SOME VARIANTS OF THE MAX NÖTHER AND MACAULAY THEOREMS

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ABSTRACT. Let $F_1, F_2, \ldots, F_m$ be polynomials on $\mathbb{C}^n$ that lack common zeroes at infinity. By means of the Koszul complex, we present a method to transfer the problem of estimating degree of polynomials $Q_1, Q_2, \ldots, Q_m$ such that $\sum F_j Q_j = \Phi$, to $\bar{\partial}$-cohomology for line bundles over $\mathbb{P}^n$. We get back certain variants of results of Macaulay and Max Nöther.

Further, we apply the method via the compactification $\mathbb{C}^n \hookrightarrow \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_\mu}$, where $n = n_1 + n_2 + \ldots + n_\mu$, and retrieve analogous results to the ones of Macaulay and Max Nöther, applicable in cases when the original ones are not.

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1. Introduction

Let $F_1, F_2, \dots, F_m$ be polynomials on $\mathbb{C}^n$. If these polynomials lack common zeroes, then Hilbert's Nullstellensatz guarantees existence of polynomials $Q_1, Q_2, \dots, Q_m$ such that

$$\sum F_j Q_j \equiv 1. \tag{1.1}$$

More generally stated, if Φ is a polynomial that vanishes on the common zero set of the F_j's, then there exist polynomials $Q_1, Q_2, \dots, Q_m$ such that

$$\sum F_j Q_j = \Phi^\nu, \tag{1.2}$$

for some power Φ^ν of Φ. However, the common proof of this result leaves no information about Q_j, e.g., gives no bound on its degree. A lot of attention has been paid in order to find such effective versions of the Nullstellensatz, in terms of the degrees of the F_j's.

The lower bound of the degree may in general be quite high. Considering the following n polynomials, encountered in Brownawell [3],

$$z_1^d, \qquad z_1 - z_2^d, \qquad \dots, \qquad z_{n-2} - z_{n-1}^d, \qquad 1 - z_{n-1} z_n^{d-1}.$$

They lack common zeroes in $\mathbb{C}^n$. For polynomials $Q_1, Q_2, \dots, Q_n$ to satisfy (1.1), especially, they will have to do so along the curve $\gamma(t) = (t^{(d-1)d^{n-2}}, t^{(d-1)d^{n-3}}, \dots, t^{d-1}, t^{-1})$. By cancelation, (1.1) takes the form $t^{(d-1)d^{n-1}} Q_1(\gamma(t)) = 1$ on $\gamma(t)$, and we see that it is necessary that

$$\deg_{z_n} Q_1 \geq d^n - d^{n-1}.$$

A major breakthrough estimating a sufficient upper bound was made by Brownawell in [3]. He used a combination of algebraic methods and estimates from complex analysis in several variables, which resulted in a bound not far from best possible. Shortly after, Kollár [10] obtained with purely algebraic methods the following result:

Theorem 1.1 (J. Kollár, 1988)**.** *Let $F_1, F_2, \dots, F_m$ be polynomials of degree $d_1, d_2, \dots, d_m$ ($d_j \neq 2$) in $\mathbb{C}^n$, ordered so that $d_1 \geq d_2 \geq \dots \geq d_m$, and let Φ be a polynomial that vanishes on the common zeroes of the F_j's. Then one can find polynomials $Q_1, Q_2, \dots, Q_m$ and a natural number ν satisfying (1.2) such that*

$$\nu \leq N'(n, d_1, \dots, d_m) \quad \textit{and} \quad \deg F_j Q_j \leq (1 + \deg \Phi) N'(n, d_1, \dots, d_m),$$

where

$$N'(n, d_1, \dots, d_m) = \begin{cases} d_1 d_2 \dots d_m & \text{if } m \leq n \\ d_1 d_2 \dots d_{n-1} d_m & \text{if } m > n > 1 \\ d_1 + d_m - 1 & \text{if } m > n = 1 \end{cases}.$$

In particullar, if F_j lack common zeroes, then there are polynomials $Q_1, Q_2, \dots, Q_m$ satisfying (1.1) such that $\deg F_j Q_j \leq N'(n, d_1, \dots, d_m)$, and this bound is sharp. The restriction $d_j \neq 2$ were recently removed by Jelonek [9].

Though the general bound is of the order of the product of the degrees, significant improvements can be made by the stronger condition of no common zeroes at infinity. Considering the compactification $\mathbb{P}^n$ of $\mathbb{C}^n$, we have the following classical result due to Macaulay [11].

Theorem 1.2 (F. S. Macaulay, 1916)**.** *Let $F_1, F_2, \ldots, F_m$ be polynomials in $\mathbb{C}^n$, ordered so that $d_1 \geq d_2 \geq \ldots \geq d_m$, and such that they lack common zeroes even at infinity. Then there exist polynomials $Q_1, Q_2, \ldots, Q_m$ satisfying (1.1) such that* $\deg(F_\alpha Q_\alpha) \leq \sum_{k=1}^{n+1} d_j - n$.

Allowing certain zeroes, there is a related sharp result due to Max Nöther [12].

Theorem 1.3 (Max Nöther, 1873)**.** *Let $F_1, F_2, \ldots, F_n$ be polynomials in $\mathbb{C}^n$ that lack common zeroes at infinity, and let Φ be a polynomial in the ideal (F). Then there exist polynomials $Q_1, Q_2, \ldots, Q_n$ satisfying*

$$\sum F_j Q_j = \Phi$$

such that $\deg F_j Q_j \leq \deg \Phi$.

Since long, in several complex variables, a method introduced by Hörmander in [8] to solve ideal problems uses the Koszul complex and solvability of $\bar{\partial}$-equations. This method can be extended to cover polynomial ideal problems of the kind above. Via homogenization, tuples of polynomials correspond to holomorphic sections of certain holomorphic vector bundles over $\mathbb{P}^n$. Therefore, the ideal problem can be reformulated as existence of holomorphic sections of such vector bundles.

In Andersson [1], this was formulated in terms of global residue currents of $\mathbb{P}^n$. But, in case of F_j, or rather their homogenizations, to lack common zeroes, or only have discrete ones, all reference to currents can be neglected. Remaining is a purely geometric and algebraic framework, left with the only global obstruction of $\bar{\partial}$-cohomology for line bundles over $\mathbb{P}^n$. In Section 8 we develop this method, and in Theorem 9.1 and Theorem 12.2 we get, by applying the method, back the results of Macaulay, Max Nöther and certain variants.

Furthermore, one can think of other compactifications of $\mathbb{C}^n$ such that sections still relates to polynomials. With the corresponding assumption of no common zeroes at infinity, it should be possible to obtain analogous results to the Max Nöther and Macaulay theorems by the very same method. As a initial study, we investigate products of projective space, and especially the case $\mathbb{C}^n \hookrightarrow \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$, where $n_1 + n_2 = n$. The results are stated in Theorem 10.2 and Theorem 12.3, and discussed in Section 11. Especially, these results can be used in certain cases when the ones of Macaulay or Max Nöther cannot. We close with an exhaustive result with respect to $\bar{\partial}$-cohomology for line bundles over projective space, for results of Max Nöther and Macaulay type derived via presented method.

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2. Preliminaries

2.1. **Several complex variables.** The differential of a smooth function $f : \mathbb{C}^n \to \mathbb{C}$ is the linear $(1, 1)$-form

$$df := \sum_j \frac{\partial f}{\partial x_j}(z)dx_j + \frac{\partial f}{\partial y_j}(z)dy_j,$$

where $z = x+iy$. In particullar, $dz = dx+idy$ and $d\bar{z} = dx-idy$, thus

$$df = \sum_j \frac{\partial f}{\partial z_j}(z)dz_j + \frac{\partial f}{\partial \bar{z}_j}(z)d\bar{z}_j = \partial f + \bar{\partial} f.$$

We say that f is *holomorphic* if $\bar{\partial} f = 0$.

This generalizes to a linear mapping of a (p, q)-form w to a $(p+1, q+1)$-form dw. Especially, ∂w is a $(p+1, q)$-form and $\bar{\partial} w$ a $(p, q+1)$-form. We say that w is *$\bar{\partial}$-closed* if $\bar{\partial} w = 0$.

Since $0 = d^2 = (\partial + \bar{\partial})^2$, then we must have $\partial^2 = 0$, $\bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$, by degree reasons. Thus, for the equation $\bar{\partial} w = v$ to be solvable, it is necessary that the right-hand side v is $\bar{\partial}$-closed.

The following two results are well known. For proof, see any introductory text on several complex variables, e.g.,[2].

Theorem 2.1 (Taylor series)**.** *Let f be a holomorphic function in a polydisc $\{z \in \mathbb{C}^n | \, |z_j - a_j| < r_j\}$. Then f has a series expansion in this polydisc*

$$f = \sum_\alpha c_\alpha (z-a)^\alpha.$$

Here $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ denotes a multi-index, and $(z - a)^\alpha = (z_1 - a_1)^{\alpha_1}(z_2 - a_2)^{\alpha_2} \ldots (z_n - a_n)^{\alpha_n}$.

