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A PALEY-WIENER THEOREM FOR THE ANALYTIC WAVE FRONT SET*

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1. Introduction. Let u be a hyperfunction with compact support in $\mathbf{R}^n$ and let $i_{\hat{u}}$ be the indicator function of its Fourier-Laplace transform $\hat{u}$,

$$\hat{u}(\zeta) = u(e^{-i\langle\cdot,\zeta\rangle}),\ \zeta \in \mathbf{C}^n; \quad i_{\hat{u}}(\zeta) = \varlimsup_{\tilde{\zeta}\to\zeta} \varlimsup_{t\to+\infty} \log|\hat{u}(t\tilde{\zeta})|/t,\ \zeta \in \mathbf{C}^n. \tag{1.1}$$

(One can think of u as an analytic functional supported by a compact set in $\mathbf{R}^n$; in particular u could be a distribution in $\mathcal{E}'(\mathbf{R}^n)$.) We recall that $i_{\hat{u}}$ is plurisubharmonic and positively homogeneous of degree 1. By an extension of the Paley-Wiener theorem

$$i_{\hat{u}}(\zeta) \le H(\operatorname{Im}\zeta), \tag{1.2}$$

where H is the supporting function of (the convex hull of) $\operatorname{supp} u$,

$$H(\eta) = \sup_{x\in\operatorname{supp} u} \langle x, \eta\rangle, \tag{1.3}$$

and there is equality in (1.2) on $\mathbf{CR}^n = \{z\xi; z \in \mathbf{C}, \xi \in \mathbf{R}^n\}$. Since $i_{\hat{u}}$ is convex on every complex line through the origin we have

$$-H(-\operatorname{Im}\zeta) \le i_{\hat{u}}(\zeta) \le H(\operatorname{Im}\zeta), \tag{1.2$'$}$$

and in particular $i_{\hat{u}}$ vanishes in $\mathbf{R}^n$.

By Theorem 2.3.1 of Sigurdsson [6], if $\xi, \eta \in \mathbf{R}^n \setminus \{0\}$, then $i_{\hat{u}}(\xi + i\eta) = H(\eta)$ if and only if $(x, \xi) \in WF_A(u)$ for some $x \in \mathbf{R}^n$ with $\langle x, \eta\rangle = H(\eta)$, that is, there is an analytic singularity of u with frequency ξ in the supporting plane of $\operatorname{supp} u$ with exterior conormal η. In this paper we shall prove that one can in fact determine the convex hull of $WF_A(u)$ for fixed frequency by means of the asymptotic behavior of $i_{\hat{u}}$ at $\mathbf{R}^n$ and obtain Sigurdsson's theorem as a corollary. Let H_ξ denote the supporting function of the fiber of $WF_A(u)$ for the frequency ξ,

$$H_\xi(\eta) = \sup\{\langle x, \eta\rangle; (x, \xi) \in WF_A(u)\}, \quad \xi, \eta \in \mathbf{R}^n,\ \xi \ne 0. \tag{1.4}$$

That $WF_A(u)$ is closed means that $H_\xi(\eta)$ is upper semicontinuous in (ξ, η). Our main result is:

THEOREM 1.1. *If u is a hyperfunction with compact support in $\mathbf{R}^n$ and $i_{\hat{u}}$ is the indicator function of the Fourier-Laplace transform $\hat{u}$, then*

$$\begin{aligned} H_\xi(\eta) &= \lim_{\delta\to+0} \varlimsup_{t\to+0} \sup_{|\tilde{\xi}-\xi|<\delta} i_{\hat{u}}(\tilde{\xi} + it\eta)/t \\ &= \varlimsup_{t\to+0,\tilde{\xi}\to\xi} i_{\hat{u}}(\tilde{\xi} + it\eta)/t = \varlimsup_{t\to+0,\tilde{\xi}\to\xi,\tilde{\eta}\to\eta} i_{\hat{u}}(\tilde{\xi} + it\tilde{\eta})/t, \end{aligned} \tag{1.5}$$

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where $H_\xi(\eta)$ *is the supporting function for* $WF_A(u)$ *defined by* (1.4), *and* $\xi, \eta \in \mathbf{R}^n$, $\xi \neq 0$.

Theorem 1.1 will be proved in Section 2. Here is a simple example:

Example. Let χ be the characteristic function for the unit ball in $\mathbf{R}^n$ and set $u = \chi + \delta_a$, where $a \in \mathbf{R}^n$. Since the indicator function for $\hat{\chi}$ is $\zeta \mapsto |\operatorname{Im}\sqrt{\langle\zeta,\zeta\rangle}|$ (see e.g. [6, p. 278]), it follows easily that

$$i_{\hat{u}}(\zeta) = \max\{|\operatorname{Im}\sqrt{\langle\zeta,\zeta\rangle}|, \langle a, \operatorname{Im}\zeta\rangle\}.$$

For $\xi \in \mathbf{R}^n \setminus \{0\}$ and $\eta \in \mathbf{R}^n$ we have

$$\frac{1}{t}|\operatorname{Im}\sqrt{\langle\xi + it\eta, \xi + it\eta\rangle}| = \frac{1}{t}|\operatorname{Im}\sqrt{|\xi|^2 + 2it\langle\xi,\eta\rangle + \mathcal{O}(t^2)}| \to \frac{|\langle\xi,\eta\rangle|}{|\xi|}, \quad \text{as } t \to 0,$$

with uniform convergence for ξ in compact subsets of $\mathbf{R}^n \setminus \{0\}$ and η bounded. Hence (1.5) gives

$$H_\xi(\eta) = \max\{|\langle\xi,\eta\rangle|/|\xi|,\ \langle a,\eta\rangle\},$$

which is the supporting function of the set consisting of the three points $\pm\xi/|\xi|$ and a, in agreement with (1.4).

The theorem of Sigurdsson [6] already mentioned follows easily from Theorem 1.1:

COROLLARY 1.2. *If* $\xi, \eta \in \mathbf{R}^n \setminus \{0\}$ *then* $i_{\hat{u}}(\xi + i\eta) = H(\eta)$ *if and only if* $H_\xi(\eta) = H(\eta)$.

Proof. If $i_{\hat{u}}(\xi + i\eta) = H(\eta)$ then the nonpositive subharmonic function $z \mapsto i_{\hat{u}}(\xi + z\eta) - \operatorname{Im} z H(\eta)$ in the upper half plane of $\mathbf{C}$ vanishes identically since it vanishes at i. Thus $i_{\hat{u}}(\xi + it\eta) = tH(\eta)$ for $t > 0$, so $H_\xi(\eta) \geq H(\eta)$ by (1.5). The opposite inequality is obvious so $H_\xi(\eta) = H(\eta)$. The proof that $H_\xi(\eta) = H(\eta)$ implies $i_{\hat{u}}(\xi + i\eta) = H(\eta)$ requires a small part of Lemma 3.2 so it will be postponed until the end of Section 3.

