooth positive weight

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functions, for which local injectivity does not hold, is also dense. In [3] we proved local injectivity for a class of smooth weight functions that was introduced by S. Gindikin [12]. In this note we shall extend the result from [3] by replacing the family of straight lines by a 2-dimensional family of curves satisfying a certain condition, thus obtaining a statement that is invariant under local diffeomorphisms in the xy space and in the $\xi\,\eta$ space of curves.

On the surface our Radon transform (2) looks like a parametric Radon transform in the sense of Ehrenpreis [7]. However, replacing dx by $ds/\sqrt{1+\xi^2}$, where ds denotes arc length measure on the line, we can of course just as well regard our transform as a restriction of a non-parametric Radon transform, expressed in affine coordinates.

With a suitable choice of coordinates in xy space and $\xi\,\eta$ space an arbitrary curve family in a neighborhood of the origin in xy space can be written $y = u(\xi,\eta,x)$ for (ξ,η) near $(0,0)$, where

$$u(\xi,\eta,x) = \eta + \xi x + \mathscr{O}(x^2) \quad \text{as } x \to 0\,.$$

The *dual* curve family is the family of curves in $\xi\,\eta$ space that is defined by the equation $y - u(x,\xi,\eta) = 0$ with x and y playing the role of parameters. The condition on the curve family in our main theorem (Theorem 2) is somewhat implicit and reads as follows: the curves in the dual family are solution curves to a second order differential equation of the form

$$\eta'' = \Psi(\xi,\eta,\eta')\,,$$

where $\Psi(\xi,\eta,p)$ is a polynomial of degree at most 3 in p. This condition is known to be independent of the choice of coordinates in the $\xi\,\eta$ plane (Proposition 2). These curve families have been known for a long time. In fact, Eli Cartan showed in 1924 that those curves are the geodesics of torsion free projective connections. But the condition has also played a role in integral geometry. I. M. Gelfand and his collaborators studied what they called *admissible families* of lines or curves, in some cases curve families with densities, in real or complex spaces in a long series of papers (see [10], [8], [11] and references given there). In the case of curve families in $\mathbf{R}^2$ it was found that a curve family with densities was admissible precisely when both curve family and densities satisfied the conditions considered here, [10]. It would be interesting to understand why our local injectivity problem leads to the same conditions on curve families and densities as the admissibility property of Gelfand, Gindikin, and Shapiro.

In Section 2 we introduce the double fibration defined by a hypersurface Z in the product of two 2-dimensional manifolds, we define a weighted Radon transform on the 2-dimensional manifold of curves defined by this fibration, and we formulate the problem of local injectivity for this transform. In Section 3 we state and prove a local injectivity theorem (Theorem 1) under assumptions that are expressed in local coordinate systems in X and Γ. In Section 4 we prove that our condition on the weight function (actually a *density* on Z) is independent of the chosen coordinate systems by expressing it in coordinate free terms. In Section 5 we give a couple of auxiliary results on curve families defined by an ordinary differential equation of the second order, and in Section 6 we extend Theorem 1 by replacing the condition on Z there by a weaker condition that is invariant under coordinate changes in X and Γ (Theorem 2).

2 A weighted Radon transform associated to $Z \subset X \times \Gamma$

To define our Radon transform in an invariant way we shall use the double fibration setup introduced by Helgason and Gelfand, [14], [9]. Let X and Γ be 2-dimensional manifolds and let $Z \subset X \times \Gamma$ be a smooth hypersurface in $X \times \Gamma$ such that both projections

$$\pi_X : Z \to X, \qquad \pi_\Gamma : Z \to \Gamma$$

have surjective differential at a point $(\mathbf{x}^0, \gamma^0) \in Z$. Using local coordinate systems in X and Γ we can define Z near $(\mathbf{x}^0, \gamma^0)$ by

$$F(x, y, \xi, \eta) = 0, \tag{3}$$

where $\mathbf{x} = (x, y)$ denotes points of X and $\gamma = (\xi, \eta)$ denotes points of Γ, and F is a smooth function satisfying

$$d_{(x,y)}F \neq 0 \quad \text{and} \quad d_{(\xi,\eta)}F \neq 0 \quad \text{at} \quad (\mathbf{x}^0, \gamma^0). \tag{4}$$

Furthermore we assume that the natural projections from the conormal

$$\begin{gathered} N^*(Z) \to T^*(X) \quad \text{and} \quad N^*(Z) \to T^*(\Gamma) \quad \text{are local diffeomorphisms} \\ \text{near the point } (\mathbf{x}^0, \gamma^0; dF(\mathbf{x}^0, \gamma^0)) \in N^*(Z). \end{gathered} \tag{5}$$

We may assume that $\mathbf{x}^0 = (x^0, y^0) = (0,0)$ and $\gamma^0 = (\xi^0, \eta^0) = (0,0)$. We can also rotate the coordinate systems in X and Γ so that

$$F'_x = 0, \quad F'_y \neq 0, \quad F'_\xi = 0, \quad F'_\eta \neq 0 \quad \text{at the origin.} \tag{6}$$

