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## EXAMPLES OF NON-UNIQUENESS FOR THE COMBINATORIAL RADON TRANSFORM MODULO THE SYMMETRIC GROUP

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**1.** In a mathematical entertainment column the following problem was found. A farmer asked his helper to weigh five sacks of wheat. However, the helper weighed the five sacks in pairs of two, in all possible combinations, and wrote down the resulting ten sums on a paper, without order. The question is if the farmer could recover the weights of the sacks from this information.

It is natural to pose a more general problem as follows. Let  $X_n$  be the factor space  $\mathbb{R}^n/S_n$ , where  $S_n$  is the symmetric group on  $n$  elements. The elements of  $X_n$  will be called *bags*. A bag will be denoted  $x = (x_1, \dots, x_n)$  and can be thought of as a list of  $n$  real numbers, where repetitions are allowed and the order is irrelevant. For any integer  $n \geq 2$  we define an operator  $W_n: X_n \rightarrow X_{n(n-1)/2}$  by  $W_n x = u$ , where  $u = (u_{ij})$ ,  $u_{ij} = x_i + x_j$  for  $i < j$ . The problem is to decide for which  $n$  the operator  $W_n$  is injective. The answer was given in [SS]:  $W_n$  is injective if and only if  $n$  is not a power of 2.

More generally, if  $1 \leq k \leq n$  we can define an operator

$$W_n^k: X_n \rightarrow X_{\binom{n}{k}}$$

corresponding to “weighing  $k$  sacks at a time”, that is,  $W_n^k x = u \in X_{\binom{n}{k}}$  consists of all  $u_{i_1, \dots, i_k} = x_{i_1} + \dots + x_{i_k}$ ,  $i_1 < \dots < i_k$ .  $W_n^k$  is called the *combinatorial Radon transform modulo the symmetric group*, see [BBO]. Here we will study the case  $k = 3$ .

The following has been known for a long time ([SS]; see also [FGS]).

$W_n^3$  is injective if  $n \geq 3$  is different from 3, 6, 27, and  $486 = 2 \cdot 3^5$ . Conversely, if  $n = 3$  or 6,  $W_n^3$  is not injective.

Thus the problem to decide if  $W_n^3$  is injective has been left open for precisely two values of  $n$ , namely  $n = 27$  and  $n = 486$ . The purpose of this note is to settle those

two remaining cases by exhibiting examples showing non-injectivity. We can then conclude the following theorem:

**THEOREM.**  $W_n^3$  is injective if and only if  $n \geq 3$  is different from 3, 6, 27, and  $486 = 2 \cdot 3^5$ .

What we need to prove is the proposition:

**PROPOSITION.**  $W_{27}^3$  and  $W_{486}^3$  are not injective.

**PROOF.** We first consider the case  $n = 27$ . Let  $x \in X_{27}$  be the bag

$$\{-4, -1^{16}, 2^{10}\};$$

this notation means that the element  $-4$  occurs once, the element  $-1$  occurs 16 times etc. We will show that  $x$  and  $-x = \{-2^{10}, 1^{16}, 4\}$  are mapped to the same element by  $W_{27}^3$ . A calculation show that  $z = W_{27}^3 x$  is the following element of  $X_{(27)} = X_{2925}$ :

$$\{-6^p, -3^q, 0^r, 3^q, 6^p\},$$

where  $p$ ,  $q$ , and  $r$  are the integers

$$p = \binom{10}{3} = \binom{16}{2} = 120$$

$$q = \binom{10}{2} \cdot 16 = \binom{16}{3} + 16 \cdot 10 = 720$$

$$r = \binom{10}{2} + \binom{16}{20} \cdot 10 = 1245.$$

Thus  $z$  is symmetric, that is,  $z = -z$ , although  $x$  is not symmetric! Therefore  $-x \neq x$  and  $W_{27}^3 x = W_{27}^3(-x)$ . This proves the first statement.

To prove the second statement we choose  $x \in X_{486}$  as follows

$$x = \{-7, -4^{56}, -1^{231}, 2^{176}, 5^{22}\}.$$

We are going to verify that  $z = W_{486}^3 x$  is symmetric, so that  $z = W_{486}^3(-x) = W_{486}^3 x$ . Since  $x$  is clearly not symmetric, this will finish the proof. The bag  $z$  consists of the numbers  $-15, -12, \dots, 12, 15$  in quantities described by the integers  $p_1, p_2, \dots, p_{11}$ , where the value of  $p_6$  is inessential and

$$p_1 = \binom{56}{2} = 1540$$

$$p_2 = 56 \cdot 231 + \binom{56}{3} = 40656$$

$$p_3 = \binom{231}{2} + \binom{56}{2} \cdot 231 + 56 \cdot 176 = 392161$$

$$p_4 = 56 \cdot \binom{231}{2} + 231 \cdot 176 + 56 \cdot 22 + \binom{56}{2} \cdot 176 = 1800568$$

