Internal Parametricity for Cubical Type Theory

Evan Cavallo
Robert Harper
Carnegie Mellon University

Martin-Löf type theory

$$\Gamma \gg \boldsymbol{A}\ \textbf{type} \qquad \Gamma \gg \boldsymbol{A} = \boldsymbol{B}\ \textbf{type}$$

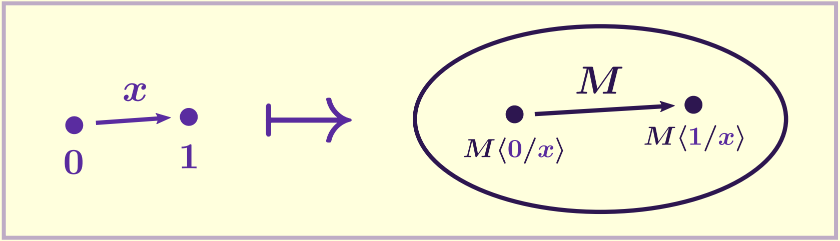
$$\Gamma \gg \boldsymbol{M} \in \boldsymbol{A} \qquad \Gamma \gg \boldsymbol{M} = \boldsymbol{N} \in \boldsymbol{A}$$

- **Martin-Löf ’82:** judgments explained by as specifications on untyped programs ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{M}$, $\boldsymbol{N}$) e.g. “$\boldsymbol{M} \in \textbf{nat}$” is “$\boldsymbol{M}$ computes a natural number”

- follow work of Angiuli, Favonia, & Harper ’18 on computational cubical type theory

Cubical type theories

$$\Gamma, x : \mathbb{I}, \Gamma' \gg M \in A$$



$$\lambda x.M \in \textbf{Path}_{x.A}(M\langle 0/x\rangle, M\langle 1/x\rangle)$$

- **coercion** operation ensures everything respects paths
- **univalence:** type paths are equivalences

Cubical type theories

$$\Gamma, x : \mathbb{I}, \Gamma' \gg M \in A$$

- intervals might be given by:

De Morgan cubes
Cohen, Coquand, Huber, & Mörtberg 2015

Cartesian cubes
Angiuli, Favonia, & Harper 2018
Angiuli, Brunerie, Coquand, Favonia, Licata, & Harper 2018

structural

Substructural cubes
Bezem, Coquand, & Huber 2013&2017

no contraction (diagonals)

Cartesian cubical type theory

FACES

$$\frac{\Gamma, x : \mathbb{I} \gg M \in A \qquad \varepsilon \in \{0,1\}}{\Gamma \gg M\langle \varepsilon / x \rangle \in A\langle \varepsilon / x \rangle}$$

SYMMETRIES

$$\frac{\Gamma, x : \mathbb{I}, y : \mathbb{I} \gg M \in A}{\Gamma, y : \mathbb{I}, x : \mathbb{I} \gg M \in A}$$

DEGENERACIES

$$\frac{\Gamma \gg M \in A}{\Gamma, x : \mathbb{I} \gg M \in A}$$

DIAGONALS

$$\frac{\Gamma, x : \mathbb{I}, y : \mathbb{I} \gg M \in A}{\Gamma, y : \mathbb{I} \gg M\langle y / x \rangle \in A\langle y / x \rangle}$$

Cartesian cubical type theory

Faces + Diagonals = Substitution

$$\frac{\Gamma, x : \mathbb{I} \gg M \in A \qquad \Gamma \gg r : \mathbb{I}}{\Gamma \gg M\langle r/x\rangle \in A\langle r/x\rangle}$$

Symmetries

$$\frac{\Gamma, x : \mathbb{I}, y : \mathbb{I} \gg M \in A}{\Gamma, y : \mathbb{I}, x : \mathbb{I} \gg M \in A}$$

