HELGASON'S SUPPORT THEOREM FOR RADON TRANSFORMS – A NEW PROOF AND A GENERALIZATION

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1. Denote by Rf the Radon transform of the function f, i. e. $Rf(H) = \int_H f ds_H$ for, say, continuous functions f on $\mathbb{R}^n$ that decay at least as $|x|^{-n}$ as $|x| \to \infty$, and $H \in G_n$, the set of hyperplanes in $\mathbb{R}^n$; the surface measure on H is denoted ds_H. The well-known support theorem of Helgason [He1], [He2] states that if $Rf(H) = 0$ for all H not intersecting the compact convex set K and $f(x) = O(|x|^{-m})$ as $|x| \to \infty$ for all m, then f must vanish outside K. In [He3] Helgason extended this theorem to the case of Riemannian manifolds with constant negative curvature. Helgason's proofs depend in an essential way on the strong symmetry properties of the Radon transform. Here we will extend the theorem just cited to the case when a real analytic weight function depending on H as well as x is allowed in the definition of the Radon transform (see below), a situation without symmetry. For C^∞ weight functions analogous theorems are not true in general [B2]. Our method depends on the microlocal regularity properties of the Radon transform, a method we have already used in [BQ1] and [BQ2]. The case when f is assumed to have compact support was considered in [BQ1]. For rotation invariant (not necessarily real analytic) Radon transforms support theorems were given by E. T. Quinto [Q3].

2. Let $\rho = \rho(x, H)$ be a smooth function on the set Z of all pairs (x, H) of $H \in G_n$ and $x \in H$. We define the generalized Radon transform R_ρ by

$$R_\rho f(H) = \int_H f(\cdot)\rho(\cdot, H) ds_H, \quad H \in G_n.$$

If ρ is constant, R_ρ is of course the classical Radon transform. To describe our assumptions on ρ we shall consider $\mathbb{R}^n$ as sitting inside the projective space $\mathbb{P}^n(\mathbb{R})$ in the usual way:

$$\mathbb{R}^n \ni (x_1, \dots, x_n) \mapsto (1, x_1, \dots, x_n) \in \mathbb{P}^n(\mathbb{R}). \tag{1}$$

The manifold Z then becomes imbedded in the manifold

$$\tilde{Z} = \{(x, H); x \in H,\ H \text{ hyperplane in } \mathbb{P}^n(\mathbb{R})\}.$$

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Note that $\mathbb{P}^n(\mathbb{R})$ and $\tilde{Z}$ are compact real analytic manifolds. Our assumption will be that ρ can be extended to a positive real analytic function on $\tilde{Z}$. This assumption is of course fulfilled for the constant function, which is the case considered by Helgason in [He1].

theorem. *Assume ρ is a positive, real analytic function on Z that can be extended to a positive, real analytic function on $\tilde{Z}$. Let K be a compact, convex subset of $\mathbb{R}^n$. Let f be continuous on $\mathbb{R}^n$ and*

$$f(x) = O(|x|^{-m}) \quad \textit{as} \quad |x| \to \infty \tag{2}$$

for all m, and assume $R_\rho f(H) = 0$ for all H disjoint from K. Then $f = 0$ outside K.

As pointed out by Helgason, the assumption that f tends to zero rapidly as $|x| \to \infty$ cannot be omitted, even in the case of constant ρ.

The assumption that ρ is analytic at infinity cannot be omitted, even if the decay assumption is considerably strengthened. In fact, using ideas from [B2] one can construct examples where ρ is real analytic on Z, $1 \leq \rho \leq C$, f not identically zero, $R_\rho f = 0$, and, for instance, $f(x) = \mathcal{O}(\exp(-e^{|x|}))$ as $|x| \to \infty$.

The approach adopted here is suggested by the following facts. First, if ρ is analytic and different from zero, it is known that any solution f to $R_\rho f = 0$ must be real analytic, hence vanish if it vanishes in some open set [B1], [BQ1]. Second, if one could prove that f, considered as a function on the projective space $\mathbb{P}^n(\mathbb{R})$, must be analytic at infinity, then the assumption that f decays rapidly at infinity would imply that f vanishes identically. Third, the examples showing that the decay assumption cannot be omitted, in two dimensions the functions $\mathrm{Re}(x_1 + ix_2)^{-m}$, are in fact analytic at infinity. An advantage with this approach is, in addition to the fact that the weight function ρ is allowed to be non-constant, that the role of the decay assumption on f is "explained".

We prove the theorem by considering R_ρ as an operator on functions on $\mathbb{P}^n(\mathbb{R})$. The crucial fact is that f, considered as a function on $\mathbb{P}^n(\mathbb{R})$, must have a certain regularity property along the hyperplane at infinity, H_∞; in precise terms, the conormal manifold to H_∞, $N^*(H_\infty)$, must be disjoint from the analytic wavefront set of f (Proposition 1).

