MATH. SCAND. 78 (1996), 207–212

EXAMPLES OF NON-UNIQUENESS FOR THE COMBINATORIAL RADON TRANSFORM MODULO THE SYMMETRIC GROUP

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1. In a mathematical entertainment column the following problem was found. A farmer asked his helper to weigh five sacks of wheat. However, the helper weighed the five sacks in pairs of two, in all possible combinations, and wrote down the resulting ten sums on a paper, without order. The question is if the farmer could recover the weights of the sacks from this information.

It is natural to pose a more general problem as follows. Let X_n be the factor space R^n/S_n, where S_n *is the symmetric group on* n elements. The elements of X_n will be called *bags*. A bag will be denoted $x = (x_1, \ldots, x_n)$ and can be thought of as a list of n real numbers, where repetitions are allowed and the order is irrelevant. For any integer $n \geqq 2$ we define an operator $W_n : X_n \to X_{n(n-1)/2}$ by $W_n x = u$, where $u = (u_{ij})$, $u_{ij} = x_i + x_j$ for $i < j$. The problem is to decide for which n the operator W_n is injective. The answer was given in [SS]: *W_n is injective if and only if n is not a power of* 2.

More generally, if $1 \leqq k \leqq n$ we can define an operator

$$W_n^k : X_n \to X_{\binom{n}{k}}$$

corresponding to "weighing k sacks at a time", that is, $W_n^k x = u \in X_{\binom{n}{k}}$ consists of all $u_{i_1,\ldots,i_k} = x_{i_1} + \ldots + x_{i_k}$, $i_1 < \ldots < i_k$. W_n^k is called the *combinatorial Radon transform modulo the symmetric group*, see [BBO]. Here we will study the case $k = 3$.

The following has been known for a long time ([SS]; see also [FGS]).

> *W_n^3 is injective if $n \geqq 3$ is different from* 3, 6, 27, *and* $486 = 2 \cdot 3^5$. *Conversely, if $n = 3$ or 6, W_n^3 is not injective.*

Thus the problem to decide if W_n^3 is injective has been left open for precisely two values of n, namely $n = 27$ and $n = 486$. The purpose of this note is to settle those

Received October 12, 1994.

two remaining cases by exhibiting examples showing non-injectivity. We can then conclude the following theorem:

THEOREM. *W_n^3 is injective if and only if $n \geqq 3$ is different from* 3, 6, 27, *and* $486 = 2 \cdot 3^5$.

What we need to prove is the proposition:

PROPOSITION. *W_{27}^3 and W_{486}^3 are not injective.*

PROOF. We first consider the case $n = 27$. Let $x \in X_{27}$ be the bag

$$\{-4, -1^{16}, 2^{10}\};$$

this notation means that the element -4 occurs once, the element -1 occurs 16 times etc. We will show that x and $-x = \{-2^{10}, 1^{16}, 4\}$ are mapped to the same element by W_{27}^3. A calculation show that $z = W_{27}^3 x$ is the following element of $X_{\binom{27}{3}} = X_{2925}$:

$$\{-6^p, -3^q, 0^r, 3^q, 6^p\},$$

where p, q, and r are the integers

$$p = \binom{10}{3} = \binom{16}{2} = 120$$

$$q = \binom{10}{2} \cdot 16 = \binom{16}{3} + 16 \cdot 10 = 720$$

$$r = \binom{10}{2} + \binom{16}{20} \cdot 10 = 1245.$$

Thus z is symmetric, that is, $z = -z$, although x is not symmetric! Therefore $-x \neq x$ and $W_{27}^3 x = W_{27}^3(-x)$. This proves the first statement.

To prove the second statement we choose $x \in X_{486}$ as follows

$$x = \{-7, -4^{56}, -1^{231}, 2^{176}, 5^{22}\}.$$

We are going to verify that $z = W_{486}^3 x$ is symmetric, so that $z = W_{486}^3(-x) = W_{486}^3 x$. Since x is clearly not symmetric, this will finish the proof. The bag z consists of the numbers $-15, -12, \ldots, 12, 15$ in quantities described by the integers $p_1, p_2, \ldots, p_{11}$, where the value of p_6 is inessential and

$$p_1 = \binom{56}{2} = 1540$$

$$p_2 = 56 \cdot 231 + \binom{56}{3} = 40656$$

$$p_3 = \binom{231}{2} + \binom{56}{2}\cdot 231 + 56\cdot 176 = 392161$$

$$p_4 = 56\cdot\binom{231}{2} + 231\cdot 176 + 56\cdot 22 + \binom{56}{2}\cdot 176 = 1800568$$