Theorem 2.2 (Hartogs' phenomenon)**.** *Let K be a compact subset of $\Omega \subset \mathbb{C}^n$, such that $\Omega \setminus K$ is connected. If f is a holomorphic function in $\Omega \setminus K$, then there is a unique holomorphic extention of f to Ω.*

2.2. **Differential geometry.** We will denote an arbitrary topological structure by $\mathcal{S}$. We are only interested in the smooth and the holomorphic structures, denoted $\mathcal{E}$ and $\mathcal{O}$, respectively. An $\mathcal{S}$-morphism between open sets in $\mathbb{C}^m$ and $\mathbb{C}^n$ simply means a smooth or a holomophic function, and by an $\mathcal{S}$-isomorphism, a diffeomorphism or a biholomorphism, i.e., a holomorphic bijection with holomorphic inverse, respectively.

Definition 1 (Manifold)**.** Let M be a Hausdorff space with a countable basis of open sets. We call M a *manifold* of dimension k if M is locally homeomorphic with $\mathbb{C}^k$. A local homeomorphism $(\mathcal{U}_\alpha, \varphi_\alpha)$ is called a *chart*, and an $\mathcal{S}$*-atlas* is a family of charts that covers M and also are $\mathcal{S}$*-compatible*, i.e., such that the compositions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are $\mathcal{S}$-morphisms on the overlaps.

A manifold equiped with an atlas, we will call *smooth* or *complex*, depending on the structure of the atlas.

We say that a function $f : \mathcal{U} \to f(\mathcal{U})$, between two $\mathcal{S}$-manifold is an $\mathcal{S}$*-morphism* or an $\mathcal{S}$*-isomorphism*, if f expressed in charts of enclosed atlases so are.

Definition 2 (Vector bundle)**.** Let E and X be two $\mathcal{S}$-manifolds and $\pi : E \to X$ a projection, which is an $\mathcal{S}$-morphism. We say that $\pi : E \to X$ is a complex $\mathcal{S}$*-vector bundle* of rank k, if

a) each *fibre* $E_x := \pi^{-1}(x)$ is a k-dimensional complex vector space.
b) for each $x \in X$ there is an neighbourhood $\mathcal{U}$ of x and an $\mathcal{S}$-isomorphism $h : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{C}^k$ such that $h(E_x) = \{x\} \times \mathbb{C}^k$ and $h^x : E_x \xrightarrow{h} \{x\} \times \mathbb{C}^k \xrightarrow{proj.} \mathbb{C}^k$ is a vector space isomorphism. We call the tuple $(\mathcal{U}, h)$ a *local trivialization.*

We sometimes say that $\pi : E \to X$ is an $\mathcal{S}$-bundle or that E is a vector bundle over X. The manifold X will be referred to as the *base space.* A vector bundle of rank 1 will be referred to as a *line* bundle.

An important observation concerning the property of the trivializations being vector space isomorphisms on each fibre, is that the compositions

$$h_\alpha \circ h_\beta^{-1} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathbb{C}^k \to \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathbb{C}^k,$$

induces a map

$$g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(k, \mathbb{C}), \tag{2.1}$$

such that

$$g_{\alpha\beta}(x) = h_\alpha^x \circ (h_\beta^x)^{-1} : \mathbb{C}^k \to \mathbb{C}^k,$$

i.e., $g_{\alpha\beta}$ is a $\mathbb{C}^k$-isomorphism at each point, since the inverse and the composition of vector space isomorphisms remains isomorphisms. The functions $g_{\alpha\beta}$ are called *transition functions* and it follows directly from the trivialization functions that they satisfy the compatibility conditions:

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = I_{k\times k} \quad \text{on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma, \tag{2.2a}$$

and

$$g_{\alpha\alpha} = I_{k\times k} \quad \text{on } \mathcal{U}_\alpha. \tag{2.2b}$$

Moreover, since the trivialization functions are $\mathcal{S}$-isomorphisms, the transition functions will be $\mathcal{S}$-morphisms.

Definition 3. Let $\pi : E \to X$ be an $\mathcal{S}$-bundle. An *$\mathcal{S}$-section* of this vector bundle is an $\mathcal{S}$-morphism $s : \mathcal{U} \to E$ on an open subset $\mathcal{U}$ of X, such that $\pi \circ s = \mathrm{id}_{\mathcal{U}}$.

We will denote the family of smooth sections by $\mathcal{E}(\mathcal{U}, E)$, and the holomorphic sections of a holomorphic vector bundle by $\mathcal{O}(\mathcal{U}, E)$.

Let $s : \mathcal{U} \to E$ be an $\mathcal{S}$-section and $\{(\mathcal{U}_\alpha, h_\alpha)\}$ a family of local trivializations covering $\mathcal{U}$. We can create a family of trivializations $\{(\mathcal{U}_\alpha, s_\alpha)\}$ of s, defined as

$$s_\alpha := h_\alpha \circ s : \mathcal{U}_\alpha \to \mathcal{U}_\alpha \times \mathbb{C}^k \xrightarrow{proj.} \mathbb{C}^k.$$

Hence, s can be seen as $\mathcal{S}$-morphisms into $\mathbb{C}^k$, satisfying the compatibility conditions:

$$s_\alpha = g_{\alpha\beta}s_\beta \quad \text{on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta. \tag{2.3}$$

Conversely, given $\mathcal{S}$-morphisms $s_\alpha : \mathcal{U}_\alpha \to \mathbb{C}^k$ satisfying (2.3), then the family of pull-backs by the local trivializations each defines local $\mathcal{S}$-sections. These piece together to an $\mathcal{S}$-section $s : \mathcal{U} \to E$, well-defined because of the compatibility condition (2.3).

Definition 4. Let $\pi : E \to X$ be an $\mathcal{S}$-bundle. A set of $\mathcal{S}$-sections, $e_j : \mathcal{U} \to E$, is called an *$\mathcal{S}$-frame* of the given bundle, if for each $x \in \mathcal{U}$, the set $\{e_j(x)\}$ gives a basis for the fibre E_x.

If for a vector bundle there exists a global frame, the bundle is called *trivial*.

For each $\mathcal{S}$-bundle of rank k, there always exists a family of local $\mathcal{S}$-frames covering X. To see this, it suffices, for any basis $\{e_j\}$ of $\mathbb{C}^k$ and any family of local trivializations $\{(\mathcal{U}_\alpha, h_\alpha)\}$, to choose $\{(\mathcal{U}_\alpha, \{h_\alpha^{-1}(e_j)\})\}$.

2.3. **Tensor product.** Let $\mathcal{V}$ and $\mathcal{W}$ be n- and m-dimensional complex vector spaces.

Definition 5. The *tensor space* $\mathcal{V} \otimes \mathcal{W}$ is the nm-dimensional vector space such that if $v \in \mathcal{V}$ and $w \in \mathcal{W}$, then $v \otimes w \in \mathcal{V} \otimes \mathcal{W}$, and the product $v \otimes w$ is bilinear.

We can further construct multiple tensor products, especially $\bigotimes^k \mathcal{V}$, and by mixing degrees we get the *tensor algebra* $T(\mathcal{V}) = \bigoplus_{k=0}^{\infty}(\bigotimes^k \mathcal{V})$.

Definition 6. Denote by $I(\mathcal{V})$ the ideal of $T(\mathcal{V})$ generated by elements of the form $v \otimes v$. We define $\Lambda^*\mathcal{V}$, the *exterior algebra* of $\mathcal{V}$, to be the quotient

$$\pi : T(\mathcal{V}) \to T(\mathcal{V})/I(\mathcal{V}).$$

Especially, $\Lambda^p\mathcal{V} = \pi(\bigotimes^p \mathcal{V})$ is called the *p-fold exterior power* of $\mathcal{V}$.

If $\alpha = \pi(a) \in \Lambda^p\mathcal{V}$ and $\beta = \pi(b) \in \Lambda^q\mathcal{V}$, then we define the *wedge product*

$$\alpha \wedge \beta = \pi(a \otimes b).$$

Since $v \otimes v$, $(v + w) \otimes (v + w) \in I(\mathcal{V})$, then $v \wedge v = 0$ and $0 = (v + w) \wedge (v + w) = v \wedge w + w \wedge v$, i.e., $v \wedge w = -w \wedge v$. Moreover, if $\{v_j\}$ is a basis for $\mathcal{V}$, then $\{v_{j_1} \wedge v_{j_2} \wedge \ldots \wedge v_{j_p}\}$ will be a basis for $\Lambda^p\mathcal{V}$.

If $\tau \in \mathcal{V}^*$, then we define *interior multiplication* by τ as the map $\delta_\tau : \Lambda^p\mathcal{V} \to: \Lambda^{p-1}\mathcal{V}$ such that, for $w \in \Lambda^p\mathcal{V}$ and a basis $\{v_j\}$ for $\mathcal{V}$,

$$\begin{aligned}
\delta_\tau w = \delta_\tau \sum a_{J_p} v_{J_p} &= \delta_\tau \sum a_{J_p}\, v_{j_1} \wedge v_{j_2} \wedge \ldots \wedge v_{j_p} \\
&= \sum (-1)^{k+1} a_{J_p} \tau(v_{j_k})\, v_{j_1} \wedge v_{j_2} \wedge \ldots \wedge \widehat{v}_{j_k} \wedge \ldots \wedge v_{j_p} \\
&= \sum (-1)^{k+1} a_{J_p} \tau(v_{j_k}) v_{J_p \setminus \{j_k\}},
\end{aligned}$$

where $\widehat{v}_{j_k}$ means that v_{j_k} is omitted. Further application of δ_τ yields, for $k < \ell$,

$$\begin{aligned}
\delta_\tau^2 w =& \delta_\tau \sum (-1)^{k+1} a_{J_p} \tau(v_{j_k}) v_{J_p \setminus \{j_k\}} \\
=& \sum (-1)^{k+1} (-1)^{\ell+1} a_{J_p} \tau(v_{j_\ell}) \tau(v_{j_k}) v_{J_p \setminus \{j_\ell, j_k\}} \\
&+ \sum (-1)^{k+1} (-1)^{\ell} a_{J_p} \tau(v_{j_k}) \tau(v_{j_\ell}) v_{J_p \setminus \{j_k, j_\ell\}} \\
=& 0,
\end{aligned}$$

where the two sums appear by first applying τ on v_{j_ℓ}, then v_{j_k}, and conversely. Thus, $\delta_\tau^2 = 0$.