Then we can solve y or η from (3) and obtain respectively

$$y = u(\xi, \eta, x), \qquad \eta = \rho(x, y, \xi). \tag{7}$$

Differentiating the identity $F(x, u(\xi, \eta, x), \xi, \eta) = 0$ we obtain

$$F'_x + F'_y u'_x = 0, \quad F'_\xi + F'_y u'_\xi = 0, \quad F'_\eta + F'_y u'_\eta = 0, \tag{8}$$

in a neighborhood of the origin. From here and (6) we obtain

$$u'_x = u'_\xi = 0 \quad \text{and} \quad u'_\eta \neq 0 \quad \text{at the origin,} \tag{9}$$

and similarly,

$$\rho'_x = \rho'_\xi = 0 \quad \text{and} \quad \rho'_y \neq 0 \quad \text{at the origin.}$$

If we choose (x,y,ξ) as coordinates on Z and (x,y,ξ,λ) as coordinates on $N^*(Z)$ and denote the conormal by $\lambda\, dF$, then the projection $N^*(Z) \to T^*(X)$ can be represented

$$(x,y,\xi,\lambda) \mapsto (x,y,\theta_x,\theta_y) = (x,y,\lambda F'_x,\lambda F'_y)\,.$$

This map has non-singular differential if and only if the map

$$(\xi,\lambda) \mapsto (\theta_x,\theta_y) = (\lambda F'_x,\lambda F'_y)$$

has non-singular differential, which is the case if and only if

$$\det\begin{vmatrix}\lambda F''_{x\xi} & F'_x \\ \lambda F''_{y\xi} & F'_y\end{vmatrix} = \lambda \det\begin{vmatrix}F''_{x\xi} & F'_x \\ F''_{y\xi} & F'_y\end{vmatrix} \neq 0\,.$$

Hence the assumption that the projection $N^*(Z) \to T^*(X)$ has non-singular differential is equivalent to

$$F''_{x\xi} \neq 0 \quad \text{at the origin.} \tag{10}$$

The fact that the condition (10) is symmetric with respect to interchange of the spaces X and Γ shows that both projections (5) have non-singular differentials if one of them does and (4) holds.

Differentiating the second equation of (8) with respect to x and using (9) and (10) we find that

$$u''_{x\xi} \neq 0 \quad \text{at the origin.} \tag{11}$$

This shows that the maps

$$\begin{aligned}(\xi,\eta) &\mapsto (u(\xi,\eta,0),u'_x(\xi,\eta,0)), \quad \text{and} \\ (x,\xi,\eta) &\mapsto (x,u(\xi,\eta,x),u'_x(\xi,\eta,x))\end{aligned} \tag{12}$$

are local diffeomorphisms near the origin.

Let μ be a given positive, smooth density on Z. For instance, in the (ξ,η,x) coordinates on Z let the density μ be given as

$$\mu = m(\xi,\eta,x)d\xi\, d\eta\, dx\,.$$

If f is a continuous function on X, then $f \circ \pi_X$ is a function on Z, so $(f \circ \pi_X)\mu$ is a density on Z. If φ is a compactly supported continuous function on Γ, then $\varphi \circ \pi_\Gamma$ is a compactly supported continuous function on Z, so we can form

$$\langle (f\circ\pi_X)\mu, \varphi\circ\pi_\Gamma\rangle = \iiint f(x,u(\xi,\eta,x))m(\xi,\eta,x)\,\varphi(\xi,\eta)dx d\xi\, d\eta\,,$$

and we can define Rf on Γ as the measure

$$\varphi \mapsto \langle (f\circ\pi_X)\mu, \varphi\circ\pi_\Gamma\rangle = \langle \mu, (f\circ\pi_X)(\varphi\circ\pi_\Gamma)\rangle\,. \tag{13}$$

The situation is symmetric in f and φ, so if we define $R^*\varphi$ as the measure on X

$$f \mapsto \langle \mu, (f \circ \pi_X)(\varphi \circ \pi_\Gamma) \rangle, \tag{14}$$

we have $\langle Rf, \varphi \rangle = \langle f, R^*\varphi \rangle$. To obtain an explicit expression for $R^*\varphi$ we express the density μ and the function $\varphi \circ \pi_\Gamma$ in terms of the (x,y,ξ) coordinates on Z,

$$\mu = n(x,y,\xi)dx\,dy\,d\xi, \qquad (\varphi \circ \pi_\Gamma)(x,y,\xi) = \varphi(\xi, \rho(x,y,\xi)).$$

This gives

$$\begin{aligned}\langle Rf, \varphi \rangle = \langle f, R^*\varphi \rangle &= \langle \mu, (f \circ \pi_X)(\varphi \circ \pi_\Gamma) \rangle \\ &= \iiint f(x,y)\varphi(\xi, \rho(x,y,\xi))n(x,y,\xi)dx\,dy\,d\xi\,.\end{aligned}$$