$$p_5 = \binom{231}{3} + 56\cdot 231\cdot 176 + 231\cdot 22 + \binom{176}{2} + \binom{56}{2}\cdot 22 = 4358893$$

$$p_7 = 22\cdot\binom{231}{2} + 22\cdot 176\cdot 56 + \binom{22}{2} + \binom{176}{2}\cdot 231 = 4358893$$

$$p_8 = \binom{176}{3} + 22\cdot 176\cdot 231 + \binom{22}{2}\cdot 56 = 1800568$$

$$p_9 = 22\cdot\binom{176}{2} + \binom{22}{2}\cdot 231 = 392161$$

$$p_{10} = \binom{22}{2}\cdot 176 = 40656$$

$$p_{11} = \binom{22}{3} = 1540.$$

As seen from this list, $p_j = p_{11-j}$ for $j = 1, 2, 3, 4, 5$, hence z is symmetric. This completes the proof of the proposition and hence of the theorem.

2. We will now briefly describe how we found the examples above.

Let $x = (x_1, \dots, x_n) \in X_n$ and let $z = W_n^k x = (z_1, \dots, z_N)$, $N = \binom{n}{k}$. Following [SS] we introduce the sums of powers

$$s_r = \sum x_i^r \quad\text{and}\quad S_r = \sum z_i^r. \tag{1}$$

Since S_r, as a function of $x_1, \dots, x_n$, is a symmetric polynomial of degree r, it must be expressible as a polynomial in $s_1, \dots, s_r$. Obviously s_r can only appear to first order in this polynomial; hence there exists a constant $A(k, n, r)$ and a polynomial Q such that

$$S_r = A(k, n, r)s_r + Q(s_1, \dots, s_{r-1}). \tag{2}$$

$A(k, n, r)$ can be computed (see [FGS, page 188]):

$$A(k, n, r) = \sum_{i=1}^{k} (-1)^{i-1}\binom{n}{k-i} i^{r-1}.$$

In particular, for $k = 2$ and $k = 3$ the constant $A(k, n, r)$ has the values

(3a) $$A(2,n,r) = n - 2^{r-1}$$

(3b) $$A(3,n,r) = \tfrac{1}{2}(n^2 - (2^r + 1)n + 2\cdot 3^{r-1}).$$

If, for given k and n, the constant $A(k,n,r)$ is different from zero for $1 \leqq r \leqq t$, then $s_1, \ldots, s_t$ can be determined inductively from $S_1, \ldots, S_t$ by means of the equations (2), and hence so can x if $t \geqq n$. In particular, if $A(k,n,r) \neq 0$ for $1 \leqq r \leqq n$, then W_n^k must be injective. This together with (3a) proves that W_n^2 is injective if n is not a power of 2. In the same way the "if"-part of the Theorem follows from (3b) and the following lemma.

LEMMA. *The equation*

$$n^2 - (2^r + 1)n + 2\cdot 3^{r-1} = 0$$

has the following solutions in integers $n \geqq 3$ and $r \geqq 1$ and no others:

$$r = 3: \quad n = 3, \quad n = 6$$

$$r = 5: \quad n = 6, \quad n = 3^3$$

$$r = 9: \quad n = 3^3, \quad n = 2\cdot 3^5.$$

The proof of this lemma is given in [SS].

We decided to look for examples x where $x \neq -x$ and $W_n^3 x = W_n^3(-x)$. Assume x has those properties, and let s_r and S_r be defined by (1). Let n be 27 or 486 and let r_0 be the smallest r such that $A(3,n,r) = 0$, that is, $r_0 = 5$ or $r_0 = 9$, respectively. Then the equations (2) imply that s_r is uniquely determined by S_r for $r < r_0$, hence $\sum x_j^r = \sum(-x_j)^r$ for all $r < r_0$, hence $s_r = 0$ for all odd r. It is natural to try to find an example where x has many repetitions, in other words x consists of n_j copies of the number a_j for $j = 1, 2, \ldots, J$, where J is a rather small number. In the case $n = 27$ we have $r_0 = 5$, and taking $J = 3$ and noting that $s_r = \sum n_j a_j^r$ we get the following requirements on a_j and n_j

$$n_1 + n_2 + n_3 = 27$$

$$n_1 a_1 + n_2 a_2 + n_3 a_3 = 0$$

(4) $$n_1 a_1^3 + n_2 a_2^3 + n_3 a_3^3 = 0.$$

We are going to choose a_j and then solve (4) for n_j. It is natural to choose the a_j as an arithmetic progression. The following argument shows that the difference in the progression should be divisible by 3.