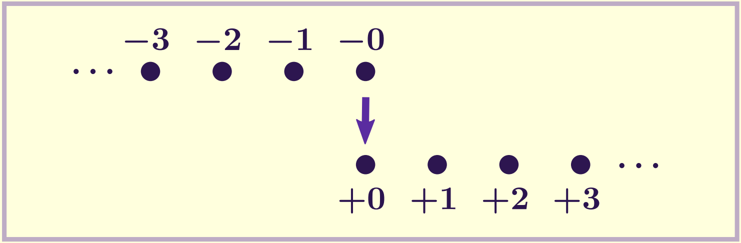
Degeneracies

$$\frac{\Gamma \gg M \in A}{\Gamma, x : \mathbb{I} \gg M \in A}$$

Higher inductive types

- combine **inductive definitions** and **quotients**

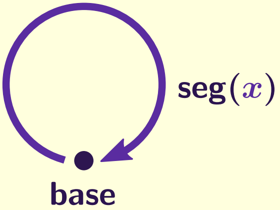
data int **where**
$\mid \mathsf{neg}(n : \mathsf{nat}) \in \mathsf{int}$
$\mid \mathsf{pos}(n : \mathsf{nat}) \in \mathsf{int}$
$\mid \mathsf{seg}(x : \mathbb{I}) \in \mathsf{int}\ [x = 0 \hookrightarrow \mathsf{neg}(0) \mid x = 1 \hookrightarrow \mathsf{pos}(0)]$



Higher inductive types

- define higher-dimensional objects (**synthetic homotopy theory**)

```
data circle where
| base ∈ circle
| loop$(x : \mathbb{I})$ ∈ circle $[x = 0 \hookrightarrow$ base $\mid x = 1 \hookrightarrow$ base$]$
```



Higher inductive types

- when combined, require >1-d reasoning

  e.g., smash product (HoTT Book §6.8)

$$- \wedge - \in \mathcal{U}_* \to \mathcal{U}_* \to \mathcal{U}_*$$

Higher inductive types

- when combined, require >1-d reasoning

  e.g., smash product (HoTT Book §6.8)

$$- \wedge - \in \mathcal{U}_* \to \mathcal{U}_* \to \mathcal{U}_*$$

  - associative?

$$(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} : \mathcal{U}_*) \to (\boldsymbol{X} \wedge \boldsymbol{Y}) \wedge \boldsymbol{Z} \to \boldsymbol{X} \wedge (\boldsymbol{Y} \wedge \boldsymbol{Z})$$

Higher inductive types

- when combined, require >1-d reasoning

  e.g., smash product (HoTT Book §6.8)

$$- \wedge - \in \mathcal{U}_* \to \mathcal{U}_* \to \mathcal{U}_*$$

  - associative?

$$(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} : \mathcal{U}_*) \to \overbrace{(\boldsymbol{X} \wedge \boldsymbol{Y}) \wedge \boldsymbol{Z}}^{\text{2-d}} \to \boldsymbol{X} \wedge (\boldsymbol{Y} \wedge \boldsymbol{Z})$$

Higher inductive types

- when combined, require >1-d reasoning

  e.g., smash product (HoTT Book §6.8)

  $$- \wedge - \in \mathcal{U}_* \to \mathcal{U}_* \to \mathcal{U}_*$$

  - associative?

  $$(X, Y, Z : \mathcal{U}_*) \to \overbrace{(X \wedge Y) \wedge Z}^{\text{2-d}} \to X \wedge (Y \wedge Z)$$

  - is the associator an isomorphism? 3-d

Higher inductive types

- when combined, require >1-d reasoning

  e.g., smash product (HoTT Book §6.8)

$$- \wedge - \in \mathcal{U}_* \to \mathcal{U}_* \to \mathcal{U}_*$$

  - associative?

$$(X, Y, Z : \mathcal{U}_*) \to \overbrace{(X \wedge Y) \wedge Z}^{\text{2-d}} \to X \wedge (Y \wedge Z)$$

  - is the associator an isomorphism? 3-d
  - Mac Lane's pentagon? 4-d

Higher inductive types

- when combined, require >1-d reasoning

  e.g., smash product (HoTT Book §6.8)

$$- \wedge - \in \mathcal{U}_* \to \mathcal{U}_* \to \mathcal{U}_*$$

  - associative?

$$(X, Y, Z : \mathcal{U}_*) \to \overbrace{(X \wedge Y) \wedge Z}^{\text{2-d}} \to X \wedge (Y \wedge Z)$$

  - is the associator an isomorphism? 3-d
  - Mac Lane's pentagon? 4-d

  (van Doorn 2018, Brunerie 2018)

Higher inductive types

- when combined, require >1-d reasoning

  e.g., smash product (HoTT Book §6.8)

  $$- \wedge - \in \mathcal{U}_* \to \mathcal{U}_* \to \mathcal{U}_*$$

  - associative?

  $$(X, Y, Z : \mathcal{U}_*) \to \overbrace{(X \wedge Y) \wedge Z}^{\text{2-d}} \to X \wedge (Y \wedge Z)$$