Since the Jacobian of the transformation $x = x', \xi = \xi', \eta = \rho(x',y,\xi')$ is ρ'_y we have

$$n(x,y,\xi) = m(\xi, \rho(x,y,\xi), x)\,\rho'_y(x,y,\xi), \tag{15}$$

so

$$R^*\varphi(x,y) = \int \varphi(\xi, \rho(x,y,\xi))n(x,y,\xi)d\xi\,.$$

It will be convenient to extend the definition of R and R^* as follows. If u is a continuous function on Z, then we define $R(uf)$ and $R^*(u\varphi)$ by replacing the right hand sides of (13) and (14) by

$$\langle \mu, (f \circ \pi_X)(\varphi \circ \pi_\Gamma)u \rangle.$$

Then we have

$$\langle R(uf), \varphi \rangle = \langle f, R^*(u\varphi) \rangle. \tag{16}$$

In geometric terms the transform R integrates over the fibers $\pi_X(\pi_\Gamma^{-1}(\gamma)) \subset X$, which we will sometimes (by abuse of language) denote by γ or $\gamma_{(\xi,\eta)}$. The adjoint R^* integrates over the fibers $\pi_\Gamma(\pi_X^{-1}(\mathbf{x})) \subset \Gamma$.

Our problem can now be formulated in invariant terms as follows. Let $(\mathbf{x}^0, \gamma^0) \in Z \subset X \times \Gamma$ and let f be a continuous function defined in an open neighborhood U of $\mathbf{x}^0$. After shrinking U, if needed, we may assume that $U \setminus \gamma^0$ has precisely two components, which we denote by U_+ and U_-. Assume f is supported in $U_+ \cup \{\mathbf{x}^0\}$ and that $Rf = 0$ in some neighborhood of γ^0. We want to conclude that $f = 0$ in some neighborhood of $\mathbf{x}^0$ (under suitable conditions on μ and Z).

3 Local injectivity

We now formulate our assumptions on the density μ and the manifold Z (which determines the curve family) in terms of the coordinates (x,y,ξ) on Z. We assume as always that the manifold Z is defined by an equation $F(x,y,\xi,\eta)=0$ for which (4) and (5) hold, and that coordinates are chosen so that (6) holds.

The assumption on the density $\mu = n(x,y,\xi)dx\,dy\,d\xi$ will be as follows:

$$\begin{gathered}\text{there exist two functions } a_1 \text{ and } b_1 \text{ that are constant on the fibers } \pi_\Gamma^{-1}(\gamma),\\ \text{such that} \quad n'_\xi = (a_1\rho'_\xi + b_1)n\,.\end{gathered} \tag{17}$$

Recall that a function $a(\xi,\eta,x)$ on Z expressed in terms of the coordinates (ξ,η,x) is constant on the fibers $\pi_\Gamma^{-1}(\gamma)$ if it is independent of x. If the function is expressed in terms of the coordinates (x,y,ξ), then it is constant on the fibers $\pi_\Gamma^{-1}(\gamma)$ if it is of the form $a(x,y,\xi) = a_0(\xi, y-\xi x)$ for some function $a_0(\xi,\eta)$. The assumption on Z will be that Z is given by an equation $\eta = \rho(x,y,\xi)$, where ρ satisfies a differential equation

$$\rho''_{\xi\xi} = a_2(\rho'_\xi)^2 + b_2\rho'_\xi + c_2\,, \tag{18}$$

for some functions a_2, b_2, and c_2 that are constant on the fibers $\pi_\Gamma^{-1}(\gamma)$.

As usual we shall assume that coordinates in X and Γ are chosen such that $\mathbf{x}^0 = (0,0)$ and $\gamma^0 = (0,0)$. In this section we shall also assume that coordinates in X are chosen so that γ^0 is equal to the x-axis, in other words,

$$u(0,0,x) = 0, \quad \text{or equivalently} \quad \rho(x,0,0) = 0\,, \tag{19}$$

in some neighborhood of $x = 0$.

Theorem 1. *Assume that Z is defined by $\eta = \rho(x,y,\xi)$, where ρ satisfies* (18) *and* (19)*, and that the positive measure μ on Z satisfies* (17)*. Let f be a continuous function defined in some neighborhood of* $(0,0) \in X$ *and supported in a compact set contained in* $\{(x,y); y>0\} \cup \{(0,0)\}$*, and assume $Rf(\xi,\eta) = 0$ in some neighborhood of* $(0,0)$*. Then $f = 0$ in some neighborhood of* $(0,0)$.

In the special case considered in [3] the manifold Z is defined by $y = \xi x + \eta$, so $\rho(x,y,\xi) = y - \xi x$, which gives $\rho'_\xi = -x$ and $\rho''_{\xi\xi} = 0$, so (18) holds with $a_2 = b_2 = c_2 = 0$. The weight function $m(\xi,\eta,x)$ in [3] was assumed to satisfy

$$m'_\xi(\xi,\eta,x) - x m'_\eta(\xi,\eta,x) = \big(x a(\xi,\eta) - b(\xi,\eta)\big) m(\xi,\eta,x) \tag{20}$$

for some $a(\xi,\eta)$ and $b(\xi,\eta)$ that are independent of x. By (15) we have in this case $n(x,y,\xi) = m(\xi, y-\xi x, x)$, hence $n'_\xi = m'_\xi - x m'_\eta$, which shows that for this Z, (17) is equivalent to (20).