3. Projective space and line bundles

In $\mathbb{C}^{n+1}$ we define the projective space to be

$$\mathbb{P}^n := \{\text{one dimensional subspaces of } \mathbb{C}^{n+1}\}.$$

Each point in $\mathbb{C}^{n+1} \setminus \{0\}$ lies in precisely one of the subspaces, hence, we have the projection

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n,\ z \mapsto [\text{subspace spanned by } z \in \mathbb{C}^{n+1}].$$

We give $\mathbb{P}^n$ the induced quotient topology letting $\mathcal{U} \subset \mathbb{P}^n$ be open if and only if $\pi^{-1}(\mathcal{U})$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$. This makes π continuous and $\mathbb{P}^n$ a Hausdorff space with a countable basis. Moreover, one can see that $\pi|_{\mathcal{S}^n} : \mathcal{S}^n \to \mathbb{P}^n$ is onto, hence, $\mathbb{P}^n$ is compact.

We want to turn $\mathbb{P}^n$ into a complex manifold. If one representative $(z_0, z_1, \ldots, z_n)$ of $[z] \in \mathbb{P}^n$ has its jth coordinate nonzero, all its representatives will. Moreover, two representatives will differ by a multiple. We can therefore define an atlas $\{(\mathcal{U}_j, \varphi_j)\}$ of $\mathbb{P}^n$ through the charts

$$\mathcal{U}_j = \pi(\{z \in \mathbb{C}^{n+1} | \, z_j \neq 0\}),$$

so that $\bigcup_j \mathcal{U}_j = \mathbb{P}^n$, and

$$\varphi_j : \mathcal{U}_j \to \mathbb{C}^n, \ [z] \mapsto \left(\frac{z_0}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j}\right),$$

which does not depend on the choice of representative, hence, are well-defined. Each φ_j is 1-1 and onto. The compositions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are of the form $z \mapsto \frac{1}{z_j} z$, for $z \in \{z \in \mathbb{C}^n | \, z_j \neq 0\}$. Hence, are $\mathcal{O}$-compatible, which turns $\mathbb{P}^n$ into a complex manifold.

To motivate the compactification of $\mathbb{C}^n$ with $\mathbb{P}^n$, we shall see that there is a natural identification of $\mathbb{P}^n$ with $\mathbb{C}^n \cup \mathbb{P}^{n-1}$.

Since the elements of $\mathbb{P}^n$ are of the form $[z_0] = \{z \in \mathbb{C}^{n+1} | \, z = \lambda z_0\}$ for some z_0 in the particular subspace $[z_0]$, then, if a representative in $\mathbb{C}^{n+1} \backslash \{0\}$ of $[z_0]$ has first coordinate nonzero, then all its representatives will. In particular, one and only one of its representatives will have first coordinate equal to 1. Hence, we can identify these elements of $\mathbb{P}^n$ with the set $\{1\} \times \mathbb{C}^n$, which is naturally identifiable with $\mathbb{C}^n$.

The other elements in $\mathbb{P}^n$ are those whose representatives have their first coordinate equal to zero. These subspaces of $\mathbb{C}^{n+1}$ will then be the set of one dimentional subspaces of $\{0\} \times \mathbb{C}^n$, i.e., the set of one dimentional subspaces of $\mathbb{C}^n$, which are $\mathbb{P}^{n-1}$. The identification of $\mathbb{P}^n$ with $\mathbb{C}^n \cup \mathbb{P}^{n-1}$ follows.

Definition 7. Let $\mathcal{O}(\ell)$ denote the holomorphic line bundle over $\mathbb{P}^n$ with transition functions

$$g_{\alpha\beta}(z) = \left(\frac{z_\beta}{z_\alpha}\right)^\ell,$$

on the intersection of $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ defined as above.

We have to justify what this means, and show that it really defines a vector bundle. Actually, we will show that given an n-dimensional $\mathcal{S}$-manifold X, an open cover $\{\mathcal{U}_\alpha\}$ and $\mathcal{S}$-morphisms $g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(k, \mathbb{C})$ satisfying the compatibility conditions (2.2), we can define an $\mathcal{S}$-bundle of rank k over X with transition functions $g_{\alpha\beta}$.

We form the disjoint union $\widetilde{E} = \coprod_\alpha \mathcal{U}_\alpha \times \mathbb{C}^k$, on which we introduce the equivalence relation

$$(x, w) \sim (y, w) \Longleftrightarrow x = y \text{ and } v = g_{\alpha\beta}(x) w,$$

where $(x, v) \in \mathcal{U}_\alpha \times \mathbb{C}^k$ and $(y, w) \in \mathcal{U}_\beta \times \mathbb{C}^k$. That this defines an equivalence relation follows directly from the compatibility conditions

(2.2). Let $E := \widetilde{E}/\!\sim$ denote the set of equivalence clases, equiped with the quotient topology.

We define local trivializations of E

$$h_\alpha : h_\alpha^{-1}(\mathcal{U}_\alpha \times \mathbb{C}^k) \to \mathcal{U}_\alpha \times \mathbb{C}^k,$$

through

$$[(x,v)] = h_\alpha^{-1}(x,v), \quad (x,v) \in \mathcal{U}_\alpha \times \mathbb{C}^k.$$

These trivializations are onto and, since two distinct points in $\mathcal{U}_\alpha \times \mathbb{C}^k$ are unrelated, 1-1. If $A_\alpha \subset \mathcal{U}_\alpha \times \mathbb{C}^k$ is open, then $h_\alpha^{-1}(A_\alpha) \subset E$ also will be, since it is if the pull-back to $\coprod_\alpha \mathcal{U}_\alpha \times \mathbb{C}^k$, i.e.,

$$\coprod_\alpha \{(y,w) \in \mathcal{U}_\beta \times \mathbb{C}^k \,|\, \exists\, (x,v) \in A_\alpha : \; x = y, \; v = g_{\alpha\beta}(x)w\},$$

is open.

Conversely, A_α will be open if $h_\alpha^{-1}(A_\alpha)$ is, since this then pulls back to an open set in $\coprod_\alpha \mathcal{U}_\alpha \times \mathbb{C}^k$, which restricted to $\mathcal{U}_\alpha \times \mathbb{C}^k$ is, merely, A_α and open. Hence, each h_α is a homeomorphism, and clearly, $h_\alpha^x : E_x \to \mathbb{C}^k$ a vector space isomorphism.

Given an $\mathcal{S}$-atlas $\{(\mathcal{V}_\alpha, \psi_\alpha)\}$ of X, we can define charts for E:

$$\varphi_{\alpha\beta} : E|_{\mathcal{U}_\alpha \cap \mathcal{V}_\beta} \xrightarrow{h_\alpha} \mathcal{U}_\alpha \cap \mathcal{V}_\beta \times \mathbb{C}^k \xrightarrow{\psi_\beta} \mathbb{C}^n \times \mathbb{C}^k.$$

On the overlaps, the compositions takes the form

$$\varphi_{\alpha\beta} \circ \varphi_{\gamma\delta}^{-1}(z,v) = (\psi_\beta \circ \psi_\delta^{-1}(z), g_{\alpha\gamma}(\psi_\delta^{-1}(z))v),$$

and are $\mathcal{S}$-morphism, thus, turns E into an $\mathcal{S}$-manifold. Moreover, they assures that each h_α is an $\mathcal{S}$-isomorphism.

Finally, let $\pi : E \to X$ be the projection that for an equivalence class $[(x,v)]$ returns $x \in X$. Since $\pi^{-1}(x)$ has the structure of a k-dimentional complex vector space, we conclude that we have a well-defined $\mathcal{S}$-bundle $\pi : E \to X$.

4. Duals, tensor products and direct sums

Given vector spaces $\mathcal{V}$ and $\mathcal{W}$, there is an amount of further vector space constructions to be done. The dual space $\mathcal{V}^*$, the tensor product $\mathcal{V} \otimes \mathcal{W}$ and the direct sum $\mathcal{V} \oplus \mathcal{W}$, are a few examples. We would like to generalize these constructions to include even vector bundles. Since for a vector bundle we have to each point in the base space an associated vector space, i.e., the fibre over this point, we are lead to construct dual-, tensor- and direct sum bundles through these vector space constructions of the fibres. We will see that this is possible.