By Theorem 3.1.4 of Sigurdsson [6], for every plurisubharmonic function p in $\mathbf{C}^n$ which is positively homogeneous of degree 1 and vanishes in $\mathbf{R}^n$, there exists a distribution $u \in \mathcal{E}'(\mathbf{R}^n)$ such that $p = i_{\hat{u}}$. Hence (1.5) with $i_{\hat{u}}$ replaced by p, assigns to every such function p a supporting function H_ξ depending on $\xi \in \mathbf{R}^n$, which corresponds to a closed conic set in $T^*(\mathbf{R}^n)$. It is in no way obvious that the three definitions of $H_\xi(\eta)$ in (1.5) are equivalent and give an upper semicontinuous function. However, in Section 3 we shall give a direct proof which is applicable to arbitrary plurisubharmonic functions q in $\mathbf{C}^n$ satisfying an analogue of (1.2),

$$q(\zeta) \leq C|\operatorname{Im}\zeta|, \quad \zeta \in \mathbf{C}^n; \quad q(\xi) = 0, \quad \xi \in \mathbf{R}^n. \tag{1.6}$$

Although Theorem 1.1 allows one to determine the convex hull of $\{x; (x,\xi) \in WF_A(u)\}$, which may be much smaller than the convex hull of $\operatorname{supp} u$ as in the example above, it gives no non-trivial information on the analytic singular support of u. In fact, in Section 4 we shall prove that

$$\bigcup_{\xi \neq 0} \operatorname{ch}\{x; (x,\xi) \in WF_A(u)\} = \operatorname{ch}\operatorname{supp} u, \tag{1.7}$$

where ch denotes the convex hull. We are grateful to Michael Atiyah for a lemma on vector fields on the sphere which is the main point in the proof.

2. Proof of Theorem 1.1. Since it is obvious that

$$\lim_{\delta\to+0}\varliminf_{t\to+0}\sup_{|\tilde\xi-\xi|<\delta} i_{\hat u}(\tilde\xi+it\eta)/t \le \varlimsup_{t\to+0,\tilde\xi\to\xi} i_{\hat u}(\tilde\xi+it\eta)/t \le \varlimsup_{t\to+0,\tilde\xi\to\xi,\tilde\eta\to\eta} i_{\hat u}(\tilde\xi+it\tilde\eta)/t,$$

it is sufficient to prove that

$$\varlimsup_{t\to+0,\tilde\xi\to\xi,\tilde\eta\to\eta} i_{\hat u}(\tilde\xi+it\tilde\eta)/t \le H_\xi(\eta) \le \lim_{\delta\to+0}\varliminf_{t\to+0}\sup_{|\tilde\xi-\xi|<\delta} i_{\hat u}(\tilde\xi+it\eta)/t.$$

Using the homogeneity of $i_{\hat u}$ we can write these inequalities in the form

$$\varlimsup_{r\to+\infty,\tilde\xi\to\xi,\tilde\eta\to\eta} i_{\hat u}(r\tilde\xi+i\tilde\eta) \le H_\xi(\eta), \tag{2.1}$$

$$H_\xi(\eta) \le \lim_{\delta\to+0}\varliminf_{r\to+\infty}\sup_{|\tilde\xi-\xi|<\delta} i_{\hat u}(r\tilde\xi+i\eta). \tag{2.2}$$

To prove (2.1) we must show that if $\xi,\eta\in\mathbf{R}^n\setminus\{0\}$ and $\gamma>H_\xi(\eta)$ then there are open neighborhoods V_ξ and V_η of ξ and η such that for some R

$$i_{\hat u}(r\tilde\xi+i\tilde\eta)\le\gamma \quad \text{when } \tilde\xi\in V_\xi,\ \tilde\eta\in V_\eta,\ r>R. \tag{2.3}$$

This will follow if we prove that there exist constants C_r such that
(2.4)

$$|\hat u\big(\lambda(r\tilde\xi+i\tilde\eta)\big)| = |u(e^{\lambda(\langle\cdot,\tilde\eta\rangle - ir\langle\cdot,\tilde\xi\rangle)})| \le C_r e^{\lambda\gamma} \quad \text{when } \lambda>0,\ r>R,\ \tilde\xi\in V_\xi,\ \tilde\eta\in V_\eta.$$

To express the hypothesis on the analytic wave front set we shall use the Fourier-Bros-Iagolnitzer (FBI) transform

$$U(\lambda,y,\theta) = u\big(\exp(-\lambda(\tfrac12|\cdot-y|^2+i\langle\cdot,\theta\rangle))\big). \tag{2.5}$$

If K is a compact set and $\langle x,\eta\rangle > H_\xi(\eta)$, $x\in K$, then $(K\times\{\xi\})\cap WF_A(u)=\emptyset$, hence

$$|U(\lambda,y,\tilde\xi)|\le Ce^{-c\lambda}, \quad \text{if } \lambda\ge0,\ y\in K,\ \tilde\xi\in V_\xi, \tag{2.6}$$

where $c>0$ and V_ξ is an open bounded neighborhood of ξ. (Cf. [1, Theorem 9.6.3].) We shall estimate $\hat u(\lambda(r\tilde\xi+i\tilde\eta))$, that is, the action of u on the function $x\mapsto\exp(\lambda\langle x,\tilde\eta\rangle - i\lambda r\langle x,\tilde\xi\rangle)$, by expressing it in terms of the FBI transform using the formula

$$e^{\lambda\langle x,\tilde\eta\rangle} = (\lambda r/2\pi)^{\frac12 n} e^{-\lambda|\tilde\eta|^2/2r}\int_{\mathbf{R}^n} e^{-\frac12\lambda r|x-y|^2+\lambda\langle y,\tilde\eta\rangle}\,dy, \tag{2.7}$$

which follows since $\int e^{-\lambda r|y-x-\tilde\eta/r|^2/2}\,dy = (2\pi/\lambda r)^{n/2}$.

If $K_\varrho = \operatorname{supp} u + \{y\in\mathbf{R}^n; |y|\le\varrho\}$ and ϱ is sufficiently large, then

$$\tfrac12|x-y|^2 > |\tilde\eta||y|+1 \quad \text{when } x\in K_1,\ y\notin K_\varrho, \text{ thus}$$
$$e^{-\frac12\lambda r|x-y|^2+\lambda\langle y,\tilde\eta\rangle} < e^{-\frac12\lambda r}e^{-\frac14\lambda r|x-y|^2}, \quad \text{when } x\in K_1,\ y\notin K_\varrho,\ r\ge2,$$

if $|\tilde\eta|<2|\eta|$. For the entire analytic function

$$F(z) = (\lambda r/2\pi)^{\frac12 n} e^{-\lambda|\tilde\eta|^2/2r}\int_{\complement K_\varrho} e^{-\frac12\lambda r\langle z-y,z-y\rangle+\lambda\langle y,\tilde\eta\rangle}\,dy$$

we therefore obtain the estimate

$$|F(z)|\le 2^{\frac12 n}e^{-\frac12\lambda r+\frac12\lambda r|\operatorname{Im} z|^2}, \quad \text{if } \operatorname{Re} z\in K_1. \tag{2.8}$$

From (2.7) it follows that

$$\hat{u}(\lambda(r\tilde{\xi}+i\tilde{\eta})) = u(Fe^{-ir\lambda\langle\cdot,\tilde{\xi}\rangle}) + (\lambda r/2\pi)^{\frac{1}{2}n}e^{-\lambda|\tilde{\eta}|^2/2r}\int_{K_\varrho} U(\lambda r, y, \tilde{\xi})e^{\lambda\langle y,\tilde{\eta}\rangle}\,dy.$$

Here $|F(z)e^{-ir\lambda\langle z,\tilde{\xi}\rangle}| \le 2^{\frac{1}{2}n}e^{-\frac{1}{4}\lambda r}$ if $\operatorname{Re} z \in K_1$ and $|\operatorname{Im} z||\tilde{\xi}| + \frac{1}{2}|\operatorname{Im} z|^2 < \frac{1}{4}$. When $r + 4\gamma \ge 0$ it follows that $|u(Fe^{-ir\lambda\langle\cdot,\tilde{\xi}\rangle})| \le Ce^{\gamma\lambda}$.