Using the coordinates (ξ,η,x) on Z we introduce the function $q(\xi,\eta,x)$ by

$$q(\xi,\eta,x) = \frac{u'_\xi(\xi,\eta,x)}{u'_\eta(\xi,\eta,x)}\,. \tag{21}$$

Since $u'_\eta(0,0,0) \neq 0$, the function q is well defined and smooth in some neighborhood of $(0,0,0) \in Z$. Choose a neighborhood U of $(0,0) \in X$ and a neighborhood

V of $(0,0) \in \Gamma$ such that $u'_\eta \neq 0$ in $(U \times V) \cap Z$. Let f be a continuous function supported in U and set

$$G_k(\xi,\eta) = (-1)^k \int q(\xi,\eta,x)^k f(x,u(\xi,\eta,x))m(\xi,\eta,x)dx, \quad k = 0,1,\ldots. \tag{22}$$

Note that all $G_k(\xi,\eta)$ are well defined in V and that $G_0(\xi,\eta) = Rf(\xi,\eta)$. The idea of the proof of Theorem 1 is to show that all G_k vanish in a fixed neighborhood of the origin. We shall see that this easily implies that $f = 0$ in some neighborhood of the origin. In the special case when $u(\xi,\eta,x) = \xi x + \eta$, we have $q = -\rho'_\xi = x$.

The main ingredient in the proof of Theorem 1 is the following fact.

Proposition 1. *Let f be a continuous function supported in U and let the functions $G_k(\xi,\eta)$ be defined in V by* (22) *as described above. Assume that the density μ and the hypersurface Z satisfy* (17) *and* (18). *Then there exist functions $a(\xi,\eta)$, $b(\xi,\eta)$, $c(\xi,\eta)$, such that the following differential equations are satisfied in distribution sense in V:*

$$\begin{gathered}(\partial_\eta - a)G_1 + (\partial_\xi - b)G_0 = 0, \quad \textit{and} \\ (\partial_\eta - a)G_{k+1} + (\partial_\xi - b)G_k - cG_{k-1} = 0, \quad k \geq 1\,. \end{gathered} \tag{23}$$

Proof. If φ is an arbitrary smooth function on Γ that is supported in V and $n(x,y,\xi)$ is the density on Z as above, we have for every $k \geq 0$ the trivial identity

$$\int \partial_\xi \Big((\rho'_\xi(x,y,\xi))^k \varphi(\xi,\rho(x,y,\xi)) n(x,y,\xi) \Big) d\xi = 0\,. \tag{24}$$

If $k \geq 1$ the integrand can be written

$$k(\rho'_\xi)^{k-1} \rho''_{\xi\xi}\, \varphi\, n + (\rho'_\xi)^k (\varphi'_\xi + \varphi'_\eta \rho'_\xi) n + (\rho'_\xi)^k \varphi\, n'_\xi\,.$$

Inserting the expressions for $\rho''_{\xi\xi}$ and n'_ξ from (17) and (18) and rearranging terms we obtain

$$(\varphi'_\xi + \varphi'_\eta \rho'_\xi)(\rho'_\xi)^k n + (\rho'_\xi)^k \varphi n (a\rho'_\xi + b) + c(\rho'_\xi)^{k-1} \varphi n\,,$$

where $a = ka_2 + a_1$ and $b = kb_2 + b_1$, and $c = kc_2$. Note that this expression for the integrand is also valid for $k = 0$, since $c = 0$ then. Thus (24) can be written

$$R^*\big((\rho'_\xi)^k \varphi'_\xi + (\rho'_\xi)^{k+1} \varphi'_\eta + (\rho'_\xi)^{k+1} a\varphi + (\rho'_\xi)^k b\varphi + (\rho'_\xi)^{k-1} c\varphi \big) = 0\,.$$

Multiplying by $f(x,y)$, integrating, and using (16) with $u = (\rho'_\xi)^j = (-q)^j$ we obtain

$$\begin{aligned}\langle R((-q)^k f), \varphi'_\xi \rangle + \langle R((-q)^{k+1} f), \varphi'_\eta \rangle + \langle R((-q)^{k+1} f), a\varphi \rangle \\ + \langle R((-q)^k f), b\varphi \rangle + \langle R((-q)^{k-1} f), c\varphi \rangle = 0\,.\end{aligned}$$

By virtue of the definition of G_k this means the same as (23).

The following simple observation will also be needed in the proof of Theorem 1.