3. We are now going to express R_ρ in terms of a Radon transform on $\mathbb{P}^n(\mathbb{R})$. For this purpose we need to introduce some more notation. Set $X = \mathbb{R}^n$, $Y = G_n$, denote $\mathbb{P}^n(\mathbb{R})$ by $\tilde{X}$ and the set of hyperplanes in $\mathbb{P}^n(\mathbb{R})$ by $\tilde{Y}$. Denote the map (1) from X to $\tilde{X}$ by α. This map induces maps $Y \to \tilde{Y}$ and $X \times Y \to \tilde{X} \times \tilde{Y}$, which we will also denote by α. As models for $\tilde{X}$ and $\tilde{Y}$ we shall use the unit sphere S^n with opposite points identified. Thus a model for $\tilde{Z} \subset \tilde{X} \times \tilde{Y}$ will consist of all pairs $(u, \omega) \in S^n \times S^n$ such that $u \cdot \omega = 0$, all four pairs $(\pm u, \pm\omega)$ identified. On the plane $L(\omega) = \{u;\ u \cdot \omega = 0\}$ we choose the measure ds_L equal to the (push-forward of) $n-1$-dimensional surface measure on S^n. We use the notation $\omega = (\omega_0, \omega')$, and we note that the plane at infinity, L_∞, is represented by $\omega = (\pm 1, 0, \dots, 0)$.

Examples of functions ρ satisfying the hypothesis of the theorem can easily be constructed as follows. Let $a(z, \omega)$ be real analytic and positive on $\{(z, \omega); z \cdot \omega = 0\} \subset$

$\mathbb{R}^{2n+2}$, even and homogeneous of degree zero in both variables (separately), let $H_{\theta,p}$ be the plane $x \cdot \theta = p$, $\theta \in S^{n-1}$, and set

$$\rho(x, H_{\theta,p}) = a(1, x, -p, \theta), \tag{3}$$

for $x \in H_{\theta,p}$. Then a (restricted to $S^n \times S^n$) represents the extension, $\tilde{\rho}$, of ρ. Conversely, every ρ satisfying our assumptions can obviously be represented in the form (3).

Let τ be a positive real analytic function on $\tilde{Z}$. For continuous functions g on $\tilde{X}$ we define the generalized Radon transform $\tilde{R}_\tau$ by

$$\tilde{R}_\tau(g)(L) = \int_L g(\cdot)\tau(\cdot, L)\, ds_L, \quad L \in \tilde{Y}.$$

If f is a function on X, sufficiently small at infinity, $\tilde{f} = f \circ \alpha^{-1}$, $\tilde{\rho} = \rho \circ \alpha^{-1}$, and $L = \alpha(H)$, then

$$\begin{aligned} R_\rho f(H) &= \int_H f(\cdot)\rho(\cdot, H)\, ds_H = \int_L \tilde{f}(\cdot)\tilde{\rho}(\cdot, L)\, \alpha_*(ds_H) \\ &= \int_L \tilde{f}(\cdot)\tilde{\rho}(\cdot, L) b(\cdot, L)\, ds_L; \end{aligned} \tag{4}$$

here $\alpha_*(ds_H) = b(u, L) ds_L$ is the push-forward of the measure ds_H under α. It is very important for us that the density $b(u, L)$ can be factored, $b(u, L) = b_0(u)b_1(L)$, into a function depending only on u and one depending only on the plane L. This fact is well known; see e. g. [GGG], pp. 64-66.

Lemma 1. *The measure $\alpha_*(ds_H)$ is equal to $b(u, L)\, ds_L$, where*

$$b(u, \omega) = b(u, L(\omega)) = c|u_0|^{-n}\sqrt{1 - \omega_0^2} = c|u_0|^{-n}|\omega'|, \quad u_0 \neq 0, \quad |\omega'| \neq 0. \tag{5}$$

for some positive constant c.

Formula (5) shows that the measure $\alpha_*(ds_H)$ has a strong singularity along the plane at infinity. However, the fact that the density function $b(u, L)$ factors as expressed by (5), $b(u, L) = b_0(u)b_1(L)$, implies that this singularity is harmless in our context. In fact, using (4) and (5) we can write

$$R_\rho f(H) = b_1(L) \int_L \tilde{f}(u) b_0(u) \tilde{\rho}(u, L)\, ds_L = b_1(L)(\tilde{R}_\tau g)(L),$$

where $g(u) = \tilde{f}(u)b_0(u) = \tilde{f}(u)|u_0|^{-n}$, and $\tau(u, L) = \tilde{\rho}(u, L)$. Note that $g(u)$ tends to zero as $u_0 \to 0$, since f is rapidly decreasing; hence g is extendible to a continuous function on all of $\tilde{X}$, which vanishes on L_∞. Thus, in particular, if $R_\rho f(H) = 0$ for all H not intersecting K, then $\tilde{R}_\tau g$ must vanish in some neighbourhood of L_∞.