Choose γ_0 with $H_\xi(\eta) < \gamma_0 < \gamma$ and let K be the compact set $\{y \in K_\varrho; \langle y,\eta\rangle \ge \gamma_0\}$. Since $(K \times \{\xi\}) \cap WF_A(u) = \emptyset$ we can then apply (2.6) and obtain

$$\lambda^{\frac{1}{2}n}\Big|\int_K U(\lambda r, y, \tilde{\xi})e^{\lambda\langle y,\tilde{\eta}\rangle}\,dy\Big| \le Ce^{\lambda\gamma}, \quad \text{if } rc + \gamma > 2|\eta|\sup_{y\in K}|y|.$$

For every $\delta > 0$ we have

$$|U(\lambda, y, \xi)| \le C_\delta e^{\delta\lambda}, \quad \text{when } y \in K_\varrho.$$

Since $\langle y,\eta\rangle \le \gamma_0$ in $K_\varrho \setminus K$ we can choose a neighborhood V_η of η such that $|\tilde{\eta}| < 2|\eta|$ as above when $\tilde{\eta} \in V_\eta$ and in addition $\langle y,\tilde{\eta}\rangle \le (\gamma_0+\gamma)/2$ when $\tilde{\eta} \in V_\eta$ and $y \in K_\varrho \setminus K$. Then we obtain

$$\lambda^{\frac{1}{2}n}\Big|\int_{y\in K_\varrho\setminus K} U(\lambda r, y, \tilde{\xi})e^{\lambda\langle y,\tilde{\eta}\rangle}\,dy\Big| \le C'_\delta e^{\delta\lambda(r+1)+\lambda(\gamma_0+\gamma)/2} \le C'_\delta e^{\lambda\gamma},$$

if $\delta = \frac{1}{2}(\gamma-\gamma_0)/(r+1)$. This completes the proof of (2.4) and of (2.1).

To prove (2.2) assume that the right-hand side is smaller than Γ. Then we can find $\delta > 0$ and an unbounded set $R \subset (1,\infty)$ such that $i_{\hat{u}}(r\tilde{\xi}+i\eta) < \Gamma$ when $|\tilde{\xi}-\xi| \le \delta$ and $r \in R$. By a classical theorem of Hartogs (see e.g. [2, Theorem 3.2.13] or (3.6) in Lemma 3.1 below, with $u_t(\zeta) = t^{-1}\log|\hat{u}(t\zeta)|$, $K = \{r\tilde{\xi}+i\eta; |\tilde{\xi}-\xi| \le \delta\}$), it follows that when $r \in R$ one can find Λ_r such that

$$|\hat{u}(\lambda(r\tilde{\xi}+i\eta))| < e^{\lambda\Gamma}, \quad \text{if } r \in R,\ |\tilde{\xi}-\xi| \le \delta,\ \lambda > \Lambda_r. \tag{2.9}$$

We must prove that $H_\xi(\eta) \le \Gamma$, that is, that $WF_A(u) \cap \{(y,\xi); \langle y,\eta\rangle > \Gamma\} = \emptyset$. To do so we shall estimate the FBI transform (2.5).

The hypothesis (2.9) can be written

$$|\hat{u}_\lambda(\lambda r\tilde{\xi})| < e^{\lambda\Gamma}, \quad \text{if } r \in R,\ |\tilde{\xi}-\xi| \le \delta,\ \lambda > \Lambda_r, \text{ where } u_\lambda = e^{\lambda\langle\cdot,\eta\rangle}u. \tag{2.9$'$}$$

We must prove that $U(\lambda, y, \tilde{\xi})$ is exponentially decreasing when $\lambda \to \infty$ if $\langle y,\eta\rangle > \Gamma$ and $\tilde{\xi}$ is sufficiently close to ξ. We have

$$U(\lambda r, y, \tilde{\xi}) = u(e^{\lambda r(-\frac{1}{2}|\cdot-y|^2 - i\langle\cdot,\tilde{\xi}\rangle)}),$$

which is the Fourier transform at $\lambda r\tilde{\xi}$ of $u_\lambda e^{-Q}$ where

$$Q(x) = \lambda(\tfrac{1}{2}r|x-y|^2 + \langle x,\eta\rangle) = \tfrac{1}{2}\lambda r|x-y+\eta/r|^2 - \lambda|\eta|^2/2r + \lambda\langle y,\eta\rangle.$$

Hence the absolute value of the Fourier transform of e^{-Q} is

$$\theta \mapsto (2\pi/\lambda r)^{n/2}\exp(\lambda|\eta|^2/2r - \lambda\langle y,\eta\rangle - |\theta|^2/2\lambda r),$$

which gives

(2.10)
$$|U(\lambda r, y, \tilde\xi)| \le (2\pi\lambda r)^{-n/2}\int |\hat u_\lambda(\lambda r\tilde\xi - \theta)| \exp\big(\lambda|\eta|^2/2r - \lambda\langle y,\eta\rangle - |\theta|^2/2\lambda r\big)\, d\theta.$$

When $|\xi - \tilde\xi| < \delta/2$ and $r \in R$, $\lambda > \Lambda_r$, the integral for $|\theta| < \lambda r\delta/2$ can be estimated using $(2.9)'$, so this contribution to $U(\lambda r, y, \tilde\xi)$ is bounded by

$$\exp\big(\lambda(\Gamma - \langle y,\eta\rangle + |\eta|^2/2r)\big). \tag{2.11}$$

If $\langle y_0, \eta\rangle > \Gamma$ and r is sufficiently large, it is exponentially decreasing when $\lambda \to +\infty$, uniformly for all y in a compact neighborhood V_0 of y_0. What remains is to estimate the integral for $|\theta| > \lambda r\delta/2$. To do so we note that

$$|\hat u_\lambda(\theta)| \le C_\varepsilon e^{\kappa\lambda + \varepsilon|\theta|}, \quad \theta \in \mathbf{R}^n,$$

if $\kappa > \sup\{\langle y,\eta\rangle; y \in \operatorname{supp} u\}$ and ε is an arbitrary positive number. For the remaining part of the right-hand side of (2.10) we obtain the bound