Lemma 1. *If coordinate systems in X and Γ are chosen so that* (6) *holds, then*

$$\partial_x(u'_\xi/u'_\eta) \neq 0 \quad \textit{at the origin}.$$

Proof. Since $u'_\eta \neq 0$, it is sufficient to observe that

$$u'_\eta u''_{x\xi} - u'_\xi u''_{x\eta} \neq 0,$$

which follows from (11) and the fact that $u'_\xi = 0$ at the origin.

Proof of Theorem 1. We are going to prove that all the functions $G_k(\xi,\eta)$ vanish in a fixed neighborhood of the origin. To do this we first need to fix a region in (ξ,η)-space for which the curve $\gamma_{(\xi,\eta)}$ does not meet the support of f.

Choose $\varepsilon > 0$ such that equations (23) hold in

$$\Omega_\varepsilon = \{(\xi,\eta); |\xi| < \varepsilon, |\eta| < \varepsilon\}.$$

By (6) we know that $u'_\eta(0,0,0) \neq 0$, and we may assume that $u'_\eta(0,0,0) > 0$. Hence we can choose $\delta > 0$ and κ such that $u'_\eta(0,0,x) \geq \kappa > 0$ for $|x| \leq \delta$. By possibly replacing ε by a smaller number and recalling the assumption (19) we can then achieve that

$$u(\xi,\eta,x) \leq -d < 0, \quad \text{for } -\varepsilon/2 < \eta < -\varepsilon,\ |\xi| < \varepsilon,\ |x| < \delta.$$

Set $K = \operatorname{supp} f$. Since $\gamma_{(0,0)} \cap K = \{(0,0)\}$ it is clear that

$$\gamma_{(\xi,\eta)} \cap K \cap \{(x,y); |x| > \delta\} = \emptyset,$$

if $|\xi|$ and $|\eta|$ are sufficiently small. Hence, by possibly choosing ε still smaller we can achieve that

$$\gamma_{(\xi,\eta)} \cap K = \emptyset, \quad \text{if } -\varepsilon/2 < \eta < -\varepsilon \text{ and } |\xi| < \varepsilon.$$

We now prove that all G_k vanish in Ω_ε. By the assumption $G_0 = 0$ in Ω_ε. By Proposition 1 this implies that the function $\eta \mapsto G_1(\xi,\eta)$ satisfies the ordinary differential equation

$$\partial_\eta G_1(\xi,\eta) - a(\xi,\eta)G_1(\xi,\eta) = 0$$

for $|\xi| < \varepsilon$. But $G_1(\xi,\eta)$ is obviously equal to zero whenever the curve $y = u(\xi,\eta,x)$ does not meet the support of f, which as we have just seen is certainly the case if $(\xi,\eta) \in \Omega_\varepsilon$ and $\eta < -\varepsilon/2$. Hence $G_1 = 0$ in Ω_ε.

To proceed we shall use induction over k. Assume that $G_k = 0$ in Ω_ε for all $k \leq p$, where $p \geq 1$. By Proposition 1 the function $\eta \mapsto G_{p+1}(\xi,\eta)$ must then satisfy the differential equation

$$\partial_\eta G_{p+1}(\xi,\eta) - a(\xi,\eta)G_{p+1}(\xi,\eta) = 0$$

in Ω_ε. Reasoning as before we can conclude that $G_{p+1} = 0$ in Ω_ε, which proves the assertion.

To complete the proof of Theorem 1 we note that in particular

$$G_k(0,\eta) = (-1)^k \int q(0,\eta,x)^k f(x,u(0,\eta,x))m(0,\eta,x)dx = 0 \tag{25}$$

for all k and all $\eta < \varepsilon$. Since $q'_x \neq 0$ by Lemma 1 we can make the change of variable $q(0,\eta,x) = t$ in the integral (25) for an arbitrary fixed η with $\eta < \varepsilon$. It follows that (with obvious notation)

$$\int t^k f(x(t,\eta),u(0,\eta,x(t,\eta)))m(0,\eta,x(t,\eta))\frac{dx}{dt}dt = 0$$

for all k and for all η with $\eta < \varepsilon$. Since $m > 0$ and $dx/dt \neq 0$ it follows that f vanishes on the curve $y = u(0,\eta,x)$, and since those curves certainly cover a neighborhood of the origin in the (x,y) plane, we can conclude that $f = 0$ in a neighborhood of the origin. This completes the proof of Theorem 1.