$$p_5 = \binom{231}{3} + 56 \cdot 231 \cdot 176 + 231 \cdot 22 + \binom{176}{2} + \binom{56}{2} \cdot 22 = 4358893$$

$$p_7 = 22 \cdot \binom{231}{2} + 22 \cdot 176 \cdot 56 + \binom{22}{2} + \binom{176}{2} \cdot 231 = 4358893$$

$$p_8 = \binom{176}{3} + 22 \cdot 176 \cdot 231 + \binom{22}{2} \cdot 56 = 1800568$$

$$p_9 = 22 \cdot \binom{176}{2} + \binom{22}{2} \cdot 231 = 392161$$

$$p_{10} = \binom{22}{2} \cdot 176 = 40656$$

$$p_{11} = \binom{22}{3} = 1540.$$

As seen from this list,  $p_j = p_{11-j}$  for  $j = 1, 2, 3, 4, 5$ , hence  $z$  is symmetric. This completes the proof of the proposition and hence of the theorem.

2. We will now briefly describe how we found the examples above.

Let  $x = (x_1, \dots, x_n) \in X_n$  and let  $z = W_n^k x = (z_1, \dots, z_N)$ ,  $N = \binom{n}{k}$ . Following [SS] we introduce the sums of powers

$$(1) \quad s_r = \sum x_i^r \quad \text{and} \quad S_r = \sum z_i^r.$$

Since  $S_r$ , as a function of  $x_1, \dots, x_n$ , is a symmetric polynomial of degree  $r$ , it must be expressible as a polynomial in  $s_1, \dots, s_r$ . Obviously  $s_r$  can only appear to first order in this polynomial; hence there exists a constant  $A(k, n, r)$  and a polynomial  $Q$  such that

$$(2) \quad S_r = A(k, n, r)s_r + Q(s_1, \dots, s_{r-1}).$$

$A(k, n, r)$  can be computed (see [FGS, page 188]):

$$A(k, n, r) = \sum_{i=1}^k (-1)^{i-1} \binom{n}{k-i} i^{r-1}.$$

In particular, for  $k = 2$  and  $k = 3$  the constant  $A(k, n, r)$  has the values

$$(3a) \quad A(2, n, r) = n - 2^{r-1}$$

$$(3b) \quad A(3, n, r) = \frac{1}{2}(n^2 - (2^r + 1)n + 2 \cdot 3^{r-1}).$$

If, for given  $k$  and  $n$ , the constant  $A(k, n, r)$  is different from zero for  $1 \leq r \leq t$ , then  $s_1, \dots, s_t$  can be determined inductively from  $S_1, \dots, S_t$  by means of the equations (2), and hence so can  $x$  if  $t \geq n$ . In particular, if  $A(k, n, r) \neq 0$  for  $1 \leq r \leq n$ , then  $W_n^k$  must be injective. This together with (3a) proves that  $W_n^2$  is injective if  $n$  is not a power of 2. In the same way the “if”-part of the Theorem follows from (3b) and the following lemma.

LEMMA. *The equation*

$$n^2 - (2^r + 1)n + 2 \cdot 3^{r-1} = 0$$

*has the following solutions in integers  $n \geq 3$  and  $r \geq 1$  and no others:*

$$r = 3: \quad n = 3, \quad n = 6$$

$$r = 5: \quad n = 6, \quad n = 3^3$$

$$r = 9: \quad n = 3^3, \quad n = 2 \cdot 3^5.$$

The proof of this lemma is given in [SS].

We decided to look for examples  $x$  where  $x \neq -x$  and  $W_n^3 x = W_n^3(-x)$ . Assume  $x$  has those properties, and let  $s_r$  and  $S_r$  be defined by (1). Let  $n$  be 27 or 486 and let  $r_0$  be the smallest  $r$  such that  $A(3, n, r) = 0$ , that is,  $r_0 = 5$  or  $r_0 = 9$ , respectively. Then the equations (2) imply that  $s_r$  is uniquely determined by  $S_r$  for  $r < r_0$ , hence  $\sum x_j^r = \sum (-x_j)^r$  for all  $r < r_0$ , hence  $s_r = 0$  for all odd  $r$ . It is natural to try to find an example where  $x$  has many repetitions, in other words  $x$  consists of  $n_j$  copies of the number  $a_j$  for  $j = 1, 2, \dots, J$ , where  $J$  is a rather small number. In the case  $n = 27$  we have  $r_0 = 5$ , and taking  $J = 3$  and noting that  $s_r = \sum n_j a_j^r$  we get the following requirements on  $a_j$  and  $n_j$

$$n_1 + n_2 + n_3 = 27$$

$$n_1 a_1 + n_2 a_2 + n_3 a_3 = 0$$

$$(4) \quad n_1 a_1^3 + n_2 a_2^3 + n_3 a_3^3 = 0.$$

We are going to choose  $a_j$  and then solve (4) for  $n_j$ . It is natural to choose the  $a_j$  as an arithmetic progression. The following argument shows that the difference in the progression should be divisible by 3.