4.1. **Duals.** Given a vector bundle $\pi : E \to X$ of rank m, with transition functions $g_{\alpha\beta}$, we want to create the dual bundle $\pi^* : E^* \to X$, over the same base space X, with the property that the fibres E_x^* of the dual bundle are the dual spaces of the corresponding fibres E_x of the given vector bundle. Obviously, E^* would be defined as the disjoint union $E^* = \coprod_{x \in X} E_x^*$, and π^* will be defined through $(\pi^*)^{-1}(x) = E_x^*$. A trivialization $(\mathcal{U}_\alpha, h_\alpha)$ of the given bundle imposes naturally a trivialization $(\mathcal{U}_\alpha, h_\alpha^*)$ of the dual bundle

$$h_\alpha^* : E^*|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times (\mathbb{C}^m)^* \simeq \mathcal{U}_\alpha \times \mathbb{C}^m,$$

where h_α^* comes given by h_α^{-1}, i.e., for $f \in E_x^*$,

$$h_\alpha^*(f) = f \circ h_\alpha^{-1} = f \circ (h_\alpha^x)^{-1}.$$

For a functional $f \in E_x^*$ on E_x and $e \in E_x$ expressed with trivializations, we have the connection

$$f(e) = f_\alpha^t e_\alpha,$$

with f_α and e_α written as column vectors. This must be valid regardless of the choice of trivialization, i.e.,

$$f_\beta^t e_\beta = f_\alpha^t e_\alpha = f_\alpha^t g_{\alpha\beta} e_\beta \quad \text{on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta.$$

Hence,

$$f_\alpha^t g_{\alpha\beta} = f_\beta^t \quad \Longleftrightarrow \quad g_{\alpha\beta}^t f_\alpha = f_\beta \quad \Longleftrightarrow \quad f_\alpha = (g_{\alpha\beta}^t)^{-1} f_\beta = (g_{\alpha\beta}^{-1})^t f_\beta$$

So, the dual bundle must have transition functions $g_{\alpha\beta}^* = (g_{\alpha\beta}^{-1})^t$.

We now know what conditions the dual bundle must satisfy in order to be a vector bundle. Conversely, defining the dual bundle in this way gives us a well-defined vector bundle with desired property. Moreover, if E is holomorphic, we see that the dual trivialization functions will be holomorphic, hence, E^* will be a holomorphic vector bundle.

Proposition 4.1. *Let $\pi : E \to X$ be an $\mathcal{S}$-bundle. Given a local $\mathcal{S}$-frame $\{e_j\}$ of E, then we can always find a dual $\mathcal{S}$-frame $\{e_j^*\}$ of the dual bundle E^*, such that $e_i^* \, e_j = \delta_{ij}$, the Kronecker delta.*

Proof. Without loss of generality, we may assume that the local frame of E is the trivialization frame $\{h^{-1}(e_j)\}$, where $\{e_j\}$ is an ortonormal basis of $\mathbb{C}^m$, and $m = \operatorname{rank} E$.

With the trivialization frame $\{(h^*)^{-1}(e_j)\}$ for E^*, we have

$$(h^*)^{-1}(e_i) \cdot h^{-1}(e_j) = e_i \circ h \cdot h^{-1}(e_j) = e_i \cdot e_j = \delta_{i,j},$$

which completes the proof. □

The dual of the line bundle $\mathcal{O}(\ell)$ is of particular interest.

Proposition 4.2. *The dual bundle of $\mathcal{O}(\ell)$ is $\mathcal{O}(-\ell)$.*

Proof. The transition functions of $\mathcal{O}(\ell)$ are $g_{\alpha\beta} = (z_\beta/z_\alpha)^\ell$. The construction of the dual bundle tells us that the dual bundle will have transition functions $(g_{\alpha\beta}^{-1})^t = (z_\beta/z_\alpha)^{-\ell}$, since the $g_{\alpha\beta}$ are one-dimensional scalar transformations. But the vector bundle over $\mathbb{P}^n$ constructed with these transition functions is $\mathcal{O}(-\ell)$. □

4.2. **Direct sums.** Suppose that we are given two vector bundles $\pi^E : E \to X$ and $\pi^F : F \to X$ of rank m and n, respectively, over the same base space X. We want to define the vector bundle $\pi^{E\oplus F} : E \oplus F \to X$ such that the fibre at a given point $x \in X$ is the direct sum of the fibres of the given bundles, i.e., $(E\oplus F)_x = E_x \oplus F_x$. We define $E \oplus F = \coprod_{x\in X} E_x \oplus F_x$ and the projection $\pi^{E\oplus F}$ through $(\pi^{E\oplus F})^{-1}(x) = E_x \oplus F_x$. Given trivialization functions

$$h_E : E|_\mathcal{U} \to \mathcal{U} \times \mathbb{C}^m, \quad h_F : F|_\mathcal{U} \to \mathcal{U} \times \mathbb{C}^n$$

we can define

$$h_{E\oplus F} : E \oplus F|_\mathcal{U} \to \mathcal{U} \times (\mathbb{C}^m \oplus \mathbb{C}^n),$$

where the image set is, importantly, isomorphic to $\mathcal{U} \times \mathbb{C}^{m+n}$, given by the rule

$$h_{E\oplus F}(u\oplus v) = (x, h_E^x(u) \oplus h_F^x(v) \simeq (x, h_E^x(u), h_F^x(v))),$$

where $u \in E_x$ and $v \in F_x$. The transition functions of the direct sum bundle become

$$g_{\alpha\beta}^{E\oplus F} = \begin{pmatrix} g_{\alpha\beta}^E & 0 \\ 0 & g_{\alpha\beta}^F \end{pmatrix}.$$

Clearly, $E \oplus F$ will be holomorphic if E and F are.

4.3. **Tensor products.** As above, we want to define the vector bundle $\pi^{E\otimes F} : E \otimes F \to X$ such that the fibre at a given point $x \in X$ is the tensor product of the fibres of the given bundles, i.e., $(E \otimes F)_x = E_x \otimes F_x$. We have $E \otimes F = \coprod_{x\in X} E_x \otimes F_x$ and $\pi^{E\otimes F}$ defined through $(\pi^{E\otimes F})^{-1}(x) = E_x \otimes F_x$. We define trivialization functions

$$\begin{aligned} h_{E\otimes F} :& E \otimes F|_\mathcal{U} \to \mathcal{U} \times (\mathbb{C}^m \otimes \mathbb{C}^n) \simeq \mathcal{U} \times \mathbb{C}^{mn} \\ & u \otimes v \mapsto (x, h_E^x(u) \otimes h_F^x(v)), \end{aligned}$$

where $u \in E_x$ and $v \in F_x$. These generate the transition functions $g_{\alpha\beta}^{E\otimes F} = g_{\alpha\beta}^E \otimes g_{\alpha\beta}^F$ of the tensor bundle, and accordingly, $E \otimes F$ will be holomorpic if E and F are.

Proposition 4.3. *The tensor bundle $\mathcal{O}(p) \otimes \mathcal{O}(q)$ equals $\mathcal{O}(p + q)$.*

Proof. From the construction, we have trivialization functions which take their values in $\mathcal{U} \times (\mathbb{C} \otimes \mathbb{C})$. But, since $\mathbb{C} \otimes \mathbb{C}$ is naturally isomorfic with $\mathbb{C}$, we can identify them. Then the transition functions of $\mathcal{O}(p) \otimes \mathcal{O}(q)$ become

$$g_{\alpha\beta} = \left(\frac{z_\beta}{z_\alpha}\right)^p \otimes \left(\frac{z_\beta}{z_\alpha}\right)^q = \left(\frac{z_\beta}{z_\alpha}\right)^p \left(\frac{z_\beta}{z_\alpha}\right)^q = \left(\frac{z_\beta}{z_\alpha}\right)^{p+q}.$$

Hence, $\mathcal{O}(p) \otimes \mathcal{O}(q) = \mathcal{O}(p+q)$. □

Further tensor operations generate multiple tensor bundles. Especially, we get $\bigotimes^k E$, and from this we can construct the k-fold bundle $\Lambda^k E$. Each fibre takes the form $\Lambda^k E_x = \pi(\bigotimes^k E_x)$, where $\pi : \bigotimes^k E \to \Lambda^k E$ is the quotient projection, and trivializations functions can be defined as

$$\begin{aligned} h_{\Lambda^k E} :& \Lambda^k E|_{\mathcal{U}} \to \mathcal{U} \times \Lambda^k \mathbb{C}^m \simeq \mathcal{U} \times \mathbb{C}^{\binom{m}{k}} \\ & e_{J_k} \mapsto (x,\, h^x_E(e_{j_1}) \wedge h^x_E(e_{j_2}) \wedge \ldots \wedge h^x_E(e_{j_k})), \end{aligned}$$

where $e_j \in \Lambda^k E_x$. The generated transition functions $g^{\Lambda^k E}_{\alpha\beta}$ merely become $g^E_{\alpha\beta}$ multiplied to each compartment, and $\Lambda^k E$ will be holomorphic if E is.

Further we may construct $\Lambda^* E = \bigoplus_{k=0}^{\infty} \Lambda^k E$. Especially, we let $\Lambda^k E \wedge \Lambda^q T^*_{0,1}$ be the powers generated from $\Lambda^*(E \oplus T^*_{0,1})$.