4 The condition on the density μ.

We now describe the condition (17) on the density μ in intrinsic terms. Let V be a vector field on Z that is everywhere tangent to the fibers $\pi_X^{-1}(\mathbf{x})$. In the coordinates (ξ,η,x) this vector field can be written $V = V_1\partial_\xi + V_2\partial_\eta$, where V_1 and V_2 are functions on Z. In fact, since the equation for the fiber $\pi_X^{-1}(x^0,y^0)$ is

$$y^0 = u(\xi,\eta,x^0),$$

we see that the condition for $v_1\partial_\xi + v_2\partial_\eta + v_3\partial_x$ to be tangent to this curve is $v_3 = 0$ and $v_1u'_\xi + v_2u'_\eta = 0$. A vector field V on Z can be invariantly defined as a linear map from $C^\infty(Z)$ into itself such that $V(\varphi\psi) = \varphi V(\psi) + \psi V(\varphi)$. The operation of V on the density μ will be defined by $\langle V(\mu),\varphi\rangle = -\langle\mu,V(\varphi)\rangle$. If $\sigma = \sigma_1 d\xi + \sigma_2 d\eta + \sigma_3 dx$ is a 1-form on Z and $V = V_1\partial_\xi + V_2\partial_\eta + V_3\partial_x$ is a vector field, then the "contraction" $\sigma \lrcorner V = \sigma_1V_1 + \sigma_2V_2 + \sigma_3V_3$ is a function on Z. If σ is a 1-form on Γ, then we denote by $\pi_\Gamma^*(\sigma)$ the pullback of σ to Z under the projection π_Γ.

Consider the following condition on μ:

$$\text{there exists a 1-form } \sigma \text{ on } \Gamma \text{ such that } V(\mu) = (\pi_\Gamma^*(\sigma) \lrcorner V)\mu \tag{26}$$

for all vector fields V that are tangent to the fibers $\pi_X^{-1}(\mathbf{x})$.

If the condition (26) holds for one non-vanishing vector field V that is everywhere tangent to the fibers $\pi_X^{-1}(\mathbf{x})$, then it holds for all such vector fields. In fact, if $\tilde{V}$ is another vector field that is everywhere tangent to those fibers, then $\tilde{V} = \phi V$ for some function ϕ on Z, so multiplying (26) by ϕ we obtain $\tilde{V}(\mu) = (\pi_\Gamma^*(\sigma) \lrcorner \tilde{V})\mu$. In this way we also see that the 1-form σ is independent of the vector field V.

We now show that (17) is the same as (26). In the (x,y,ξ) coordinates $V=\partial_\xi$ is tangent to the fibers $\pi_X^{-1}(\mathbf{x})$. Thus

$$\begin{aligned}\langle V(\mu),\varphi\rangle = -\langle \mu, V(\varphi)\rangle &= -\iiint n(x,y,\xi)\partial_\xi \varphi(x,y,\xi)\,dx\,dy\,d\xi \\ &= \iiint n'_\xi(x,y,\xi)\varphi(x,y,\xi)\,dx\,dy\,d\xi\,. \qquad (27)\end{aligned}$$

Let $\sigma = a(\xi,\eta)d\xi + b(\xi,\eta)d\eta$ be a 1-form on Γ. To compute $\pi_\Gamma^*(\sigma)$ on Z we note that

$$\pi_\Gamma :\ (x,y,\xi) \mapsto (\xi,\eta)\,,$$

where $\eta = \rho(x,y,\xi)$. Thus

$$\pi_\Gamma^*(\sigma) = a\,d\xi + b\,d\rho = a\,d\xi + b(\rho'_x dx + \rho'_y dy + \rho'_\xi d\xi)\,.$$

With $V=\partial_\xi$ this gives

$$\pi_\Gamma^*(\sigma) \lrcorner\, V = a + b\rho'_\xi\,.$$

Hence

$$\langle (\pi_\Gamma^*(\sigma) \lrcorner\, V)\mu,\varphi\rangle = \iiint (a+b\rho'_\xi) n(x,y,\xi)\varphi(x,y,\xi)\,dx\,dy\,d\xi\,. \qquad (28)$$

Combining (27) and (28) we now see that (26) is equivalent to (17).

It is possible to prove Theorem 1 using the coordinates (ξ,η,x) instead of the (x,y,ξ) coordinates. Recall that the function $q(\xi,\eta,x)$ is defined by (21). The assumptions on $\mu = m(\xi,\eta,x)d\xi\,d\eta\,dx$ and Z then read as follows. The function $m(\xi,\eta,x)$ must satisfy

$$\partial_\eta(qm) - \partial_\xi m = (a_1 q + b_1)m\,, \qquad (29)$$

where a_1 and b_1 are functions that depend only on (ξ,η). The condition on Z is expressed by the following condition on the function q: there exist functions a_2, b_2, and c_2 that depend only on (ξ,η) such that

$$q\,\partial_\eta q - \partial_\xi q = a_2 q^2 + b_2 q + c_2\,. \qquad (30)$$

The proof of Theorem 1 using those assumptions is quite parallel to the proof of Theorem 1 in [3], the factor q playing the role of the factor x in [3]. Recall that $q(\xi,\eta,x) = x$, if $u(\xi,\eta,x) = \xi x + \eta$.