If $n_1, n_J \geqq 3$ then $3a_1$ and $-3a_J$ are the smallest elements in $W_n^3(x)$ and $W_n^3(-x)$ respectively, so then $a_1 = -a_J$ and $n_1 = n_J$. Now the second lowest element is $2a_1 + a_2$ and $-2a_J - a_{J-1}$ respectively, hence $a_2 = -a_{J-1}$ and $n_2 = n_{J-1}$. Repeating this argument shows that $x = -x$. In (4) we have $J = 3$ and it is easily seen that the case n_1 and $n_3 < 3$ is not possible. We may hence assume that $n_1 = 1$ *or* 2 and $n_3 \geqq 3$, so $a_1 + 2a_2 = 3(-a_3)$ or $2a_1 + a_2 = 3(-a_3)$ and hence $a_2 - a_1 = 3(a_3 + a_2)$ or $a_2 - a_1 = -3(a_3 + a_1)$ respectively. In any case the difference in the progression should be divisible by 3.

Since we did not want x to be symmetric, we chose the set $\{a_1, a_2, a_3\}$ non-symmetric about the origin. Thus we arrived at the choice $a_1 = -4$, $a_2 = -1, a_3 = 2$. With those choices of a_j the system (4) has the unique solution

$$n_1 = 1, \quad n_2 = 16, \quad n_3 = 10,$$

which gives our first example above.

When $n = 486$ we have $r_0 = 9$, so we can get four equations corresponding to $r = 1, 3, 5, 7$ and one more equation, five equations in all. This means that we can allow five unknowns n_j, so we need to choose five a_j:s. Reasoning as before we were led to trying for a_j the numbers

$$-7, \quad -4, \quad -1, \quad 2, \quad 5.$$

This leads to the system of equations

$$\begin{aligned} n_1 + n_2 + n_3 + n_4 + n_5 &= 486 \\ (-7)n_1 + (-4)n_2 + (-1)n_3 + 2n_4 + 5n_5 &= 0 \\ (-7)^3 n_1 + (-4)^3 n_2 + (-1)^3 n_3 + 2^3 n_4 + 5^3 n_5 &= 0 \\ (-7)^5 n_1 + (-4)^5 n_2 + (-1)^5 n_3 + 2^5 n_4 + 5^5 n_5 &= 0 \\ (-7)^7 n_1 + (-4)^7 n_2 + (-1)^7 n_3 + 2^7 n_4 + 5^7 n_5 &= 0, \end{aligned}$$

which has the unique solution

$$n_1 = 1, \quad n_2 = 56, \quad n_3 = 231, \quad n_4 = 176, \quad n_5 = 22.$$

This gives the second example above.

NOTE 1. Another way of searching for examples is to look for binomial identities of the type $\binom{10}{3} = \binom{16}{2}$. For example $\binom{5}{3} = \binom{10}{1}$ led us to look for an example of type $\{a^2, b^{10}, \ldots, c^5\}$ and we found $\{-5^2, -2^{10}, 1^{10}, 4^5\}$ which also has symmetric image in $X_{\binom{27}{3}}$. All examples we know of when $n = 27$ are (modulo translations and scaling):

$$\{-4^1, -1^{10}, 2^{16}\}$$
$$\{-5^2, -2^{10}, 1^{10}, 4^5\}$$
$$\{-7^1, -4^5, -1^{10}, 2^6, 5^5\}$$

$$\{-8^2, -5^4, -2^6, 1^8, 4^3, 7^4\}$$

$$\{-14^2, -11^1, -8^3, -5^3, -2^3, 1^6, 4^2, 7^3, 10^1, 13^3\}$$

NOTE 2. When $k = 4$ it is known (see [SS]) that W_n^4 is injective if $n \neq 4, 8$ and 12, and that if $n = 4$ or 8, W_n^4 is not injective. The case when $n = 12$ is still unsettled. When $1 \leqq r \leqq 12$, $A(4, 12, r) = 0$ if and only if $r = 6$. This means that if there is an example $x \neq y \in X_{12}$ with $W_{12}^4 x = W_{12}^4 y$, then $\sum x_i^6 \neq \sum y_i^6$ since otherwise equations (2) would imply $x = y$. Hence we must have $x \neq -y$, so an example of the above type does not exist.

ADDED IN PROOF. W_{12}^4 has in fact been shown to be injective by John A. Ewell [E]. The authors want to thank Melkamu Zekele for this information.

ACKNOWLEDGEMENT. The second author would like to thank Mats Boij and Mattias Jonsson for many stimulating discussions.

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