  - is the associator an isomorphism? 3-d
  - Mac Lane's pentagon? 4-d

  (van Doorn 2018, Brunerie 2018)

- **can we avoid this complexity?**

Parametricity

- “Parametric” functions are uniform in type variables:

$$\lambda a.a \in X \to X$$

$$\lambda a.\lambda b.a \in X \to Y \to X$$

- Contrast with “ad-hoc” polymorphic functions:

$$\lambda a. \left[\begin{array}{ll} \textbf{true}, & \text{if } X = \textbf{bool} \\ a, & \text{otherwise} \end{array} \right] \in X \to X$$

A Restrict ourselves to write only parametric functions

B Parametric functions satisfy many properties “automatically”

Reynolds' abstraction theorem ('83)

- **Def:** A family of (set-theoretic) functions is **parametric** when it acts on relations. **e.g.**,

$$F_X \in X \to X:$$

$$\text{for all sets } A, B \text{ and } R \subseteq A \times B,$$

$$R(a,b) \text{ implies } R(F_A(a), F_B(b))$$

- **Abstraction theorem**: the denotation of any term in simply-typed λ-calculus (with ×, bool) is parametric.
- Key idea: λ-calculus has a **relational interpretation**.

Reynolds' abstraction theorem ('83)

- **Def:** A family of (set-theoretic) functions is **parametric** when it acts on relations. **e.g.**,

$$F_X \in X \to X :$$

for all sets A, B and $R \subseteq A \times B$,

$R(a, b)$ implies $R(F_A(a), F_B(b))$

$$\Downarrow$$

$$F_A(a) = a$$

"theorems for free" (Wadler '89)

Internal parametricity (Bernardy & Moulin ’12)

- Make relational interp. visible **inside** type theory

Internal parametricity (Bernardy & Moulin ’12)

- Make relational interp. visible **inside** type theory

cubical type theory	**parametric type theory**
constructions act on **isomorphisms**	constructions act on **relations**
$\mathbf{Path}_{x.A}(M_0, M_1)$ equal over iso $\boldsymbol{x.A}$	$\mathbf{Bridge}_{\underline{x}.A}(M_0, M_1)$ related by rel $\underline{\boldsymbol{x}}\boldsymbol{.A}$
univalence: $\mathbf{Path}_{\mathcal{U}}(A, B) \simeq (A \simeq B)$	**relativity:** $\mathbf{Bridge}_{\mathcal{U}}(A, B) \simeq A \times B \to \mathcal{U}$

Parametric Cubical Type Theory

$$\Gamma,\ \underline{\underline{x} : \mathbf{2}},\ \Gamma',\ \underline{y : \mathbb{I}},\ \Gamma'' \gg M \in A$$

bridge dimensions
substructural

path dimensions
structural

Parametric Cubical Type Theory

$$\Gamma,\ \underline{\underline{x} : \mathbf{2}},\ \Gamma',\ \underline{y : \mathbb{I}},\ \Gamma'' \gg M \in A$$

bridge dimensions
substructural

path dimensions
structural

$\boxed{\text{A}}$ Use parametricity to prove results about HITs

$\boxed{\text{B}}$ Use good properties of cubical type theory to get better results from / simplify internal parametricity

$\boxed{\text{C}}$ Compare and contrast internal parametricity and cubical type theory