(2.12)
$$(2\pi\lambda r)^{-n/2} C_\varepsilon \int_{|\theta| > \lambda r\delta/2} \exp\big(\lambda(\kappa + \varepsilon|r\tilde\xi - \theta/\lambda| - \langle y,\eta\rangle + |\eta|^2/2r) - |\theta|^2/2\lambda r\big)\, d\theta$$
$$\le (\lambda r/2\pi)^{n/2} C_\varepsilon e^{\lambda b}\int_{|\theta|>\delta/2} e^{\lambda r(\varepsilon|\tilde\xi - \theta| - |\theta|^2/2)}\, d\theta, \text{ if } r > 1;\ b = \sup_{y\in V_0}(\kappa - \langle y,\eta\rangle + |\eta|^2/2).$$

If $\varepsilon(4|\tilde\xi|/\delta^2 + 2/\delta) < 1/4$ then $\varepsilon|\tilde\xi - \theta| < |\theta|^2/4$ when $|\theta| > \delta/2$, so (2.12) can be estimated by

$$(\lambda r/2\pi)^{n/2} C_\varepsilon e^{\lambda b}\int_{|\theta|>\delta/2} e^{-\lambda r|\theta|^2/4}\, d\theta < C' e^{\lambda(b - r\delta^2/17)}.$$

Hence (2.12) is exponentially decreasing as $\lambda \to +\infty$ if $r > 17b/\delta^2$, so $U(\lambda r, y, \tilde\xi)$ is uniformly exponentially decreasing when $\lambda \to +\infty$ if $|\xi - \tilde\xi| < \delta/2$ and $y \in V_0$. This proves that $(y_0, \xi) \notin WF_A(u)$ if $\langle y_0, \eta\rangle > \Gamma$ (cf. [1, Theorem 9.6.3]), that is, $H_\xi(\eta) \le \Gamma$. The proof of (2.2) and of Theorem 1.1 is now complete.

3. Remarks on plurisubharmonic functions. As mentioned in the introduction, for every plurisubharmonic function p which is positively homogeneous of degree 1 and vanishes in $\mathbf{R}^n$ there exists by Theorem 3.1.4 of Sigurdsson [6] a distribution $u \in \mathcal{E}'(\mathbf{R}^n)$ such that $i_{\hat u} = p$. Thus Theorem 1.1 shows that with every such p are associated supporting functions H_ξ in $\mathbf{R}^n$, $\xi \in \mathbf{R}^n \setminus \{0\}$, defined by (1.5) with $i_{\hat u}$ replaced by p, such that $H_\xi(\eta)$ is upper semicontinuous as a function of (ξ,η). The equivalence of the three definitions in (1.5) is not obvious but will be proved here directly under weakened hypotheses without using Theorem 1.1 and Sigurdsson's theorem.

From the three line theorem and the Phragmén-Lindelöf theorem it follows that $H(\eta) = p(i\eta)$ is a supporting function in $\mathbf{R}^n$ and that $p(\xi + i\eta) \le H(\eta)$ for $\xi, \eta \in \mathbf{R}^n$. Since the restriction to a complex line through the origin is a convex function we have

$$-H(-\eta) \le p(\xi + i\eta) \le H(\eta), \quad \xi,\eta \in \mathbf{R}^n, \tag{3.1}$$

and $p(\xi + i\eta) = H(\eta)$ when $\xi + i\eta \in \mathbf{CR}^n$, for the nonpositive subharmonic function $p(z\eta) - \operatorname{Im} z H(\eta)$ in the upper half plane vanishes on the imaginary axis, hence identically.

With $\xi \in \mathbf{R}^n$ and $\zeta \in \mathbf{C}^n$, (1.5) concerns the limits of $p(\xi + t\zeta)/t$ as $t \to +0$. However, in studying them we shall drop the homogeneity assumption on p in view of some potential applications. In what follows *we shall denote by q an arbitrary plurisubharmonic function in $\mathbf{C}^n$ such that for some constant C*

$$q(\zeta) \le C|\operatorname{Im}\zeta|, \quad \zeta \in \mathbf{C}^n; \quad q(\xi) = 0, \quad \xi \in \mathbf{R}^n. \tag{3.2}$$

We shall need some well-known facts concerning plurisubharmonic functions summed up in the following lemma.

LEMMA 3.1. *Let u_t, $t > 0$, be plurisubharmonic functions in the connected open set $\Omega \subset \mathbf{C}^n$ which are uniformly bounded above on every compact subset of Ω, and set*

$$u(\zeta) = \overline{\lim_{t\to+0}}\, u_t(\zeta). \tag{3.3}$$

Then either $u \equiv -\infty$ or else the upper semicontinuous regularization $u^*(\zeta) = \overline{\lim}_{\tilde\zeta\to\zeta} u(\tilde\zeta)$ is plurisubharmonic and equal to $u(\zeta) > -\infty$ except in a pluripolar set, which is also a Lebesgue null set for fixed $\operatorname{Im}\zeta$ (or $\operatorname{Re}\zeta$), and we have

$$u^*(\zeta) = \overline{\lim_{\mathbf{R}^n \ni \xi \to 0}}\, u(\xi + \zeta), \quad \zeta \in \Omega. \tag{3.4}$$

If U is a plurisubharmonic function in a ball $B = \{\zeta \in \mathbf{C}^n; |\zeta| < R\}$ and $U \le M$ in B, $U \le m \le M$ in $B \cap \mathbf{R}^n$, then

$$U(\xi + i\eta) \le m + 2(M - m)|\eta|/(R - |\xi|), \quad \textit{if } |\xi| + |\eta| < R. \tag{3.5}$$

If K is a compact subset of Ω and f is a continuous function on K, then

$$\overline{\lim_{t\to+0}} \sup_{\zeta\in K} (u_t(\zeta) - f(\zeta)) \le \sup_{\zeta\in K} (u^*(\zeta) - f(\zeta)). \tag{3.6}$$

Proof. With the possible exception of (3.4), (3.5) the statements are well known. Proofs can be found for example in Sections 3.2 and 4.1 of [2] apart from the basic result of pluripotential theory that negligible sets are pluripolar. (Cf. Klimek [4, Theorem 4.7.6].) To prove (3.5) we assume at first that $\xi = 0$ and that $n = 1$ so that U is a subharmonic function $\le M$ in $\{z \in \mathbf{C}; |z| < R\}$ and $u \le m$ on $(-R, R)$. Then it follows from the maximum principle that

$$U(z) \le m + (M - m)\frac{2}{\pi} \arg \frac{R + z}{R - z}, \quad |z| < R,\ \operatorname{Im} z > 0,$$

for the right-hand side is a harmonic function equal to m on $(-R, R)$ and equal to M when $|z| = R$ and $\operatorname{Im} z > 0$. Hence

$$U(iy) \le m + (M - m)\frac{4}{\pi} \arg(R + iy) \le m + 2(M - m)y/R, \quad 0 < y < R.$$

We obtain (3.5) if we apply this estimate to the subharmonic function $U(\xi + z\eta)$ when $z \in \mathbf{C}$ and $|\eta||z| + |\xi| < R$, taking $y = 1$ and replacing R by $(R - |\xi|)/|\eta|$.