It is instructive to verify that (29) is also the same as (26). Choose $V = \partial_\xi - q\partial_\eta$, where $q = u'_\xi / u'_\eta$ as above. Then

$$\begin{aligned}\langle V(\mu),\varphi\rangle &= -\langle \mu, V(\varphi)\rangle = -\langle \mu,(\partial_\xi - q\partial_\eta)\varphi\rangle\\ &= -\iiint \big((\partial_\xi - q\partial_\eta)\varphi(\xi,\eta,x)\big)m(\xi,\eta,x)d\xi\,d\eta\,dx\\ &= \iiint (\partial_\xi m - \partial_\eta(qm))\varphi\,d\xi\,d\eta\,dx\,.\end{aligned}$$

Assume that $\sigma = a(\xi,\eta)d\xi + b(\xi,\eta)d\eta$. Then

$$\pi_\Gamma^*(\sigma) \lrcorner V = aV_1 + bV_2 = a - bq\,.$$

Thus condition (26) means that

$$\iiint (\partial_\xi m - \partial_\eta(qm))\varphi\,d\xi\,d\eta\,dx = \iiint (a-bq)m\varphi\,d\xi\,d\eta\,dx$$

for all φ, or

$$\partial_\xi m - \partial_\eta(qm) = (a-bq)m$$

for some a and b, which is condition (29).

5 Curve families and ordinary differential equations

In the next section we shall see that the condition (18) on Z is not invariant with respect to smooth coordinate transformations in Γ, and we shall replace it by an invariant condition. To do this we shall use two well known results from the geometric theory of ordinary differential equations.

Proposition 2. *Consider an ordinary differential equation of the second order*

$$y'' = \Phi(x,y,p), \quad p = dy/dx\,, \tag{31}$$

where $\Phi(x,y,p)$ is a polynomial in p of degree ≤ 3 with coefficients that are smooth functions of x and y. Under an arbitrary smooth variable transformation in the plane

$$x = F(X,Y), \quad y = G(X,Y)\,, \tag{32}$$

the equation (31) *is transformed into an equation of the same form, that is,*

$$Y'' = \Psi(X,Y,P), \quad P = dY/dX\,, \tag{33}$$

where $\Psi(X,Y,P)$ is a polynomial in P of degree at most 3.

This is Theorem 2 of Chapter 1 §6 E in [1].

Corollary 1. *Let a* 2*-parameter family of curves in a neighborhood of* $(0,0)$ *in xy space be given by $F(x,y,\xi,\eta) = y - u(\xi,\eta,x) = 0$ for parameters in a neighborhood of* $(0,0)$ *in $\xi\,\eta$ space, and assume that F satisfies* (4) *and* (5)*. Assume that* *there exist coordinates in xy space in which the curves are represented as straight lines. Then the curves of the given family and its dual family both have the property that the curves of the family are solution curves of some differential equation* (31) *where $\Phi(x,y,p)$ is a polynomial in p of degree at most* 3.

Proof. The assumption means that we can choose coordinates in xy space such that $u(\xi,\eta,x) = A(\xi,\eta)x + B(\xi,\eta)$ for all (ξ,η) in some neighborhood of $(0,0)$. Since the map (12) is a local diffeomorphism we can choose coordinates in a neighborhood of the origin in $\xi\,\eta$ space such that $A(\xi,\eta) = \xi$ and $B(\xi,\eta) = \eta$, hence in the new coordinates $u(\xi,\eta,x) = \xi x + \eta$. Thus we can find coordinates so that both the given family and its dual are solution curves of the differential equation $y'' = 0$. The zero function is a polynomial of degree at most 3, so the statement now follows from Proposition 2.

Remark. The converse of the statement in the corollary is also true. In other words, the condition that both the curves of the given family and its dual family are solution curves of a differential equation (31) where $\Phi(x,y,p)$ is a polynomial in p of degree at most 3 is a necessary and sufficient condition for the given curve family to be diffeomorphic to a family of straight lines. See [1], Chapter 1 §6 G.

Proposition 3. *Assume $\Phi(x,y,p)$ is a polynomial in p of degree ≤ 3 as in Proposition 2. Then there exists a smooth variable transformation*

$$x = F(X,Y), \quad y = Y, \tag{34}$$

in some neighborhood of the origin, $F(0,0) = 0$, $F'_X(0,0) \neq 0$, such that the differential equation (31) *is transformed to an equation* (33) *where $\Psi(X,Y,P)$ is a polynomial in P of degree at most* 2.

For the proof we shall use the following lemma from [1], Chapter 1 §6 B.

Lemma 2. *The substitution* (34) *transforms equation* (31) *into the equation*

$$Y'' = \widehat{\Phi}(X,Y,P), \quad P = dY/dX, \tag{35}$$

where

$$\widehat{\Phi}(X,Y,P) = \frac{\Delta^3}{F'_X}\Phi(X,Y,P/\Delta) + \frac{P}{F'_X}(F''_{XX} + 2PF''_{XY} + P^2F''_{YY}), \tag{36}$$

and $\Delta = F'_X + PF'_Y$.

