Comment to "Remarks on modification of Helgason's support theorem. II" by T. Takiguchi

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Abstract: We slightly extend a uniqueness theorem for generalized Radon transforms $f \mapsto Rf$ proved in [B2], by giving sharp decay conditions on the function f at infinity.

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In my article [B2] I gave a uniqueness theorem for generalized Radon transforms with real analytic weights on projective space. One of the corollaries of that theorem reads in the special case of constant weight as follows (Corollary 3, p. 26). The Radon transform Rf is defined by $Rf(H) = \int_H f\, ds$, where ds is Euclidean surface measure on the hyperplane H.

Theorem A. *Let K be a compact convex subset of $\mathbf{R}^n$ and let Γ be an open conic subset of $\mathbf{R}^n$. Let f be continuous on $\mathbf{R}^n \setminus K$ and decay enough at infinity to be integrable on hyperplanes, for instance $f(x) = \mathcal{O}(|x|^{-n})$ as $|x| \to \infty$. Assume moreover that f decays faster than any negative power of $|x|$ as x tends to infinity in Γ. Assume finally $Rf(H) = 0$ for all hyperplanes H not intersecting K. Then $f = 0$ in the set*

$$\bigcap_{x \in K} (x + \Gamma \cup (-\Gamma)). \tag{1}$$

In the article [T2] in this journal, as well as in [T1], T. Takiguchi has interpreted the phrase " ... decay enough at infinity to be integrable on hyperplanes, for instance $f(x) = \mathcal{O}(|x|^{-n})$ as $|x| \to \infty$" as meaning only that the function is integrable on every hyperplane, and as a consequence he claims that my statement is incorrect. To support his claim Takiguchi gives a modification of Zalcman's well-known counterexample [Z], in this case a non-vanishing entire function f which is integrable on every line in the plane, decays faster than any negative power in some cone, and satisfies $\int_H f ds = 0$ for all lines H. But Takiguchi's function, like Zalcman's, does not "decay at infinity", but instead grows super-exponentially in a narrow strip tending to infinity; see (6) below. However, what I meant — and admittedly expressed a little too informally — was that I assume f decays at infinity, uniformly with respect to direction of course, sufficiently fast to be integrable on all hyperplanes. Since I did not want to give the weakest possible condition, I gave an example of a sufficient condition, $f(x) = \mathcal{O}(|x|^{-n})$ as $|x| \to \infty$. A sharp condition is provided by (3), (4) below.

Moreover, Takiguchi claims that my statement in Theorem A with the assumption $f(x) = \mathcal{O}(|x|^{-n})$ as $|x| \to \infty$ is left open in my article. Therefore he proves a weaker statement, assuming $f(x) = o(|x|^{-n})$ as $|x| \to \infty$ in Theorem A, and mentions my statement as an open problem. In the special case when K is empty and $\Gamma = \mathbf{R}^n$ the latter statement reads

$$\text{(2)} \qquad \begin{aligned} &\textit{Assume } f(x) = \mathcal{O}(|x|^{-n}) \textit{ as } |x| \to \infty \textit{ and} \\ &Rf(H) = 0 \textit{ for all hyperplanes } H \subset \mathbf{R}^n. \\ &\textit{Then } f = 0. \end{aligned}$$

In [T1] Takiguchi specifically raises the question whether (2) is true. Here I will first give an elementary proof of a strengthened version of this statement (Proposition 1). Then I will clarify my proof in [B2] of (a similarly strengthened version of) Theorem A.

Here, as in Takiguchi's articles, we shall only be concerned with the classical Radon transform. An important point in [B2], though, is that the results are valid for generalized Radon transforms R_ρ with variable, real-analytic weight functions $\rho(H, x)$.

The following proposition implies statement (2).

Proposition 1. *Let f be continuous on $\mathbf{R}^n$, $n \geq 2$, and satisfy*

$$\text{(3)} \qquad |f(x)| \leq \lambda(|x|), \quad x \in \mathbf{R}^n,$$

where $\lambda(t)$ is a positive, decreasing function on $t \geq 0$ such that

$$\text{(4)} \qquad \int_0^\infty t^{n-2} \lambda(t) dt < \infty.$$

Assume $Rf(H) = 0$ for all hyperplanes H. Then $f = 0$.

Note that (4) is the necessary and sufficient condition for (3) to imply that f is integrable on every hyperplane. Choosing $\lambda(t) = C/(1 + t^n)$ we obtain (2).