If  $n_1, n_J \geq 3$  then  $3a_1$  and  $-3a_J$  are the smallest elements in  $W_n^3(x)$  and  $W_n^3(-x)$  respectively, so then  $a_1 = -a_J$  and  $n_1 = n_J$ . Now the second lowest element is  $2a_1 + a_2$  and  $-2a_J - a_{J-1}$  respectively, hence  $a_2 = -a_{J-1}$  and  $n_2 = n_{J-1}$ . Repeating this argument shows that  $x = -x$ . In (4) we have  $J = 3$  and it is

easily seen that the case  $n_1$  and  $n_3 < 3$  is not possible. We may hence assume that  $n_1 = 1$  or  $2$  and  $n_3 \geq 3$ , so  $a_1 + 2a_2 = 3(-a_3)$  or  $2a_1 + a_2 = 3(-a_3)$  and hence  $a_2 - a_1 = 3(a_3 + a_2)$  or  $a_2 - a_1 = -3(a_3 + a_1)$  respectively. In any case the difference in the progression should be divisible by 3.

Since we did not want  $x$  to be symmetric, we chose the set  $\{a_1, a_2, a_3\}$  non-symmetric about the origin. Thus we arrived at the choice  $a_1 = -4$ ,  $a_2 = -1$ ,  $a_3 = 2$ . With those choices of  $a_j$  the system (4) has the unique solution

$$n_1 = 1, \quad n_2 = 16, \quad n_3 = 10,$$

which gives our first example above.

When  $n = 486$  we have  $r_0 = 9$ , so we can get four equations corresponding to  $r = 1, 3, 5, 7$  and one more equation, five equations in all. This means that we can allow five unknowns  $n_j$ , so we need to choose five  $a_j$ :s. Reasoning as before we were led to trying for  $a_j$  the numbers

$$-7, \quad -4, \quad -1, \quad 2, \quad 5.$$

This leads to the system of equations

$$\begin{aligned} n_1 + n_2 + n_3 + n_4 + n_5 &= 486 \\ (-7)n_1 + (-4)n_2 + (-1)n_3 + 2n_4 + 5n_5 &= 0 \\ (-7)^3 n_1 + (-4)^3 n_2 + (-1)^3 n_3 + 2^3 n_4 + 5^3 n_5 &= 0 \\ (-7)^5 n_1 + (-4)^5 n_2 + (-1)^5 n_3 + 2^5 n_4 + 5^5 n_5 &= 0 \\ (-7)^7 n_1 + (-4)^7 n_2 + (-1)^7 n_3 + 2^7 n_4 + 5^7 n_5 &= 0, \end{aligned}$$

which has the unique solution

$$n_1 = 1, \quad n_2 = 56, \quad n_3 = 231, \quad n_4 = 176, \quad n_5 = 22.$$

This gives the second example above.

NOTE 1. Another way of searching for examples is to look for binomial identities of the type  $\binom{1^0}{3} = \binom{1^6}{2}$ . For example  $\binom{5}{3} = \binom{1^0}{1}$  led us to look for an example of type  $\{a^2, b^{1^0}, \dots, c^5\}$  and we found  $\{-5^2, -2^{1^0}, 1^{1^0}, 4^5\}$  which also has symmetric image in  $X_{\binom{27}{3}}$ . All examples we know of when  $n = 27$  are (modulo translations and scaling):

$$\begin{aligned} &\{-4^1, -1^{1^0}, 2^{1^6}\} \\ &\{-5^2, -2^{1^0}, 1^{1^0}, 4^5\} \\ &\{-7^1, -4^5, -1^{1^0}, 2^6, 5^5\} \end{aligned}$$

$$\{-8^2, -5^4, -2^6, 1^8, 4^3, 7^4\}$$

$$\{-14^2, -11^1, -8^3, -5^3, -2^3, 1^6, 4^2, 7^3, 10^1, 13^3\}$$

NOTE 2. When  $k = 4$  it is known (see [SS]) that  $W_n^4$  is injective if  $n \neq 4, 8$  and 12, and that if  $n = 4$  or 8,  $W_n^4$  is not injective. The case when  $n = 12$  is still unsettled. When  $1 \leq r \leq 12$ ,  $A(4, 12, r) = 0$  if and only if  $r = 6$ . This means that if there is an example  $x \neq y \in X_{12}$  with  $W_{12}^4 x = W_{12}^4 y$ , then  $\sum x_i^6 \neq \sum y_i^6$  since otherwise equations (2) would imply  $x = y$ . Hence we must have  $x \neq -y$ , so an example of the above type does not exist.

ADDED IN PROOF.  $W_{12}^4$  has in fact been shown to be injective by John A. Ewell [E]. The authors want to thank Melkamu Zekele for this information.

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