4.4. **Pull-backs.** A final construction of our interest is the pull-back of a bundle $\pi : E \to X$, by a base space homomorphism $f : Y \to X$. We construct the pull-back bundle $\pi_f : f^* E \to Y$ with fibres $f^* E_y = E_{f(y)}$, where trivialization functions come given by

$$\begin{aligned} & h_{f^*E} : f^* E|_{f^{-1}(\mathcal{U})} \to f^{-1}(\mathcal{U}) \times \mathbb{C}^m \\ & h_{f^*E}(u) = (f(y), h^{f(y)}(u)), \end{aligned}$$

where $u \in f^* E_y$. The transition functions simply become the pull-backs $f^* g_{\alpha\beta} = g_{\alpha\beta} \circ f$, and the pull-back bundle $f^* E$ is holomorphic if E and f are.

5. Holomorphic sections of line bundles

5.1. **Compactifying $\mathbb{C}^n$ by $\mathbb{P}^n$.** Given a function, $\psi : \mathcal{U} \to \mathbb{C}$, defined in an open subset $\mathcal{U} \subset \mathbb{P}_n$, we can identify ψ with a 0-homogeneous function, $\Psi : \pi^{-1}(\mathcal{U}) \to \mathbb{C}$, such that $\Psi(z) = \psi([z])$, e.g., $\Psi(z) = \psi \circ \pi(z)$, where π is the projection $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{P}^n$ and $\pi^{-1}(\mathcal{U}) \subset \mathbb{C}^{n+1} \backslash \{0\}$. If ψ is smooth or holomorphic, by composition, Ψ also will be, since π is holomorphic, and ψ expressed in this structure is smooth or holomorphic, respectively.

Given a section ξ of the line bundle $\mathcal{O}(\ell)$, we can identify it with complex-valued functions on open subsets of $\mathbb{P}^n$, hence, we identify it with 0-homogeneous functions ξ_α on open subsets of $\mathbb{C}^{n+1}\backslash\{0\}$. Furthermore, these piece together to an ℓ-homogeneous complex-valued function on $\mathbb{C}^{n+1}\backslash\{0\}$, which due to the transition functions of $\mathcal{O}(\ell)$, equals $z_\alpha^\ell \xi_\alpha$ on $\pi^{-1}(\mathcal{U}_\alpha) \subset \mathbb{C}^{n+1}$. It is holomorphic if ξ is and well-defined because of the compatibility condition (2.3), i.e., on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ we have

$$z_\alpha^\ell \xi_\alpha = z_\beta^\ell \xi_\beta.$$

Conversely, given an ℓ-homogeneous function $f : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}$, we can define

$$f_\alpha := f/z_\alpha^\ell$$

on $\pi^{-1}(\mathcal{U}_\alpha)$. The collection $\{f_\alpha\}$ is 0-homogenous and satisfy the compatibility condition (2.3) on $\pi^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$. Thus, it represents merely a section of $\mathcal{O}(\ell)$, holomorphic if f is.

Especially, ℓ-homogeneous polynomials are holomorphic sections of $\mathcal{O}(\ell)$.

Proposition 5.1. *Let $\mathcal{O}(\mathbb{P}^n, \mathcal{O}(\ell))$ denote the set of global holomorphic section of $\mathcal{O}(\ell)$.*

a) If $\ell < 0$, then $\mathcal{O}(\mathbb{P}^n, \mathcal{O}(\ell)) = \{0\}$.
b) If $\ell \geq 0$, then $\mathcal{O}(\mathbb{P}^n, \mathcal{O}(\ell)) = \{\ell$-homogenous polynomials on $\mathbb{C}^{n+1}\}$.

Proof. Let $\xi \in \mathcal{O}(\mathbb{P}^n, \mathcal{O}(\ell))$. It remains to show that ξ is of claimed form. ξ can be thought of as an ℓ-homogeneous holomorphic function on $\mathbb{C}^{n+1}\backslash\{0\}$. By Hartogs' phenomenon, we have a unique holomorphic extension of ξ to $\mathbb{C}^{n+1}$. Every holomorphic function can be expanded in a Taylor series, and hence,

$$\xi(z) = \sum_\alpha a_\alpha z^\alpha,$$

where the coefficients are given by

$$a_\alpha = \frac{1}{\alpha!} \frac{\partial^\alpha \xi}{\partial z^\alpha}(0).$$

The relation $\xi(\lambda z) = \lambda^\ell \xi(z)$ yields by differentiation $\lambda \frac{\partial \xi}{\partial z}(\lambda z) = \lambda^\ell \frac{\partial \xi}{\partial z}(z)$ and generally

$$\frac{\partial^\alpha \xi}{\partial z^\alpha}(\lambda z) = \lambda^{\ell - |\alpha|} \frac{\partial^\alpha \xi}{\partial z^\alpha}(z),$$

where, if we put $z = 0$, we get

$$\frac{\partial^\alpha \xi}{\partial z^\alpha}(0) = \lambda^{\ell - |\alpha|} \frac{\partial^\alpha \xi}{\partial z^\alpha}(0),$$

for all $\lambda \in \mathbb{C}$. Therefore we have that

$$\frac{\partial^\alpha \xi}{\partial z^\alpha}(0) = 0$$

for all α such that $|\alpha| \neq \ell$. Hence, if $\ell < 0$, ξ turns out to be identically zero, and if $\ell \geq 0$

$$\xi(z) = \sum_{|\alpha|=\ell} a_\alpha z^\alpha.$$

$\square$

5.2. **Compactifying $\mathbb{C}^{n+\nu}$ by $\mathbb{P}^n \times \mathbb{P}^\nu$.** Consider the compactification $\mathbb{P}^n \times \mathbb{P}^\nu$ of $\mathbb{C}^n \times \mathbb{C}^\nu$. Defining $\mathcal{U}_\alpha := \{[z] \in \mathbb{P}^n |\, z_\alpha \neq 0\}$ and $\mathcal{V}_\gamma := \{[\zeta] \in \mathbb{P}^\nu |\, \zeta_\gamma \neq 0\}$, we have

$$\begin{aligned} \mathcal{W}_{\alpha,\gamma} \cap \mathcal{W}_{\beta,\delta} :=& \mathcal{U}_\alpha \times \mathcal{V}_\gamma \cap \mathcal{U}_\beta \times \mathcal{V}_\delta \\ =& \{([z],[\zeta]) \in \mathbb{P}^n \times \mathbb{P}^\nu |\, z_\alpha, \zeta_\gamma \neq 0\} \\ & \cap \{([z],[\zeta]) \in \mathbb{P}^n \times \mathbb{P}^\nu |\, z_\beta, \zeta_\delta \neq 0\}. \end{aligned}$$

From the line bundles $\mathcal{O}(\ell)$ over $\mathbb{P}^n$ and $\mathcal{O}(\lambda)$ over $\mathbb{P}^m$, we have the pull-back bundles $\mathcal{O}_z(\ell)$ and $\mathcal{O}_\zeta(\lambda)$ over $\mathbb{P}^n \times \mathbb{P}^\nu$. From these we form the holomorphic line bundle $\mathcal{O}_z(\ell) \otimes \mathcal{O}_\zeta(\lambda)$ over $\mathbb{P}^n \times \mathbb{P}^\nu$. Its transition functions become

$$g^{\gamma\delta}_{\alpha\beta} = (z_\beta/z_\alpha)^\ell \otimes (\zeta_\delta/\zeta_\gamma)^\lambda \simeq (z_\beta/z_\alpha)^\ell (\zeta_\delta/\zeta_\gamma)^\lambda$$

on $\mathcal{W}_{\alpha,\gamma} \cap \mathcal{W}_{\beta,\delta}$. From these transition functions, it follows that its dual is $\mathcal{O}_z(-\ell) \otimes \mathcal{O}_\zeta(-\lambda)$, and we shall see that sections of this bundle relate to polynomials in a similar way as sections of $\mathcal{O}(\ell)$ do.

We aim to generalize what has been said about sections of $\mathcal{O}(\ell)$. A function $\varphi([z],[\zeta]) : \mathcal{W} \to \mathbb{C}$, defined on an open subset $\mathcal{W}$ of $\mathbb{P}^n \times \mathbb{P}^\nu$, can be identified with a $(0,0)$-homogeneous function $\Phi(z,\zeta) : \pi^{-1}(\mathcal{W}) \to \mathbb{C}$ on the open subset $\pi^{-1}(\mathcal{W})$ of $\mathbb{C}^{n+1} \backslash \{0\} \times \mathbb{C}^{\nu+1} \backslash \{0\}$.

As before, a given $\mathcal{S}$-section ξ of $\mathcal{O}_z(\ell) \otimes \mathcal{O}_\zeta(\lambda)$, is identifiable with a family of $\mathcal{S}$-morphisms on open subsets of $\mathbb{P}^n \times \mathbb{P}^\nu$, thus, with a family of $(0,0)$-homogeneous $\mathcal{S}$-functions on open sets of $\mathbb{C}^{n+1} \backslash \{0\} \times \mathbb{C}^{\nu+1} \backslash \{0\}$. Moreover, these give rise to a (ℓ,λ)-homogeneous $\mathcal{S}$-function on $\mathbb{C}^{n+1} \backslash \{0\} \times \mathbb{C}^{\nu+1} \backslash \{0\}$, i.e., ℓ-homogeneous in z and λ-homogeneous in ζ, which restricted to $\pi_p^{-1}(\mathcal{W}_{\alpha,\gamma})$ is $z_\alpha^\ell \zeta_\gamma^\lambda \xi_{\alpha,\gamma}$. On $\mathcal{W}_{\alpha,\gamma} \cap \mathcal{W}_{\beta,\delta}$ we have

$$z_\alpha^\ell \zeta_\gamma^\lambda \xi_{\alpha,\gamma} = z_\beta^\ell \zeta_\delta^\lambda \xi_{\beta,\delta},$$

because of the compatibility conditions (2.3), hence, this is well defined.