4. We will now turn our attention to the microlocal regularity properties of solutions to the equation $\tilde{R}_\tau g = 0$. The result that we shall need, Proposition 1 below, is well known, but it is not easy to find it in the literature. The analogous statement for the C^∞ category is clearly contained in the very general theory in [H1] as well as in [GS]. The additional arguments needed for the real analytic case can be found for instance in [Bj], ch. 4. Generalized Radon transforms as Fourier integral operators are discussed in [GS]; see also [GU] and [Q2]. In particular, Radon transforms on projective spaces are considered in [Q1]. For definition and basic properties of the analytic wavefront set, see [H2], ch. 8. If E is a smooth submanifold of the manifold M, one denotes by $N^*(E)$ the conormal manifold to E, that is, the set of all $(x, \xi) \in T^*(M) \setminus 0$ such that $x \in E$ and ξ is conormal to the tangent space to E at x.

Proposition 1. *Assume $\tau(u, L)$ is real analytic and positive on $\tilde{Z}$ and that*

$$\tilde{R}_\tau g(L) = 0$$

for all L in some neighbourhood of $L_0 \in \tilde{Y}$. Then

$$N^*(L_0) \cap \mathrm{WF}_\mathrm{A}(g) = \emptyset.$$

We finally need the following lemma, related to an important theorem of Hörmander, Kawai, and Kashiwara ([H2], Theorem 8.5.6.).

Lemma 2. *Let S be the spherical surface $\{x;\ |x| = 1\}$ in $\mathbb{R}^m$, and let f be continuous in some neighbourhood of S. Assume*

$$f(x) = O((|x| - 1)^N) \quad \text{as} \quad |x| \to 1 + 0 \quad \text{for all} \quad N, \tag{6}$$

(or as $|x| \to 1 - 0$), and

$$N^*(S) \cap \mathrm{WF}_\mathrm{A}(f) = \emptyset. \tag{7}$$

Then $f = 0$ in some neighbourhood of S.

Proof. Let S_t be the sphere with radius t and define the function h by

$$h(t) = \int_{S_t} f\, ds,$$

where ds is surface measure on S_t. By basic facts about the wavefront set it follows from (7) that none of the elements above $t = 1$ can be contained in $\mathrm{WF}_\mathrm{A}(h)$, i. e. h is analytic at $t = 1$. But (6) implies that h is rapidly decreasing as $t \to 1 + 0$. Hence h must vanish in some neighbourhood of $t = 1$.

Let $\varphi(x)$ be any real analytic function defined on a neighbourhood of S. Multiplying f by φ clearly preserves the properties (6) and (7). Applying our reasoning to φf we conclude that

$$\int_{S_t} \varphi f\, ds = 0$$

for t near 1 and for all bounded and analytic functions φ. This implies that $f = 0$ in some neighbourhood of S. The lemma is proved.

Proof of the theorem. Let f be a function satisfying the hypotheses of the theorem, and consider again the function g on $\tilde{X}$ defined by

$$g(u) = |u_0|^{-n}\tilde{f}(u) = |u_0|^{-n} f(\alpha^{-1}(u)).$$

We have seen that $\tilde{R}_\tau g(L)$ must vanish for L near L_∞. But then Proposition 1 implies that

$$N^*(L_\infty) \cap \mathrm{WF}_{\mathrm{A}}(g) = \emptyset. \tag{8}$$

Now we want to use Lemma 2 to infer that g vanishes near L_∞. The fact that L_∞, considered as a hypersurface in $\tilde{X}$, is non-orientable is a slight inconvenience for us; we therefore move up to S^n, the double cover of $\tilde{X}$. We will use the same notation on S^n as on $\tilde{X}$, so that points will be denoted by u and the function g pulled back to S^n will again be denoted g; this will cause no confusion. Thus g is an even function defined in a neighbourhood of the equator $\Sigma = \{u;\ u_0 = 0\} \subset S^n$. It is clear that (8) holds with Σ in place of L_∞. Now, the stereographic map takes S^n with the north pole removed into $\mathbb{R}^n$ and Σ into an $n-1$-sphere in $\mathbb{R}^n$. An application of Lemma 2 now proves that $g = 0$ in a neighbourhood of Σ, hence $f = 0$ outside some compact set in $\mathbb{R}^n$. An application of the theorem in [BQ1] therefore finishes the proof. We prefer, however, to complete the proof with another application of the arguments already used here. Since a compact, convex set is equal to the intersection of all closed balls that contain it, we may assume that K is a ball. We may also assume that its center is the origin; let R be its radius. Let S_r be the sphere with radius r, centered at the origin, and let r_0 be the smallest r for which $f = 0$ outside S_r. Assume $r_0 > R$. Applying Proposition 1 (or the analogous statement for the Radon transform R_ρ on $\mathbb{R}^n$) to all tangentplanes to S_{r_0} we find that $N^*(S_{r_0}) \cap \mathrm{WF}_{\mathrm{A}}(f)$ must be empty. Lemma 2 now shows that f must vanish in some neighbourhood of S_{r_0}. This contradicts the definition of r_0, hence the proof is complete.

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