Proof of Proposition 3. By Proposition 2 we know that Ψ will be a polynomial in P of degree at most 3, so it will be enough to show that we can choose $F(X,Y)$ so that the coefficient of P^3 in (36) vanishes. Assume that

$$\Phi = \Phi_0 + p\Phi_1 + p^2\Phi_2 + p^3\Phi_3,$$

where Φ_j are functions of (x,y). Third degree terms in P will only occur in the terms

$$\frac{1}{F'_X}(\Delta^3\Phi_0 + P\Delta^2\Phi_1 + P^2\Delta\Phi_2 + P^3\Phi_3) + P^3\frac{F''_{YY}}{F'_X}\,.$$

The coefficient of P^3 will therefore be

$$\frac{1}{F'_X}(\Phi_0(F'_Y)^3 + \Phi_1(F'_Y)^2 + \Phi_2 F'_Y + \Phi_3) + \frac{F''_{YY}}{F'_X}\,,$$

which can be written

$$\frac{1}{F'_X}\big(\Lambda(X,Y,F'_Y) + F''_{YY}\big)\,,$$

where we have denoted the polynomial $P^3\Phi(X,Y,1/P)$ by $\Lambda(X,Y,P)$. Note that $F'_X \neq 0$, since the Jacobian of the transformation is F'_X. Now choose $H(X,Y)$ in a neighborhood of the origin as a solution of the ordinary differential equation

$$H'_Y(X,Y) = -\Lambda(X,Y,H(X,Y)), \quad H(X,0) = 0\,,$$

and then choose

$$F(X,Y) = X + \int_0^Y H(X,t)dt\,.$$

Then the coefficient of P^3 will vanish identically, and the proposition is proved.

6 Local injectivity: invariant statement

We can now formulate a statement that is invariant under separate coordinate transformations in X and Γ.

As always the manifold Z is defined by $F(x,y,\xi,\eta) = 0$ and is assumed to satisfy (4) and (5), and coordinates are assumed to be chosen so that (6) holds.

Theorem 2. *Let $Z \subset X \times \Gamma$ be defined by $F(x,y,\xi,\eta) = \eta - \rho(x,y,\xi) = 0$ in a neighborhood V of a point $(\mathbf{x}^0,\gamma^0) = (0,0,0,0) \in X \times \Gamma$. Assume that the function $\xi \mapsto \rho(x,y,\xi)$ for all (x,y) in a neighborhood of $(0,0) \in X$ satisfies a differential equation*

$$\rho''_{\xi\xi} = \Phi(\xi,\eta,\rho'_\xi)$$

in some neighborhood of $(\xi^0,\eta^0) = (0,0) \in \Gamma$, where $\Phi(\xi,\eta,p)$ is a polynomial of degree ≤ 3 in p with coefficients that are smooth functions of (ξ,η), and that the density μ on Z satisfies (17) *in V. Let U be a neighborhood of $\mathbf{x}^0$ in X such that $U \setminus \gamma^0$ has just two components that we denote by U_+ and U_-. Let f be a continuous function on U such that* $\operatorname{supp} f$ *is contained in $U_+ \cup \{\mathbf{x}^0\}$. Assume that $Rf = 0$ in some neighborhood of $\gamma^0 \in \Gamma$. Then $f = 0$ in some neighborhood $\mathbf{x}^0$.*

Proof. Choose a coordinate system in X such that (19) holds. Since this coordinate change obviously preserves the condition on $\rho(x,y,\xi)$ in the theorem, we can then use Proposition 3 to choose coordinates in Γ such that $\rho(x,y,\xi)$ satisfies (18). Those coordinate changes preserve the validity of (17), since we have proved that this condition is invariant under coordinate changes in X and Γ. The assertion now follows from Theorem 1.

We remark that Proposition 2 shows that the condition on $\rho(x,y,\xi)$ in Theorem 2 is invariant under coordinate changes in Γ, and we have already observed that it is trivially invariant under coordinate changes in X.

Finally we give an example of a curve family satisfying the condition in Theorem 2, which cannot be transformed to a family of straight lines. Following Gindikin [13] we consider for this purpose the set of horocycles in the hyperbolic plane with the right half plane as a model for the latter. Then the horocycles are the circles that are tangent to the y-axis

$$F(x,y,\xi,\eta) = (x-\xi)^2 + (y-\eta)^2 - \xi^2 = 0\,. \tag{37}$$

The full set of horocycles is unsuitable for us, since there are in general two horocycles with the same tangent direction through a given point. But if we restrict (x,y,ξ,η) to a neighborhood of for instance the point $(x^0,y^0) = (1,0)$, $(\xi^0,\eta^0) = (1,1)$, we get rid of this ambiguity, and F will satisfy (4) and (5). Differentiating (37) twice with respect to ξ and eliminating x and y leads to the differential equation

$$\eta'' = (1+\eta'^2)/2\eta\,,$$

which shows that the condition of Theorem 2 is satisfied. On the other hand, a similar computation with (x,y) and (ξ,η) interchanged shows that the curve family in the xy plane satisfies the differential equation

$$y'' = \frac{1}{x}(1+y'^2)(y' + \sqrt{1+y'^2})\,.$$

The last expression is not a polynomial in y', so by the Corollary our curve family cannot be transformed to straight lines.

For further information on the geometric theory of differential equations we refer to [6] and references given there.

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