Conversely, given a (ℓ,λ)-homogeneous $\mathcal{S}$-function $f : \mathbb{C}^{n+1} \backslash \{0\} \times \mathbb{C}^{\nu+1} \backslash \{0\} \to \mathbb{C}$, the collection $\{(\pi_p^{-1}(\mathcal{W}_{\alpha,\gamma}), f_{\alpha,\gamma})\}$, where

$$f_{\alpha,\gamma} := \frac{1}{z_\alpha^\ell \zeta_\gamma^\lambda} f,$$

satisfy the compatibility conditions (2.3) and are $(0,0)$-homogeneous. Therefore, they can be seen as $\mathcal{S}$-functions on $\mathbb{P}^n \times \mathbb{P}^\nu$, hence, as an $\mathcal{S}$-section of $\mathcal{O}_z(\ell) \otimes \mathcal{O}_\zeta(\lambda)$.

Especially, the (ℓ,λ)-homogeneous polynomials on $\mathbb{C}^{n+1}\times\mathbb{C}^{\nu+1}$ are holomorphic sections of $\mathcal{O}_z(\ell)\otimes\mathcal{O}_\zeta(\lambda)$.

Proposition 5.2. *Let $\mathcal{O}(\mathbb{P}^n\times\mathbb{P}^\nu,\mathcal{O}_z(\ell)\otimes\mathcal{O}_\zeta(\lambda))$ denote the global holomorphic sections of $\mathcal{O}_z(\ell)\otimes\mathcal{O}_\zeta(\lambda)$.*

a) If $\ell<0$ or $\lambda<0$, then $\mathcal{O}(\mathbb{P}^n\times\mathbb{P}^\nu,\mathcal{O}_z(\ell)\otimes\mathcal{O}_\zeta(\lambda))=\{0\}$.
b) If ℓ, $\lambda\geq 0$, then $\mathcal{O}(\mathbb{P}^n\times\mathbb{P}^\nu,\mathcal{O}_z(\ell)\otimes\mathcal{O}_\zeta(\lambda))$
$=\{(\ell,\lambda)$-homogeneous polynomials on $\mathbb{C}^{n+1}\times\mathbb{C}^{\nu+1}\}$.

Proof. Let ξ be a global holomorphic section of $\mathcal{O}_z(\ell)\otimes\mathcal{O}_\zeta(\lambda))$. It remains to show that ξ is of claimed form. We can see ξ as an (ℓ,λ)-homogeneous holomorphic function on $\mathbb{C}^{n+1}\setminus\{0\}\times\mathbb{C}^{\nu+1}\setminus\{0\}$. Analogously to the proof of Proposition 5.1, for fixed $\zeta\in\mathbb{C}^{\nu+1}$, $\xi(z,\zeta)$ can be extended to a holomorphic function on $\mathbb{C}^{n+1}\times\mathbb{C}^{\nu+1}\setminus\{0\}$, according to Hartogs' phenomenon. Moreover, we have the Taylor expansion

$$\xi=\sum_\alpha\frac{1}{\alpha!}\frac{\partial^\alpha\xi}{\partial z^\alpha}(0,\zeta)z^\alpha.$$

Through differentiation of $\xi(\tau z,\zeta)=\tau^\ell\xi(z,\zeta)$, we have

$$\frac{\partial^\alpha\xi}{\partial z^\alpha}(\tau z,\zeta)=\tau^{\ell-|\alpha|}\frac{\partial^\alpha\xi}{\partial z^\alpha}(z,\zeta),$$

and conclude that

$$\frac{\partial^\alpha\xi}{\partial z^\alpha}(0,\zeta)=0$$

for all α such that $|\alpha|\neq\ell$. Hence, $\xi\equiv0$ if $\ell<0$, and for $\ell\geq0$

$$\xi=\sum_{|\alpha|=\ell}\frac{1}{\alpha!}\frac{\partial^\alpha\xi}{\partial z^\alpha}(0,\zeta)z^\alpha.$$

In the same way, for fixed z, we can extend ξ holomorphically to $\mathbb{C}^{n+1}\times\mathbb{C}^{\nu+1}$. The expansion in a Taylor series gives

$$\xi=\sum_\beta\frac{1}{\beta!}\frac{\partial^\beta\xi}{\partial\zeta^\beta}(z,0)\zeta^\beta=\sum_\beta\frac{1}{\beta!}\frac{\partial^\beta}{\partial\zeta^\beta}\left(\sum_{|\alpha|=\ell}\frac{1}{\alpha!}\frac{\partial^\alpha\xi}{\partial z^\alpha}(0,\zeta)z^\alpha\right)_{\zeta=0}\zeta^\beta.$$

Moreover, homogeneity implies that

$$\frac{\partial^\beta\xi}{\partial\zeta^\beta}(z,0)=0$$

for β such that $|\beta|\neq\lambda$. Hence, $\xi\equiv0$ if $\lambda<0$, and for $\lambda\geq0$

$$\xi=\sum_{\substack{|\alpha|=\ell\\|\beta|=\lambda}}\frac{1}{\alpha!\beta!}\frac{\partial^{\beta+\alpha}\xi}{\partial\zeta^\beta\partial z^\alpha}(0,0)z^\alpha\zeta^\beta.$$

$\square$

6. Operations on holomorphic vector bundles

Let $\pi : E \to X$ be a holomorphic vector bundle, $f \in \mathcal{O}(X, E^*)$ be a global holomorphic section of the dual E^* and $\xi \in \mathcal{E}(X, E)$ be a section of this bundle.

6.1. **The $\bar\partial$-operator on holomorphic vector bundles.** We define $\bar\partial\xi$ to be differential operator $\bar\partial$ applied to local trivializations $(\mathcal{U}_\alpha, \xi_\alpha)$ of ξ. We introduce the notation $\mathcal{E}_{0,q}(X, E) := \mathcal{E}(X, E \wedge \Lambda^q T^*_{0,1})$ for the $(0, q)$-forms with values in E.

Proposition 6.1. *Let $\pi : E \to X$ be a holomorphic vector bundle and $\xi \in \mathcal{E}(X, E)$. Then $\bar\partial\xi$ is a $(0, 1)$-form with values in E, i.e., $\bar\partial\xi \in \mathcal{E}_{0,1}(X, E)$.*

Proof. Since on the overlaps $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ we have

$$\bar\partial\xi_\alpha = \bar\partial(g_{\alpha\beta}\xi_\beta) = \bar\partial g_{\alpha\beta}\xi_\beta + g_{\alpha\beta}\bar\partial\xi_\beta = g_{\alpha\beta}\bar\partial\xi_\beta,$$

it follows that the $\bar\partial\xi_\alpha$'s satisfy the compatibility conditions (2.3), and, hence, are a $(0, 1)$-form with values in E. □

This proposition justifies the above definition. Moreover, for ξ expressed in a local holomorphic frame $\{e_j\}$, $\bar\partial\xi$ takes the form

$$\bar\partial\xi = \bar\partial \sum \xi_j e_j = \bar\partial \sum \xi_j \, h_\alpha \circ e_j = \sum \bar\partial\xi_j \wedge h_\alpha \circ e_j = \sum \bar\partial\xi_j \wedge e_j,$$

where the second to last equality follows since the frame is holomorphic. Especially, for any holomorphic bundle L over X, we may exchange E for $\Lambda^k E \wedge \Lambda^q T^*_{0,1} \otimes L$, and we obtain a map $\bar\partial : \mathcal{E}_{0,q}(X, \Lambda^k E \otimes L) \to \mathcal{E}_{0,q+1}(X, \Lambda^k E \otimes L)$.

6.2. **Interior multiplication on k-fold bundles.** We define interior multiplication $\delta_f : \Lambda^k E \to \Lambda^{k-1} E$ on k-fold bundles as

$$\begin{aligned}
\delta_f \xi = \delta_f \sum \xi_J e_J &= \delta_f \sum \xi_J \, e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_k} \\
&= \sum (-1)^{\ell+1} \xi_J f(e_{j_\ell}) \, e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge \widehat{e}_{j_\ell} \wedge \ldots \wedge e_{j_k} \\
&= \sum (-1)^{\ell+1} \xi_J f(e_{j_\ell}) e_{J \setminus \{j_\ell\}},
\end{aligned}$$

for a local frame $\{e_j\}$ of E. This generalizes to a map $\delta_f : \mathcal{E}(X, \Lambda^k E) \to \mathcal{E}(X, \Lambda^{k-1} E)$ by the same rule, and further to a map $\delta_f : \mathcal{E}_{0,q}(X, \Lambda^k E \otimes L) \to \mathcal{E}_{0,q}(X, \Lambda^{k-1} E \otimes L)$ given by

$$\delta_f \xi = \delta_f \sum_{I,j} \xi^E_{I,j} \wedge d\bar z_I \otimes l_j = \sum_{I,j} \delta_f \xi^E_{I,j} \wedge d\bar z_I \otimes l_j,$$

for a local frame $\{l_j\}$ of a bundle L over X.