For the function u in (3.3) it follows for small R that

$$u^*(\zeta + \xi + i\eta) \le \sup_{|\theta|<R} u(\zeta + \theta) + C|\eta|/(R - |\xi|), \quad |\xi| + |\eta| < R,$$

where $\theta \in \mathbf{R}$. In fact, if $0 \le \chi \in C_0^\infty(\mathbf{R}^n)$, $\int \chi(\xi)\, d\xi = 1$, and $|\xi| < 1$ if $\xi \in \operatorname{supp} \chi$, then

$$u_\varepsilon(\tilde\zeta) = \int u(\tilde\zeta - \varepsilon\xi)\chi(\xi)\, d\xi = \int u^*(\tilde\zeta - \varepsilon\xi)\chi(\xi)\, d\xi$$

is plurisubharmonic where it is defined, and

$$u_\varepsilon(\zeta + \xi + i\eta) \le \sup_{|\theta|<R} u(\zeta + \theta) + C|\eta|/(R - |\xi| - \varepsilon), \quad \text{if } |\xi| + |\eta| + \varepsilon < R,$$

which implies that

$$u^*(\zeta + \xi + i\eta) \le \sup_{|\theta|<R} u(\zeta + \theta) + C|\eta|/(R - |\xi|)$$

for almost every (ξ, η) with $|\xi| + |\eta| < R$. Hence

$$u^*(\zeta) \le \sup_{|\theta|<R} u(\zeta + \theta) \downarrow \varlimsup_{\theta \to 0} u(\zeta + \theta) \quad \text{when } R \downarrow 0.$$

Remark. Using [2, Proposition 4.1.9] it is also easy to prove that $U(\zeta)$ is the limit as $R \to 0$ of the mean value over the *real* ball with center ζ and radius R if U is plurisubharmonic in a neighborhood of ζ. This implies (3.4), but we chose a proof using (3.5) since this inequality will be needed in another context below.

For an arbitrary plurisubharmonic function q in $\mathbf{C}^n$ satisfying (3.2) with some constant C we set

$$q_\xi(\zeta) = \varlimsup_{t \to +0} q(\xi + t\zeta)/t, \quad \zeta \in \mathbf{C}^n. \tag{3.7}$$

It is clear that (3.2) remains valid with q replaced by q_ξ or the upper semicontinuous plurisubharmonic regularization q_ξ^*. Since q_ξ and q_ξ^* are positively homogeneous of degree 1, it follows from the remarks at the beginning of the section that $h_\xi(\eta) = q_\xi^*(i\eta)$ is a supporting function, and that

$$-h_\xi(-\operatorname{Im}\zeta) \le q_\xi^*(\zeta) \le h_\xi(\operatorname{Im}\zeta), \quad \zeta \in \mathbf{C}^n; \quad q_\xi^*(\zeta) = h_\xi(\operatorname{Im}\zeta), \quad \zeta \in \mathbf{C}\mathbf{R}^n. \tag{3.8}$$

We have $h_\xi(\eta) = q_\xi^*(i\eta) = q_\xi(i\eta)$ for almost all $\eta \in \mathbf{R}^n$, and since h_ξ is continuous it follows that

$$h_\xi(\eta) = \varlimsup_{\tilde\eta \to \eta} q_\xi(i\tilde\eta), \quad \eta \in \mathbf{R}^n. \tag{3.9}$$

By Lemma 3.1 we also have

$$h_\xi(\eta) = \varlimsup_{\mathbf{R}^n \ni \tilde\xi \to 0} q_\xi(\tilde\xi + i\eta), \quad \eta \in \mathbf{R}^n. \tag{3.9$'$}$$

When $n = 1$ we have $q_\xi(\zeta) = h_\xi(\operatorname{Im}\zeta) > -\infty$ for all $\zeta \in \mathbf{C}$, for $q_\xi(\zeta)$ and $h_\xi(\operatorname{Im}\zeta) = q_\xi^*(\zeta)$ are both homogeneous and equal almost everywhere on every ray through the origin in $\mathbf{C}$. For general n we can apply this observation to $\mathbf{C} \ni z \mapsto q(\xi + z\eta)$, where $\xi, \eta \in \mathbf{R}^n$, and conclude that

$$q_\xi(\zeta) = \varlimsup_{t \to +0} q(\xi + t\zeta)/t > -\infty, \quad \zeta \in \mathbf{C}\mathbf{R}^n. \tag{3.10}$$

The following lemma will show that $q(\xi + t\zeta)/t$ converges to $h_\xi(\operatorname{Im}\zeta)$ on $\mathbf{C}\mathbf{R}^n$ in suitable topologies.

LEMMA 3.2. *Let v be a subharmonic function in $\mathbf{C}_+ = \{z \in \mathbf{C}; \operatorname{Im} z > 0\}$ such that $v(z) \le C \operatorname{Im} z$ when $z \in \mathbf{C}_+$ and the boundary measure $\lim_{y\to+0} v(x+iy)\,dx$ on $\mathbf{R}$ vanishes. Then*

$$\overline{\lim_{t\to+0}}\, v(tz)/t = \gamma \operatorname{Im} z, \quad z \in \mathbf{C}_+; \quad \gamma = \Gamma - \frac{1}{\pi}\int_{\mathbf{C}_+} \frac{\operatorname{Im}\zeta}{|\zeta|^2}\, d\mu(\zeta), \tag{3.11}$$

where $d\mu = \Delta v$ is a positive measure and

$$\Gamma = \overline{\lim_{y\to+\infty}}\, v(iy)/y = \lim_{y\to+\infty} \sup_x v(x+iy)/y. \tag{3.12}$$

If $\gamma > -\infty$ and $d\nu$ is a positive measure with compact support in $\mathbf{C}_+$ such that the logarithmic potential $\int \log|z-\zeta|\, d\nu(\zeta)$ is locally bounded, then

$$\lim_{t\to+0} \int t^{-1} v(tz) d\nu(z) = \gamma \int \operatorname{Im} z\, d\nu(z), \tag{3.13}$$

$$\lim_{t\to+0} \int |v(tz)/t - \gamma \operatorname{Im} z|\, d\nu(z) = 0. \tag{3.14}$$

We have $\gamma = \Gamma$ if and only if $v(z) = \Gamma \operatorname{Im} z$, $z \in \mathbf{C}_+$.