Parametric Cubical Type Theory

$$\boldsymbol{X : \mathcal{U}, a : X \gg N \in B}$$

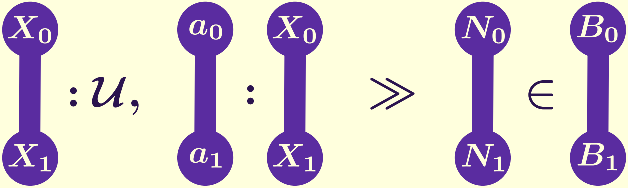
Parametric Cubical Type Theory

$$X : \mathcal{U}, a : X \gg N \in B$$

$$X : \mathcal{U}, \quad a : X \quad \gg \quad N \in B$$

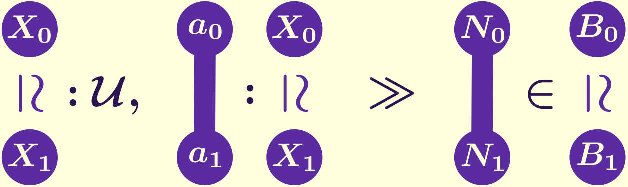
Parametric Cubical Type Theory

$$x : \mathbb{I}, X : \mathcal{U}, a : X \gg N \in B$$



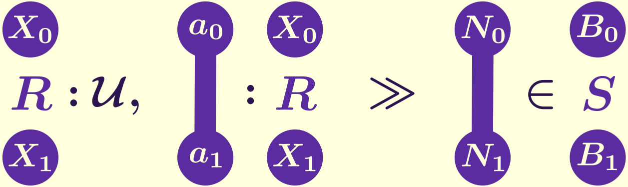
Parametric Cubical Type Theory

$$x : \mathbb{I}, X : \mathcal{U}, a : X \gg N \in B$$



Parametric Cubical Type Theory

$$\underline{x} : \mathbf{2}, X : \mathcal{U}, a : X \gg N \in B$$



Internal parametricity: affine dimensions

FACES

$$\frac{\Gamma, x : \mathbb{I} \gg M \in A \qquad \varepsilon \in \{0, 1\}}{\Gamma \gg M\langle\varepsilon/x\rangle \in A\langle\varepsilon/x\rangle}$$

SYMMETRIES

$$\frac{\Gamma, x : \mathbb{I}, y : \mathbb{I} \gg M \in A}{\Gamma, y : \mathbb{I}, x : \mathbb{I} \gg M \in A}$$

DEGENERACIES

$$\frac{\Gamma \gg M \in A}{\Gamma, x : \mathbb{I} \gg M \in A}$$

PERMUTATIONS

$$\frac{\Gamma, x : \mathbb{I}, y : \mathbb{I} \gg M \in A}{\Gamma, x : \mathbb{I}, y : \mathbb{I} \gg M\langle y \Leftrightarrow x\rangle \in A\langle y \Leftrightarrow x\rangle}$$

Internal parametricity: affine dimensions

Faces + Permutations = Fresh Substitution

$$\frac{\Gamma \gg r : \mathbb{I} \qquad \Gamma^{\backslash r}, x : \mathbb{I} \gg M \in A}{\Gamma \gg M\langle r/x\rangle \in A\langle r/x\rangle}$$

Symmetries

$$\frac{\Gamma, x : \mathbb{I}, y : \mathbb{I} \gg M \in A}{\Gamma, y : \mathbb{I}, x : \mathbb{I} \gg M \in A}$$

Degeneracies

$$\frac{\Gamma \gg M \in A}{\Gamma, x : \mathbb{I} \gg M \in A}$$

Internal parametricity: affine dimensions

FACES + PERMUTATIONS = FRESH SUBSTITUTION

$$\frac{\Gamma \gg r : \mathbb{I} \qquad \Gamma^{\backslash r}, x : \mathbb{I} \gg M \in A}{\Gamma \gg M\langle r/x\rangle \in A\langle r/x\rangle}$$

- Dimension removal ($\Gamma^{\backslash \underline{r}}$)

$$\begin{aligned}
\Gamma^{\backslash \underline{\varepsilon}} &:= \Gamma \\
(\Gamma, \underline{x} : 2)^{\backslash \underline{x}} &:= \Gamma \\
(\Gamma, \underline{y} : 2)^{\backslash \underline{x}} &:= (\Gamma^{\backslash \underline{x}}, \underline{y} : 2) \\
(\Gamma, y : \mathbb{I})^{\backslash \underline{x}} &:= (\Gamma^{\backslash \underline{x}}, y : \mathbb{I}) \\
(\Gamma, a : A)^{\backslash \underline{x}} &:= \Gamma^{\backslash \underline{x}}
\end{aligned}$$

Internal parametricity: Bridge-types

$$\frac{\underline{x} : 2 \gg A \ \textbf{type} \qquad M_0 \in A\langle \underline{0}/\underline{x}\rangle \qquad M_1 \in A\langle \underline{1}/\underline{x}\rangle}{\textbf{Bridge}_{\underline{x}.A}(M_0, M_1) \ \textbf{type}}$$

$$\frac{\underline{x} : 2 \gg M \in A}{\lambda^2 \underline{x}.M \in \textbf{Bridge}_{\underline{x}.A}(M\langle \underline{0}/\underline{x}\rangle, M\langle \underline{1}/\underline{x}\rangle)}$$

$$\frac{\Gamma \gg r : 2 \qquad \Gamma^{\setminus \underline{r}} \gg P \in \textbf{Bridge}_{\underline{x}.A}(M_0, M_1)}{\Gamma \gg P@\underline{r} \in A\langle \underline{r}/\underline{x}\rangle}$$