Proposition 6.2. *Let $\pi : E \to X$ be a holomorphic vector bundle and $f \in \mathcal{O}(X, E^*)$. Then $\bar\partial$ and δ_f anti-commute on $\mathcal{E}_{0,q}(X, \Lambda^k E \otimes L)$.*

Proof. Let $\{e_j\}$ and $\{l_j\}$ be holomorphic frames of E and L, and $\xi \in \mathcal{E}_{0,q}(X, \Lambda^k E \otimes L)$. We have

$$\begin{aligned}\delta_f \bar\partial \xi &= \delta_f \sum \bar\partial \xi_{I,J,\kappa} \wedge e_J \wedge d\bar z_I \otimes l_\kappa \\ &= \sum (-1)^\ell f(e_{j_\ell})\, \bar\partial \xi_{I,J,\kappa} \wedge e_{J\setminus\{j_\ell\}} \wedge d\bar z_I \otimes l_\kappa,\end{aligned}$$

and

$$\begin{aligned}\bar\partial \delta_f \xi &= \bar\partial \sum (-1)^{\ell-1} \xi_{I,J,\kappa} f(e_{j_\ell})\, e_{J\setminus\{j_\ell\}} \wedge d\bar z_I \otimes l_\kappa \\ &= \sum (-1)^{\ell-1} \bar\partial (f(e_{j_\ell}) \xi_{I,J,\kappa}) \wedge e_{J\setminus\{j_\ell\}} \wedge d\bar z_I \otimes l_\kappa \\ &= \sum (-1)^{\ell-1} f(e_{j_\ell})\, \bar\partial \xi_{I,J,\kappa} \wedge e_{J\setminus\{j_\ell\}} \wedge d\bar z_I \otimes l_\kappa.\end{aligned}$$

Hence, $\bar\partial \delta_f = -\delta_f \bar\partial$. □

7. The Koszul complex

Let $\pi : E \to X$ be a holomorphic vector bundle of rank m. Let f be a pointwise nonvanishing section, i.e., $f_x \neq 0$ for all $x \in X$, of the dual bundle E^*, and φ a smooth function. There exists always a global smooth solution $v \in \mathcal{E}(X, E)$ to the equation $fv = \varphi$. Given a family of local frames $\{\mathcal{U}_\alpha, \{e_{\alpha,i}\}\}$ covering E, we can express f as $f_\alpha = \sum_{i=1}^m f_{\alpha,i} e^*_{\alpha,i}$. Choosing

$$\sigma_\alpha = \sum_i \frac{\bar f_{\alpha,i}}{|f_\alpha|^2} e_{\alpha,i}, \tag{7.1a}$$

and a partition of unity $\{\psi_\alpha\}$ subordinate the covering, then

$$\sigma = \sum \psi_\alpha \sigma_\alpha \tag{7.1b}$$

is a smooth solution to $fv \equiv 1$, and $\varphi\sigma$ is a smooth solution to $fv = \varphi$.

Given a holomorphic section $f \in \mathcal{O}(X, E^*)$, which is pointwise non-vanishing, and a holomorphic function $\varphi \in \mathcal{O}(X)$, we consider the task of finding a holomorphic solution to the equation

$$fv = \varphi. \tag{7.2}$$

We shall introduce the Koszul complex, and present a method making use of it to solve such a task.

We say that a function $f : E \to F$, for vector bundles over the same base space X, is a *vector bundle homomorphism* if f preserves fibres and is linear on each fibre, i.e., $f|_{E_x} : E_x \to F_x$ and $f_x := f|_{E_x}$ is linear.

We define $\operatorname{Ker} f = \bigcup_{x\in X} \operatorname{Ker} f_x$ and $\operatorname{Im} f = \bigcup_{x\in X} \operatorname{Im} f_x$. A sequence of vector bundle homomorphisms,

$$\ldots \to E \xrightarrow{f} F \xrightarrow{g} G \to \ldots,$$

is called a *complex* if $\operatorname{Im} f \subset \operatorname{Ker} g$ at each bundle F. If $\operatorname{Im} f = \operatorname{Ker} g$ at each bundle, it is said to be *exact.*

Definition 8. Let $\pi : E \to X$ be a vector bundle of rank m, $f \in \mathcal{E}(X, E^*)$ and $\delta_f : \Lambda^{k+1}E \to \Lambda^k E$ be the interior multiplication with f. We call the following complex

$$0 \to \Lambda^m E \xrightarrow{\delta_f} \Lambda^{m-1} E \xrightarrow{\delta_f} \dots \xrightarrow{\delta_f} \Lambda^2 E \xrightarrow{\delta_f} E \xrightarrow{\delta_f} \mathbb{C} \to 0, \tag{7.3}$$

the *Koszul complex.*

Since $\delta_f^2 = 0$, it is clear that sequence (7.3) really defines a complex.

Proposition 7.1. *The Koszul complex is exact at a given point $x \Leftrightarrow f_x \neq 0$.*

Proof. ($\Rightarrow$:) Suppose that $f_x = 0$. Then $f_x : E_x \to \mathbb{C}$ is not onto, and the complex will not be exact.

($\Leftarrow$:) Let $x \in X$ be fixed. It is sufficient to show that $\operatorname{Ker} \delta_{f,\,x} \subset \operatorname{Im} \delta_f$. Since $f_x \neq 0$, then we can find $\sigma \in E_x$ such that $f_x \sigma = 1$. Then

$$\delta_f(\sigma \wedge w) = \delta_f \sigma \wedge w - \sigma \wedge \delta_f w = w - \sigma \wedge \delta_f w,$$

and introducing $\tau\, w := \sigma \wedge w$,

$$\delta_f \circ \tau + \tau \circ \delta_f = \mathrm{id}_x .$$

So for $w \in \operatorname{Ker} \delta_{f,\,x}$

$$\delta_f(\tau\, w) = (\delta_f \circ \tau + \tau \circ \delta_f) w = w.$$

Hence, $w \in \operatorname{Im} \delta_f$. ∎

We will need to consider a more general form of the Koszul complex:

$$0 \to \Lambda^m E \otimes L \xrightarrow{\delta_f} \dots \xrightarrow{\delta_f} \Lambda^2 E \otimes L \xrightarrow{\delta_f} E \otimes L \xrightarrow{\delta_f} L \to 0, \tag{7.4}$$

where L is any vector bundle over the same base space X as E. In the same way, it follows that complex (7.4) is exact at x if and only if $f_x \neq 0$. We shall now see that if the Koszul complex is exact, then the complex obtained by interchanging $\Lambda^k E \otimes L$ by $\mathcal{E}_{0,q}(X, \Lambda^k E \otimes L)$ is exact.

Proposition 7.2. *If $f \neq 0$, $v \in \mathcal{E}_{0,q}(X, \Lambda^k E \otimes L)$ and $\delta_f v \equiv 0$, then there is a smooth section $w \in \mathcal{E}_{0,q}(X, \Lambda^{k+1} E \otimes L)$ such that $\delta_f w = v$.*

Proof. Let $\{l_j\}$ be a local frame for L. For any $\sigma \in \mathcal{E}(X, E)$ such that $f\sigma \equiv 1$, and $v \in \mathcal{E}_{0,q}(X, \Lambda^k E \otimes L)$, we have

$$\begin{aligned} \delta_f(\sigma \wedge v) &= \delta_f \sum \sigma \wedge v_{I,j} \wedge d\bar{z}_I \otimes l_j = \sum \delta_f(\sigma \wedge v_{I,j}) \wedge d\bar{z}_I \otimes l_j \\ &= \sum (\delta_f \sigma \wedge v_{I,j} - \sigma \wedge \delta_f v_{I,j}) \wedge d\bar{z}_I \otimes l_j = v - \sigma \wedge \delta_f v. \end{aligned}$$

Using the notation from the proof of Proposition 7.1, then

$$\delta_f \circ \tau + \tau \circ \delta_f = \mathrm{id},$$

and if $\delta_f v \equiv 0$,

$$\delta_f(\tau\, v) = (\delta_f \circ \tau + \tau \circ \delta_f)v = v.$$

Hence, $w = \sigma \wedge v$ will do. □

8. Existence of holomorphic sections

Slightly modified, we restate our problem. Let E be a holomorphic vector bundle of rank m over an n-manifold X. Given a holomorphic section f of the dual bundle E^* which is nonvanishing at each point, and a holomorphic section $\varphi \in \mathcal{O}(X, L)$ of the holomorphic line bundle L over X, we consider the question of existence of holomorphic sections in $\mathcal{O}(X, E \otimes L)$, satisfying

$$\delta_f v = \varphi. \tag{8.1}$$

Assume that v_1 is a smooth section of E that actually solves (8.1). We are already granted existence of smooth sections, e.g., $\sigma \otimes \varphi$ for σ defined as in (7.1). From equation (8.1) we see that $\bar\partial v_1$ satisfies

$$\delta_f \bar\partial v_1 = -\bar\partial \delta_f v_1 = -\bar\partial \varphi = 0,$$

since $\bar\partial$ and δ_f anti-commute. We cannot conclude that $\bar\partial v_1 = 0$, i.e., that v_1 is holomorphic. Though, Proposition 7.2 tells us that $\bar\partial v_1$ takes the form

$$\bar\partial v_1 = \delta_f v_2 \quad \Leftrightarrow \quad \bar\partial v_1 - \delta_f v_2 = 0$$

for some $(0,1)$-form v_2 with values in $\Lambda^2 E \otimes L$. This gives a hint about how to find a holomorphic solution. If we could solve the equation $\bar\partial w_2 = v_2$, then

$$0 = \bar\partial v_1 - \delta_f \bar\partial w_2 = \bar\partial(v_1 + \delta_f w_2).$$