Proof. The hypothesis implies that we have the Riesz representation

$$v(z) = \Gamma \operatorname{Im} z + \frac{1}{2\pi}\int_{\mathbf{C}_+} \log\left|\frac{z-\zeta}{\bar z-\zeta}\right| d\mu(\zeta), \quad z \in \mathbf{C}_+. \tag{3.15}$$

(See e.g. [3, Theorem 2.2].) Hence

$$v(tz)/t = \Gamma \operatorname{Im} z + \frac{1}{2\pi t}\int_{\mathbf{C}_+} \log\left|\frac{tz-\zeta}{t\bar z-\zeta}\right| d\mu(\zeta),$$

and since

$$|t\bar z - \zeta|^2/|tz-\zeta|^2 = 1 + 4t \operatorname{Im} z \operatorname{Im}\zeta/|tz-\zeta|^2 \tag{3.16}$$

we can rewrite this formula as

$$v(tz)/t = \Gamma \operatorname{Im} z - \frac{1}{4\pi t}\int_{\mathbf{C}_+} \log(1 + 4t \operatorname{Im} z \operatorname{Im}\zeta/|tz-\zeta|^2)\, d\mu(\zeta).$$

When $t \to +0$ Fatou's lemma gives

$$\overline{\lim_{t\to+0}}\, v(tz)/t \le \Gamma \operatorname{Im} z - \frac{1}{\pi}\int_{\mathbf{C}_+} \frac{\operatorname{Im} z \operatorname{Im}\zeta}{|\zeta|^2}\, d\mu(\zeta) = \gamma \operatorname{Im} z,$$

which proves (3.11) when $\gamma = -\infty$. If $\gamma > -\infty$, that is, if the integral in (3.11) converges, we only get an inequality instead. To prove the opposite inequality we shall accept (3.13) for the moment, take $\chi \ge 0$ in $C_0(\mathbf{R}_+)$, and apply Fatou's lemma to the special case of (3.13) where $d\nu = \int \delta_{sz}\chi(s)\, ds/s$, that is,

$$\lim_{t\to+0} \int v(tsz)\chi(s)\, ds/ts = \gamma \operatorname{Im} z \int \chi(s)\, ds.$$

Since $v(tsz)\chi(s)/ts$ is bounded above we obtain

$$\gamma \operatorname{Im} z \int \chi(s)\, ds \le \int \chi(s)\, ds\, \overline{\lim_{\tau\to+0}}\, v(\tau z)/\tau,$$

hence $\overline{\lim}_{t\to+0} v(tz)/t \ge \gamma \operatorname{Im} z$. This will prove (3.11) when we have verified (3.13). To do so we write

$$\int_{\mathbf{C}_+} t^{-1}v(tz)d\nu(z) = \Gamma\int_{\mathbf{C}_+} \operatorname{Im} z\, d\nu(z) + \frac{1}{2\pi t}\int_{\mathbf{C}_+} d\mu(\zeta)\int_{\mathbf{C}_+} \log\Big|\frac{tz-\zeta}{t\bar z-\zeta}\Big|\, d\nu(z)$$
$$= \Gamma\int_{\mathbf{C}_+} \operatorname{Im} z\, d\nu(z) + \int_{\mathbf{C}_+} t^{-1}w(\zeta/t)\, d\mu(\zeta),$$

where

$$w(\zeta) = \frac{1}{2\pi}\int_{\mathbf{C}_+} \log\Big|\frac{z-\zeta}{\bar z-\zeta}\Big|\, d\nu(z) = -\frac{1}{4\pi}\int_{\mathbf{C}_+} \log(1+4\operatorname{Im} z\operatorname{Im}\zeta/|\zeta-z|^2)\, d\nu(z);$$

the equality follows from (3.16). This gives at once that

$$|w(\zeta)| \le C_\nu \frac{\operatorname{Im}\zeta}{|\zeta|^2+1}$$

outside a compact subset of $\mathbf{C}_+$, and since the logarithmic potential of $d\nu$ is locally bounded by assumption, we have such a bound in $\mathbf{C}_+$. Hence $|w(\zeta/t)|/t \le C_\nu \operatorname{Im}\zeta/|\zeta|^2$, and since $w(\zeta/t)/t \to -\pi^{-1}\operatorname{Im}\zeta|\zeta|^{-2}\int \operatorname{Im} z\, d\nu(z)$ when $t\to+0$, the dominated convergence theorem proves that

$$\frac{1}{t}\int_{\mathbf{C}_+} w(\zeta/t)\, d\mu(\zeta) \to -\frac{1}{\pi}\int_{\mathbf{C}_+} \operatorname{Im} z\, d\nu(z)\int_{\mathbf{C}_+} \frac{\operatorname{Im}\zeta}{|\zeta|^2}\, d\mu(\zeta) \quad \text{as } t\to+0,$$

which completes the proof of (3.13).

If $\delta>0$ it follows from (3.11), which is uniform for $z\in\operatorname{supp} d\nu$, that $v(tz)/t \le \gamma\operatorname{Im} z+\delta$ when $z\in\operatorname{supp} d\nu$ and t is sufficiently small. Hence

$$\int |v(tz)/t-\gamma\operatorname{Im} z-\delta|\, d\nu(z) = \int(\gamma\operatorname{Im} z+\delta-v(tz)/t)\, d\nu(z) \to \delta\int d\nu(z)$$

by (3.13) when $t\to+0$, which proves that

$$\overline{\lim_{t\to+0}}\int |v(tz)/t-\gamma\operatorname{Im} z|\, d\nu(z) \le 2\delta\int d\nu(z),$$

and completes the proof of (3.14).

Finally, $\gamma=\Gamma$ is equivalent to $d\mu=0$ by (3.11), hence equivalent to $v(z)=\Gamma\operatorname{Im} z$ by the Riesz representation. The proof of the lemma is complete.

Remark. We can take for $d\nu$ the Lebesgue measure in any compact subset of $\mathbf{C}_+$ and conclude that $v(tz)/t\to\gamma\operatorname{Im} z$ in $L^1_{\text{loc}}(\mathbf{C}_+)$ when $t\to+0$. We could also take for $d\nu$ the one dimensional Lebesgue measure on an interval $z+I\subset\mathbf{C}_+$ where I is an interval on $\mathbf{R}$ with finite length $|I|$. Then (3.13) gives

$$\lim_{t\to+0}\int_I t^{-1}v(t(s+z))\, ds = \gamma\operatorname{Im} z|I|,$$

which implies that $\gamma\operatorname{Im} z \le \underline{\lim}_{t\to+0}\, t^{-1}\sup_{s\in I} v(t(s+z))$ and in view of (3.11)

$$\lim_{t\to+0} t^{-1}\sup_{s\in I} v(t(s+z)) = \gamma\operatorname{Im} z, \quad z\in\mathbf{C}_+. \tag{3.17}$$

Let us also note that the hypotheses of Lemma 3.2 are fulfilled by $v(z) = q(\xi + z\eta)$ for arbitrary $\xi, \eta \in \mathbf{R}^n$, if q is plurisubharmonic in $\mathbf{C}^n$ and satisfies (3.2), and that $\gamma > -\infty$ then by (3.10).

We can now give a direct proof of the equalities in (1.5) for more general plurisubharmonic functions.

THEOREM 3.3. *If q is a plurisubharmonic function in $\mathbf{C}^n$ satisfying* (3.2), *then*

$$\lim_{\delta\to+0} \varliminf_{t\to+0} \sup_{|\tilde\xi-\xi|<\delta} q(\tilde\xi + it\eta)/t = \varlimsup_{t\to+0,\tilde\xi\to\xi} q(\tilde\xi + it\eta)/t = \varlimsup_{t\to+0,\tilde\xi\to\xi,\tilde\eta\to\eta} q(\tilde\xi + it\tilde\eta)/t, \quad \xi,\eta\in\mathbf{R}^n. \tag{3.18}$$

Here $\tilde\xi, \tilde\eta \in \mathbf{R}^n$.