Denote by $\mathcal{H}_n$ the set of hyperplanes in $\mathbf{R}^n$, and identify $\mathcal{H}_n$ in the usual way with $(S^{n-1} \times \mathbf{R})/(\pm)$, where (ω, p) corresponds to the plane $x \cdot \omega = p$ and $(\omega, p) \sim (-\omega, -p)$. The adjoint Radon transform R^* is defined on $C^\infty(\mathcal{H}_n)$ by $R^*\phi(x) = \int_{S^{n-1}} \phi(\omega, x \cdot \omega) d\omega$, where $d\omega$ is the Euclidean surface measure on S^{n-1}. Furthermore we shall use the notation

$$\langle f, \varphi \rangle_{\mathbf{R}^n} = \int_{\mathbf{R}^n} f(x) \varphi(x) dx,$$

$$\langle g, \psi \rangle_{\mathcal{H}_n} = \int_{\mathbf{R}} \int_{S^{n-1}} g(\omega, p) \psi(\omega, p) d\omega dp.$$

For the proof of Proposition 1 we shall need the following lemma.

Lemma 2. *Denote the volume of the sphere S^{n-2} by Ω_{n-2} for $n \geq 3$ and set $\Omega_0 = 2$. The adjoint Radon transform R^* satisfies the estimate*

$$\sup_x (1 + |x|) |R^*\psi(x)| \leq 4\pi \Omega_{n-2} \sup_{\omega, p} (1 + p^2) |\psi(\omega, p)|$$

for all $\psi \in C^\infty(\mathcal{H}_n)$ for which the quantity on the right hand side is finite.

Proof. Set $M = \sup(1+p^2)|\psi(\omega,p)|$. If the angle between ω and x is $\frac{\pi}{2} - \beta$, then $|x\cdot\omega| = |x||\sin\beta| \geq |x||\beta|/2$ for $0 \leq |\beta| \leq \pi/2$, hence

$$\begin{aligned}|R^*\psi(x)| &\leq \int_{S^{n-1}} \frac{M}{1+|x\cdot\omega|^2} d\omega \\ &\leq M\Omega_{n-2}\int_{-\pi/2}^{\pi/2}(1+|x|^2\beta^2/4)^{-1}d\beta \\ &\leq M\Omega_{n-2}\min(\pi, \frac{2}{|x|}\cdot\pi) \leq 4\pi\Omega_{n-2}M/(1+|x|),\end{aligned}$$

which proves the statement.

Proof of Proposition 1. Assume f satisfies (3), (4). Then

$$\langle Rf, \psi\rangle_{\mathcal{H}_n} = \langle f, R^*\psi\rangle_{\mathbf{R}^n} \tag{5}$$

for all $\psi \in C_0^\infty(\mathcal{H}_n)$. Assume $Rf = 0$ and let φ be arbitrary in $C_0^\infty(\mathbf{R}^n)$. By the inversion formula for the Radon transform $\varphi = c_n R^* J_{n-1} R\varphi$, where J_{n-1} is the operator defined by $\widehat{J_{n-1}h}(\tau) = |\tau|^{n-1}\widehat{h}(\tau)$, sometimes denoted $J_{n-1} = (-\partial_p^2)^{(n-1)/2}$. If we choose $\psi = c_n J_{n-1} R\varphi$ we have $\varphi = R^*\psi$. If n is odd, J_{n-1} is a differential operator, so $\psi \in C_0^\infty(\mathcal{H}_n)$. We see now from (5) that $\langle f, \varphi\rangle = 0$, and since φ is arbitrary we have proved that $f = 0$ in this case. If n is even J_{n-1} is a convolution operator, $J_{n-1}h = k_n * h$, where the kernel k_n is a distribution, homogeneous of order $-n$ and smooth outside the origin. Hence $\psi \in C^\infty(\mathcal{H}_n)$ and $\psi(\omega,p) = \mathcal{O}(|p|^{-n})$ as $|p| \to \infty$. Using Lemma 2 it is easy to see that (5) extends by continuity to all $\psi \in C^\infty$ satisfying $\psi(\omega,p) = \mathcal{O}(|p|^{-2})$ as $|p| \to \infty$. The proof can now be finished as in the case when n is odd.