$+\ \beta$-, η-, coercion rules

Internal parametricity: “relativity”

- Want: $\textbf{Bridge}_{\mathcal{U}}(A, B) \simeq A \times B \to \mathcal{U}$
- Forward:

$$C \longmapsto \lambda\langle a, b\rangle.\textbf{Bridge}_{\underline{x}.C@\underline{x}}(a, b)$$

- Backward:

$$\frac{\Gamma \gg \underline{r} : 2 \qquad \Gamma^{\backslash \underline{r}} \gg R \in A \times B \to \mathcal{U}}{\Gamma \gg \textbf{Gel}_{\underline{r}}(A, B, R) \;\; \textbf{type}}$$

$$\textbf{Gel}_{\underline{0}}(A, B, R) = A \qquad \textbf{Gel}_{\underline{1}}(A, B, R) = B$$

  - Parallels structural **Glue/V**

Internal parametricity: function types

- **paths:** function extensionality

$$\begin{aligned} \textbf{Path}_{x.A \to B}(F, G) &\simeq (a : A) \to \textbf{Path}_{x.B}(Fa, Ga) \\ \lambda^2 x.\lambda a.M &\Leftrightarrow \lambda a.\lambda^2 x.M \end{aligned}$$

- **bridges:** relational interpretation

$$\begin{gathered} \textbf{Bridge}_{\underline{x}.A \to B}(F, G) \simeq \\ (a_0 : A\langle \underline{0}/\underline{x} \rangle)(a_1 : A\langle \underline{1}/\underline{x} \rangle) \\ \to \textbf{Bridge}_{\underline{x}.A}(a_0, a_1) \to \textbf{Bridge}_{\underline{x}.B}(Fa_0, Ga_1) \end{gathered}$$

(difference invisible for paths because of **coercion**)

Internal parametricity: function types

$$\mathsf{Bridge}_{\underline{x}.A\to B}(F,G) \simeq$$
$$(a_0 : A\langle \underline{0}/\underline{x}\rangle)(a_1 : A\langle \underline{1}/\underline{x}\rangle)$$
$$\to \mathsf{Bridge}_{\underline{x}.A}(a_0,a_1) \to \mathsf{Bridge}_{\underline{x}.B}(Fa_0, Ga_1)$$

- Forward:

$$H \longmapsto \lambda a_0.\lambda a_1.\lambda \overline{a}.\lambda^2\underline{x}.(H@\underline{x})(\overline{a}@\underline{x})$$

- Backward:

$$K \longmapsto \text{“}\lambda^2\underline{x}.\lambda a.K(\qquad\qquad)(\qquad\qquad)(\qquad\qquad)@\underline{x}\text{”}$$

Internal parametricity: function types

$$\boxed{\begin{gathered}\mathsf{Bridge}_{\underline{x}.A\to B}(F,G) \ \simeq \\ (a_0 : A\langle\underline{0}/\underline{x}\rangle)(a_1 : A\langle\underline{1}/\underline{x}\rangle) \\ \to \mathsf{Bridge}_{\underline{x}.A}(a_0,a_1) \to \mathsf{Bridge}_{\underline{x}.B}(Fa_0,Ga_1)\end{gathered}}$$

- Forward:

$$H \ \longmapsto \ \lambda a_0.\lambda a_1.\lambda\overline{a}.\lambda^2\underline{x}.(H@\underline{x})(\overline{a}@\underline{x})$$

- Backward:

$$K \ \longmapsto \ \text{``}\lambda^2\underline{x}.\lambda a.K(a\langle\underline{0}/\underline{x}\rangle)(a\langle\underline{1}/\underline{x}\rangle)(\lambda^2\underline{x}.a)@\underline{x}\text{''}$$

Internal parametricity: function types

$$\boxed{\begin{gathered}\textbf{Bridge}_{\underline{x}.A\to B}(F,G) \;\simeq\\ (a_0 : A\langle \underline{0}/\underline{x}\rangle)(a_1 : A\langle \underline{1}/\underline{x}\rangle)\\ \to \textbf{Bridge}_{\underline{x}.A}(a_0,a_1) \to \textbf{Bridge}_{\underline{x}.B}(Fa_0,Ga_1)\end{gathered}}$$

- Forward:

$$H \;\longmapsto\; \lambda a_0.\lambda a_1.\lambda\overline{a}.\lambda^2\underline{x}.(H@\underline{x})(\overline{a}@\underline{x})$$