Proof. It is obvious that (3.18) is valid with $=$ replaced by $\leq$, so what must be proved is that

$$\varlimsup_{t\to+0,\tilde\xi\to\xi,\tilde\eta\to\eta} q(\tilde\xi + it\tilde\eta)/t \leq \lim_{\delta\to+0} \varliminf_{t\to+0} \sup_{|\tilde\xi-\xi|<\delta} q(\tilde\xi + it\eta)/t. \tag{3.19}$$

Let Γ be larger than the right-hand side, that is, assume that for some $\delta > 0$

$$\varliminf_{t\to+0} \sup_{|\tilde\xi|<3\delta} q(\xi + \tilde\xi + it\eta)/t < \Gamma. \tag{3.20}$$

Set, with $\tilde\xi \in \mathbf{R}^n$ as in (3.19), (3.20),

$$Q(\zeta) = \sup_{|\tilde\xi|\leq 2\delta} q(\zeta + \tilde\xi), \tag{3.21}$$

which is an upper semicontinuous, hence plurisubharmonic function satisfying (3.2). By (3.20) we have

$$\varliminf_{t\to+0} \sup_{|s|<1} Q(\xi + (s+i)t\eta)/t < \Gamma,$$

which by (3.17) implies

$$\varlimsup_{t\to+0} Q(\xi + it\eta)/t < \Gamma,$$

that is, $\varlimsup_{t\to+0} \sup_{|\tilde\xi|\leq 2\delta} q(\xi + \tilde\xi + it\eta)/t < \Gamma$. Thus we can find $T > 0$ such that

$$q(\xi + \tilde\xi + it\eta)/t < \Gamma, \quad \text{if } 0 < t < T,\ |\tilde\xi| < 2\delta. \tag{3.22}$$

Hence

$$q(\xi + \tilde\xi + t(\hat\xi + i\eta))/t < \Gamma, \quad \text{if } 0 < t < T,\ |\tilde\xi| < \delta,\ T|\hat\xi| < \delta,$$

and since we have a uniform upper bound for $q(\xi + \tilde\xi + t\zeta)/t$ when $\operatorname{Im}\zeta$ is bounded, it follows from (3.5) if $\widehat\Gamma > \Gamma$ that for some $\delta' > 0$

$$q(\xi + \tilde\xi + it\tilde\eta)/t < \widehat\Gamma, \quad \text{if } 0 < t < T,\ |\tilde\xi| < \delta,\ |\eta - \tilde\eta| < \delta'.$$

This means that

$$\varlimsup_{t\to+0,\tilde\xi\to 0,\tilde\eta\to\eta} q(\xi + \tilde\xi + it\tilde\eta)/t \leq \widehat\Gamma,$$

which proves (3.19) and the theorem.

For the function $H_\xi(\eta)$ defined by (3.18) we have

$$H_\xi(\eta) = p(i\eta), \quad \eta \in \mathbf{R}^n, \quad \text{where } p(\zeta) = \varlimsup_{t\to+0,\tilde\xi\to\xi} q(\tilde\xi + t\zeta)/t.$$

Since

$$p(\theta + i\eta) = \varlimsup_{t\to+0,\tilde\xi\to\xi} q(\tilde\xi + t\theta + it\eta)/t = p(i\eta), \quad \theta,\eta \in \mathbf{R}^n,$$

it follows from (3.4) that $p^*(\theta + i\eta) = p(i\eta)$, so $H_\xi(\eta) = p(i\eta)$ is a convex positively homogeneous function, and it is obviously upper semicontinuous as a function of (ξ,η). Hence

$$W = \{(x,\xi); \langle x,\eta\rangle \le H_\xi(\eta)\,\forall\eta \in \mathbf{R}^n\} \tag{3.23}$$

is a closed subset of $\mathbf{R}^n \times \mathbf{R}^n$ which is convex with supporting function H_ξ when ξ is fixed. If q is positively homogeneous of degree 1 then $H_\xi(\eta)$ is positively homogeneous of degree 0 in ξ, so W is conic in the second variable when $\xi \ne 0$. For $\xi = 0$ the fiber of W becomes the set with supporting function equal to $q(i\cdot)$, which agrees with the common convention to define a fiber of $WF_A(u)$ over the frequency 0 as the support of u.

For the supporting function h_ξ defined by (3.9) or (3.9)′ we have

$$h_\xi(\eta) \le H_\xi(\eta) \tag{3.24}$$

in view of the last expression for $H_\xi(\eta)$ in (3.18). However, there is no obvious reason why $h_\xi(\eta)$ should be upper semicontinuous as a function of (ξ,η). The upper semicontinuous regularization

$$h^*_\xi(\eta) = \varlimsup_{\tilde\xi\to\xi} h_{\tilde\xi}(\eta) \tag{3.25}$$

is also a supporting function for fixed ξ, and we have

$$h^*_\xi(\eta) \le H_\xi(\eta). \tag{3.26}$$

We have not been able to decide whether there is always equality in (3.26), but this question is not really relevant in the context of Theorem 1.1.

If $u \in \mathcal{E}'(\mathbf{R}^n)$ then the description of the asymptotic behavior of $\hat u$ given by the indicator function $i_{\hat u}$ can be refined by studying the set $L_\infty(u)$ of limits of the plurisubharmonic functions $\zeta \mapsto t^{-1}\log|\hat u(t\zeta)|$ as $t \to +\infty$. If H is the supporting function of $\operatorname{supp} u$ defined by (1.3), then $L_\infty(u)$ is a compact subset of the set P_H of plurisubharmonic functions q in $\mathbf{C}^n$ such that

$$q(\zeta) \le H(\operatorname{Im}\zeta), \quad \zeta \in \mathbf{C}^n; \quad u(\zeta) = H(\operatorname{Im}\zeta), \quad \zeta \in \mathbf{CR}^n. \tag{3.27}$$

(See [3], Sections 3 and 4.) We have

$$i_{\hat u}(\zeta) = \max_{q\in L_\infty(u)} q(\zeta), \quad \zeta \in \mathbf{C}^n.$$

Every $q \in P_H$ satisfies the hypotheses of Theorem 3.3. When $q \in L_\infty(u)$ the set in $T^*(\mathbf{R}^n)$ corresponding to the function $H_\xi(\eta)$ defined by q according to (3.18) is a closed subset of $WF_A(u)$. The sets so obtained might contain additional information on the analytic singularities of u.

We shall finally complete the proof of Corollary 1.2 using no results obtained in this section except the last and very elementary statement in Lemma 3.2.

Proof of Corollary 1.2. Assuming that $H_\xi(\eta) = H(\eta)$, where $\xi, \eta \in \mathbf{R}^n \setminus \{0\}$, we must prove that $i_{\hat u}(\xi + i\eta) = H(\eta)$. Using the first expression for $H_\xi(\eta)$ in (1.5) we have by the hypothesis

$$\lim_{\delta\to+0} \varlimsup_{t\to+0} \sup_{|\tilde\xi-\xi|<\delta} i_{\hat u}(\tilde\xi + it\eta)/t = H(\eta).$$

Thus we have for every $\delta > 0$

$$H(\eta) \le \varlimsup_{t\to+0} q_\delta(it\eta)/t = \gamma,\ \Gamma = \varlimsup_{t\to+\infty} q_\delta(it\eta)/t \le H(\eta), \quad \text{if } q_\delta(\zeta) = \sup_{|\tilde\xi-\xi|\le\delta} i_{\hat u}(\tilde\xi + \zeta),$$

for $i_{\hat u}(\theta + it\eta)/t \le H(\eta)$ when $\theta \in \mathbf{R}^n$ and $t > 0$. Since $\gamma \le \Gamma$ by Lemma 3.2 it follows that $\gamma = \Gamma = H(\eta)$, which proves that $q_\delta(z\eta) = H(\eta)\operatorname{Im} z$ when $\operatorname{Im} z > 0$, by the last statement in Lemma 3.2. When $z = i$ and $\delta \to 0$ we conclude that $i_{\hat u}(\xi + i\eta) \ge H(\eta)$, for $i_{\hat u}$ is upper semicontinuous. The opposite inequality is valid for all $\xi, \eta \in \mathbf{R}^n$, which completes the proof of Corollary 1.2.