I will now clarify — and somewhat extend — the proof in [B2] of my statement in Theorem A above. It will be enough to prove that f has bounded support in each cone $\Gamma_0 \subset\subset \Gamma$, because the rest of the argument is easy. Assume first that $f(x) = \mathcal{O}(|x|^{-n})$ as $|x| \to \infty$. Let α be the mapping $\mathbf{R}^n \ni x \mapsto \pm u \in S^n/(\pm) = \mathbf{P}^n$, where

$$u = (u_0, u_1, \dots, u_n) = (1, x)/\sqrt{1+|x|^2} \in S^n,$$

and let $\widetilde{f}(u)$ be the function on $\mathbf{P}^n$ defined by $\widetilde{f}(u) = f(\alpha^{-1}(u))$ for $u_0 \neq 0$. The push-forward under α of the measure ds on H can be written $b(H)|u_0|^{-n}d\sigma$, where $d\sigma$ is the (push-forward to $\mathbf{P}^n$ of) the Euclidean surface measure on S^n and $b(H)$ is a non-vanishing factor depending only on H (see Lemma 1 in [B1] and references given there). Our Radon transform on $\mathbf{R}^n$ is thereby transformed to a Radon transform on $\mathbf{P}^n$ by the formula

$$\int_H f\,ds = b(H)\int_{\widetilde{H}} |u_0|^{-n}\widetilde{f}(u)\,d\sigma,$$

where $\widetilde{H} = \alpha(H)$. The function $g(u) = |u_0|^{-n}\widetilde{f}(u)$ is continuous everywhere on $\mathbf{P}^n$ except possibly on the plane at infinity $H_\infty = \{\pm u;\, u_0 = 0\}$, where it is not defined, and the assumption that $f(x) = \mathcal{O}(|x|^{-n})$ as $|x| \to \infty$ implies that $g(u)$ is bounded. Hence $g \in L^\infty(\mathbf{P}^n)$. The integral of g over each hyperplane in $\mathbf{P}^n$ close to H_∞ is zero. The fact that we cannot say anything about the integral over

H_∞ is no problem, because the microlocal regularity theorem ([B2], Proposition 1) and the local unique continuation theorem ([B2], Proposition 2) are valid for L^1_{loc}-functions and even for distributions. (Strictly speaking I need the statement of the Theorem on page 25 for piecewize continuous functions or bounded functions, but for the reasons mentioned this is obviously no problem.) Thus, as explained in [B2], by the microlocal regularity theorem the analytic wave front set $WF_A(g)$ contains no conormal vector to $H_\infty \cap \overline{\alpha(\Gamma)}$; here $\overline{\alpha(\Gamma)}$ denotes the closure of $\alpha(\Gamma)$ in $\mathbf{P}^n$. Hence the local unique continuation theorem implies that g vanishes in some neighborhood of $H_\infty \cap \overline{\alpha(\Gamma)}$, which is the same as saying that f has bounded support in each cone $\Gamma_0 \subset\subset \Gamma$.

This argument works also for the more general case of continuous f satisfying (3), (4). In that case g will not necessarily be bounded on $\mathbf{P}^n$, but it will belong to $L^1(\mathbf{P}^n)$.

Finally I will make a comment on Takiguchi's paper. The counterexample constructed by Takiguchi is an entire function $f(z)$ on the plane $\mathbf{C}$ with the following properties. It is integrable on every line in the plane and has integral zero over every line. It is $\mathcal{O}(\exp(-(\log|z|)^2))$ as $|z| \to \infty$ outside a thin neighborhood S of one (half) branch of the hyperbola $(\mathfrak{Re}\, z)^2 - (\mathfrak{Im}\, z)^2 = 1$. The width of S is $\mathcal{O}(1/|z|)$ as $|z| \to \infty$ and S is disjoint from its asymptote. Hence f *almost* satisfies the assumptions of Theorem A with $K = \emptyset$ and Γ any open cone whose closure does not contain the asymptote to S. But only almost, because $f(z)$ is not $\mathcal{O}(|z|^{-2})$ as $|z| \to \infty$ (uniformly with respect to direction). Instead, $f(z)$ grows very fast indeed as $|z| \to \infty$ in the strip S. This is rather obvious, since $f(z)$ decays quite fast in all other directions. More exactly, using the mean value property for the subharmonic function $\log|f(z)|$ on large circles it is easy to prove that $M(r) = \sup_{|z|=r} |f(z)|$ must satisfy the estimate

$$M(r) \geq c \exp(c(r \log r)^2), \quad r > 1, \tag{6}$$

for some $c > 0$.

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