- Backward:

$$K \;\longmapsto\; \lambda^2\underline{x}.\lambda a.\textbf{extent}_{\underline{x}}(a; F, G, K)$$

"case analysis for interval terms"

Internal parametricity: function types

- Stability of abstraction under substitution relies on absence of diagonals:

$$M(\underline{x}, \underline{y}) \xmapsto{\boldsymbol{\lambda}^{\mathbf{2}} \underline{x}.-} \boldsymbol{\lambda}^{\mathbf{2}} \underline{x}.M(\underline{x}, \underline{y})$$

Internal parametricity: function types

- Stability of abstraction under substitution relies on absence of diagonals:

$$\begin{tikzcd}
M(\underline{x},\underline{y}) \arrow[r, mapsto, "\boldsymbol{\lambda}^{2}\underline{x}.-"] & \boldsymbol{\lambda}^{2}\underline{x}.M(\underline{x},\underline{y}) \arrow[d, mapsto, "\langle\underline{y}/\underline{x}\rangle"] \\
 & \boldsymbol{\lambda}^{2}\underline{x}.M(\underline{x},\underline{y})
\end{tikzcd}$$

Internal parametricity: function types

- Stability of abstraction under substitution relies on absence of diagonals:

$$\begin{tikzcd}
M(\underline{x}, \underline{y}) \arrow[r, mapsto, "\boldsymbol{\lambda}^2 \underline{x}.-"] \arrow[dd, mapsto, "\langle \underline{y} / \underline{x} \rangle"'] & \boldsymbol{\lambda}^2 \underline{x}.M(\underline{x}, \underline{y}) \arrow[d, mapsto, "\langle \underline{y} / \underline{x} \rangle"] \\
& \boldsymbol{\lambda}^2 \underline{x}.M(\underline{x}, \underline{y}) \\
M(\underline{y}, \underline{y}) &
\end{tikzcd}$$

Internal parametricity: function types

- Stability of abstraction under substitution relies on absence of diagonals:

$$\begin{tikzcd}
M(\underline{x},\underline{y}) \arrow[r, mapsto, "\lambda^2\underline{x}.-"] \arrow[dd, mapsto, "\langle\underline{y}/\underline{x}\rangle"'] & \lambda^2\underline{x}.M(\underline{x},\underline{y}) \arrow[d, mapsto, "\langle\underline{y}/\underline{x}\rangle"] \\
 & \lambda^2\underline{x}.M(\underline{x},\underline{y}) \arrow[d, phantom, "\neq" description] \\
M(\underline{y},\underline{y}) \arrow[r, mapsto, "\lambda^2\underline{x}.-"] & \lambda^2\underline{x}.M(\underline{y},\underline{y})
\end{tikzcd}$$

Cubical equality for internal parametricity

- Function extensionality & univalence
  - $\textbf{Bridge}_{\mathcal{U}}(A, B) \;\simeq\; A \times B \to \mathcal{U}$
    - obtained by Bernardy & Moulin by adding

      $$\textbf{Bridge}_{\underline{x}.\textbf{Gel}_x(A,B,R)}(a, b) = R\langle a, b\rangle \qquad \cdots$$

      but this complicates the presheaf model (Bernardy, Coquand, & Moulin ’15)
  - $(X : \mathcal{U}) \to X \to X \to X \;\simeq\; \textbf{bool}$
  - **bridge-discrete types** closed under →
    - types A for which $\textbf{Path}_A(-,-) = \textbf{Bridge}_A(-,-)$

Internal parametricity for cubical equality

- Motivating example: smash product
  - In the paper: any map

    $$(X, Y : \mathcal{U}_*) \to X \wedge Y \to X \wedge Y$$

    is constant or the polymorphic identity.
  - Implies any non-constant map

    $$(X, Y : \mathcal{U}_*) \to X \wedge Y \to Y \wedge X$$

    is an isomorphism.
  - **Key:** scales to characterize maps

    $$(X_1, \ldots, X_n : \mathcal{U}_*) \to \bigwedge_i X_i \to \bigwedge_i X_i$$

Conclusions

- Combine internal parametricity & cubical type theory
  - Parametricity is especially useful for cubical type theory because it contains inductive types with complex algebraic properties
  - As with ordinary type theory, using cubical equality smooths rough edges
- Push internal parametricity further
  - Bridge-discrete types for identity extension lemma
- Theories with interval variables
  - When are different kinds of intervals appropriate?