4. Proof of (1.7). If u is a hyperfunction in $\mathbf{R}^n$ with compact support K, and $x \in K$ is a boundary point of the convex hull $\operatorname{ch} K$, then $\langle x - y, \xi\rangle \ge 0$ when $y \in K$, for some $\xi \in \mathbf{R}^n \setminus \{0\}$. By Holmgren's uniqueness theorem (cf. [1, Theorem 8.5.6]) this implies that $(x, \pm\xi) \in WF_A(u)$. Hence (1.7) follows from the following purely geometrical fact:

Theorem 4.1. *Let $K \subset \mathbf{R}^n$ be a compact set, and define for $\xi \in \mathbf{R}^n \setminus \{0\}$*

$$K_\xi = \{x \in K; \langle x - y, \xi\rangle \ge 0 \text{ for } y \in K\}. \tag{4.1}$$

Then it follows that

$$\bigcup_{\xi\ne0} \operatorname{ch}(K_\xi \cup K_{-\xi}) = \operatorname{ch} K, \tag{4.2}$$

where ch *denotes the convex hull.*

The main point in the proof of the theorem is the following lemma which we owe to Michael Atiyah:

Lemma 4.2. *On the unit sphere $S^{n-1} = \{x; x \in \mathbf{R}^n,\ |x| = 1\}$ there is no non-vanishing continuous tangential vector field v such that $v(x) = v(-x)$, $x \in S^{n-1}$.*

Proof. If n is odd it is well known that there is no non-vanishing continuous tangential vector field at all, so we could assume that n is even. However, the following proof works for every n. Assume that there is a vector field v as in the statement. The tangent bundle T of the real projective space P^{n-1} is equal to

$$\{(x, \tau); x \in S^{n-1}, \tau \in \mathbf{R}^n, \langle x, \tau\rangle = 0\}$$

with the identification $(x, \tau) \sim (-x, -\tau)$. Since $\langle x, v(x)\rangle = 0$ and $v(x) = v(-x)$, $x \in S^{n-1}$, the vector field v generates a line subbundle H of T equal to

$$\{(x, tv(x));\ x \in S^{n-1},\ t \in \mathbf{R}\},$$

with the identification $(x, tv(x)) \sim (-x, -tv(x)) = (-x, -tv(-x))$. It is isomorphic to the line bundle over P^{n-1} defined by $S^{n-1} \times \mathbf{R}$ with the identification $(x, t) \sim (-x, -t)$.

But this is also isomorphic to the tautological line bundle on P^{n-1} obtained from the normal bundle of S^{n-1} by the identification $(x, \lambda x) \sim (-x, \lambda x) = (-x, -\lambda(-x))$. This proves that the Stiefel-Whitney class $w(H)$ of H is equal to that of the tautological line bundle, hence equal to $1 + a$ where a is the element in $H^1(P^{n-1}, \mathbf{Z}_2)$ generating $H^*(P^{n-1}, \mathbf{Z}_2)$, thus $a^{n-1} \neq 0$ and $a^n = 0$. (See e.g. Milnor and Stasheff [5].)

With $H^\perp$ denoting the subbundle of T orthogonal to H we have $T = H \oplus H^\perp$, hence

$$(1 + a)^n = w(T) = w(H) \cdot w(H^\perp) = (1 + a)w(H^\perp).$$

This implies that

$$w(H^\perp) = (1 + a)^{n-1} = 1 + \cdots + a^{n-1},$$

which is a contradiction since $a^{n-1} \neq 0$ and $H^\perp$ has rank $n - 2$.

Proof of Theorem 4.1. We shall argue in three steps.

1. Assume that K is a convex set with C^∞ boundary having strictly positive Gaussian curvature. When $\xi \in S^{n-1}$ we denote by $x(\xi)$ the boundary point with exterior conormal ξ. Then $x(\xi)$ is a C^∞ function of ξ, and $K_\xi = \{x(\xi)\}$, which means that $\operatorname{ch}(K_\xi \cup K_{-\xi})$ is the interval between $x(\xi)$ and $x(-\xi)$ on the line L_ξ through these points. The statement (4.2) will follow if we prove that $\cup_{\xi \in S^{n-1}} L_\xi = \mathbf{R}^n$. If this were not true we could find a point $x \notin \cup_{\xi \in S^{n-1}} L_\xi$. Then we denote by $y(\xi)$ the intersection of L_ξ and the hyperplane $\{y; \langle y - x, \xi\rangle = 0\}$, which is not parallel to L_ξ since $\langle x(\xi) - x(-\xi), \xi\rangle \neq 0$. It is now clear that $y(\xi)$ is a C^∞ function of ξ and that $v(\xi) = y(\xi) - x$ is a non-zero tangent vector of S^{n-1} at ξ, with $v(\xi) = v(-\xi)$ since $L_\xi = L_{-\xi}$. This contradicts Lemma 4.2 and proves Theorem 4.1 in this special case.

2. If K is just convex we can choose a sequence $K_j \downarrow K$ of convex sets with C^∞ boundaries having strictly positive Gaussian curvature. Given $x \in K$ we can choose $\xi_j \in S^{n-1}$ so that x is on the interval between the points $x_j^\pm$ on ∂K_j with exterior conormal $\pm\xi_j$. Passing to a subsequence we can assume that $x_j^\pm$ converge to limits $x^\pm \in \partial K$ and $\xi_j \to \xi$. Then $x^\pm \in K_{\pm\xi}$ and x is on the interval between x^+ and x^-, which proves the statement in the convex case.

3. If K is an arbitrary compact set we denote the convex hull by $\tilde{K}$ and note that the theorem follows from its validity for $\tilde{K}$ if we prove that

$$\operatorname{ch}(K_\xi \cup K_{-\xi}) \supset \operatorname{ch}(\tilde{K}_\xi \cup \tilde{K}_{-\xi}), \quad \xi \in S^{n-1}. \tag{4.3}$$

It suffices to prove that $\tilde{K}_\xi \subset \operatorname{ch} K_\xi$, so assume that $x \in \tilde{K}_\xi$, thus $x \in \tilde{K}$ and $\langle x - y, \xi\rangle \geq 0$, $y \in \tilde{K}$. This implies that x is in the convex hull of $M = \{y \in K; \langle y - x, \xi\rangle = 0\} \subset K_\xi$, which completes the proof.

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