Hence,

$$q = v_1 + \delta_f w_2 \tag{8.2}$$

would be a holomorphic section of $E \otimes L$. Moreover, $\delta_f q = \varphi$, and q would be the solution we are looking for. But, for a $\bar\partial$-equation to be solvable, its right hand side have to be $\bar\partial$-closed. Further application of the $\bar\partial$-operator yields

$$\delta_f \bar\partial v_2 = -\bar\partial \delta_f v_2 = -\bar\partial^2 v_1 = 0,$$

again since $\bar\partial^2 = 0$, and as before

$$\bar\partial v_2 = \delta_f v_3,$$

for some $(0,2)$-form v_3 with values in $\Lambda^3 E \otimes L$. In general, v_2 will not be $\bar\partial$-closed. But, if w_3 satisfies $\bar\partial w_3 = v_3$, then we may be able to solve the equation $\bar\partial w_2 = v_2 + \delta_f w_3$ with respect of w_2. We have now modified the $\bar\partial$-equation for w_2 and, hence, w_2 itself, assuming such solution exists. This will be of no concern, since $\bar\partial \delta_f w_2 = -\delta_f v_2 - \delta_f^2 w_3 = -\delta_f v_2$ and therefore

$$\bar\partial q = \bar\partial v_1 + \bar\partial \delta_f w_2 = \bar\partial v_1 - \delta_f v_2 = 0$$

and

$$\delta_f q = \delta_f v_1 + \delta_f^2 w_2 = \varphi.$$

Accordingly, q still solves equation (8.1).

As above, for solvability of $\bar\partial w_3 = v_3$, we need its right-hand side to be $\bar\partial$-closed. This we cannot know, and, following the same argument as above, we have the equations

$$\begin{aligned} \bar\partial v_3 &= \delta_f v_4 & \qquad \bar\partial w_3 &= v_3 + \delta_f w_4 \\ \bar\partial v_4 &= \delta_f v_5 & \qquad \bar\partial w_4 &= v_4 + \delta_f w_5 \\ &\vdots & &\vdots \end{aligned}$$

where v_k is a $(0, k-1)$-form and w_k is a $(0, k-2)$-form, both with values in $\Lambda^k E \otimes L$. As before, no modification of the $\bar\partial$-equations changes desired property of their solutions.

The left sequence will terminate. Since $\operatorname{rank} E = m$, then $v_{m+1} = 0$, since the exterior product of linearly dependent vectors is zero. We get the ending equation

$$\bar\partial v_m = 0.$$

But this equation gives rise to the equation $\bar\partial w_m = v_m$, with right hand side already $\bar\partial$-closed. And therefore the $\bar\partial w_k$-sequence of equations terminates as well. The sequence of $\bar\partial v_k$-equations can actually terminate before reaching the mth step. Since $v_k \in \mathcal{E}_{0,k-1}(X, \Lambda^k E \otimes L)$, then v_k will vanish if $k-1 > n$. Hence, each sequence terminates at $N = \min(m, n+1)$. We are left with the sufficient existence condition that for any sequence $\{v_k\}$, where $v_k \in \mathcal{E}_{0,k-1}(X, \Lambda^k E \otimes L)$, satisfying

$$\begin{aligned} \delta_f v_1 &= \varphi \\ \bar\partial v_1 - \delta_f v_2 &= 0 \\ \bar\partial v_2 - \delta_f v_3 &= 0 \\ &\vdots \\ \bar\partial v_{N-1} - \delta_f v_N &= 0 \\ \bar\partial v_N &= 0 \end{aligned} \tag{8.3}$$

the arisen $\bar\partial$-equations

$$\begin{aligned} \bar\partial w_N &= v_N \\ \bar\partial w_{N-1} &= v_{N-1} + \delta_f w_N \\ \bar\partial w_{N-2} &= v_{N-2} + \delta_f w_{N-1} \\ &\vdots \\ \bar\partial w_3 &= v_3 + \delta_f w_4 \\ \bar\partial w_2 &= v_2 + \delta_f w_3 \end{aligned} \tag{8.4}$$

are solvable.

We introduce the ∇-operator on $\bigoplus_{k=1}^{N} \mathcal{E}_{0,k-1}(X, \Lambda^k E \otimes L)$ as $\nabla = \delta_f - \bar\partial$. The system of equations (8.3), for $v = v_1 + v_2 + v_3 + \ldots + v_N$, can be abbreviated into

$$\nabla v = \varphi, \tag{8.5}$$

in view of the difference in degree of the v_k's as forms and as sections of $\Lambda^k E$. The solvability of equation (8.5) is due to the exactness of the Koszul complex, and continuing the idea from the proof of Proposition 7.2, we may obtain a concrete solution. We begin to set $u_1 = \sigma$, for any smooth section of E satisfying $f\sigma \equiv 1$. Since each u_{k+1} has to satisfy $\delta_f u_{k+1} = \bar\partial u_k$, we take

$$u_{k+1} = \tau\, \bar\partial u_k = \sigma \wedge \bar\partial u_k = \sigma \wedge (\bar\partial\sigma)^k.$$

Since $\bar\partial\delta_f\sigma = 0$, then $\delta_f(\sigma \wedge (\bar\partial\sigma)^k) = (\bar\partial\sigma)^k$, and for $u = u_1 + u_2 + \ldots + u_N$, it stands clear that $\nabla u \equiv 1$. One easily verifies that

$$\nabla(u \otimes g) = \nabla u \otimes g - u \otimes \bar\partial g, \tag{8.6}$$

for any $g \in \mathcal{E}(X, L)$, and for $v = u \otimes \varphi$, we conclude that $\nabla v = \varphi$.

Proposition 8.1. *Let E and L be holomorphic vector bundles over X, φ a global holomorphic section of L and f a pointwise nonvanishing global holomorphic section of the dual bundle E^*. Then, there exists a solution to the ∇-equation (8.5).*

In addition to what has been said, the above procedure can be used to point out existence of holomorphic sections even in cases when we allow f to have zeroes. The only difference is that the Koszul complex will not be exact, and therefore a solution to the ∇-equation (8.5) may not exist. But, when such a solution exists, then the existence of holomorphic sections can be granted through solvability of $\bar\partial$-equations.

Proposition 8.2. *Let E and L be holomorphic vector bundles over X, $\varphi \in \mathcal{O}(X, L)$ and $f \in \mathcal{O}(X, E^*)$. If there exists a solution to the ∇-equation (8.5) such that the corresponding $\bar\partial$-equations (8.4) are solvable, then there exists a global holomorphic section of $E \otimes L$, satisfying*

$$\delta_f v = \varphi.$$

More precisely, one solution is given by

$$v = v_1 + \delta_f w_2.$$

9. Macaulay's theorem

Let $F_1, F_2, \ldots, F_m$ be polynomials in $\mathbb{C}^n$. The problem we consider is to find m other polynomials $Q_1, Q_2, \ldots, Q_m$, such that

$$\sum_{j=1}^{m} F_j Q_j = 1. \tag{9.1}$$

An obviously necessary condition is that the initial F_j's lack common zeroes. According to Hilberts Nullstellensatz, this condition is also sufficient. Via the additional assumption of no common zeroes at infinity, one may also recive a good estimate for the degree of the Q_j's.

We homogenize each polynomial F by

$$f := z_0^d F(z/z_0), \tag{9.2}$$

where $d = \deg F$. That the f_j's lack common zeroes then corresponds to the F_j's to lack common zeroes on the compactification $\mathbb{P}^n$ of $\mathbb{C}^n$.

Theorem 9.1 (Macaulay, 1916)**.** *Let $F_1, F_2, \ldots, F_m$ be polynomials on $\mathbb{C}^n$, ordered so that $d_1 \geq d_2 \geq \ldots \geq d_m$, and such that the homogenizations $f_1, f_2, \ldots, f_m$ lack common zeroes in $\mathbb{C}^{n+1} \setminus \{0\}$. Then there exist polynomials $Q_1, Q_2, \ldots, Q_m$ satisfying (9.1) such that* $\deg(F_\alpha Q_\alpha) \leq \sum_{k=1}^{n+1} d_j - n$.

In the previous section we discussed a procedure using the Koszul complex in order to reduce the existence of holomorphic sections solving (8.1) to the solvability of $\bar{\partial}$-equations. Via homogenization (9.2), we can express equation (9.1) as an equation over a direct sum of line bundles of the form $\mathcal{O}(\ell)$. First we need to know something about solvability of $\bar{\partial}$-equations over this bundle.

The solvability of $\bar{\partial}$-equations is closely related to cohomology groups, and when they vanish. Observe that Proposition 5.1 says that the cohomology group $H^q(\mathbb{P}^n, \mathcal{O}(\rho)) = 0$ when $q = 0$ and $\rho < 0$. For $1 \leq q < n$ it always vanishes, and for $q = n$ it vanishes if $\rho \geq -n$. For a more detailed presentation and proof, see Demailly [4], Chapter VII:10. Formulated as solvability of $\bar{\partial}$-equations, thus:

Theorem 9.2. *Suppose that ξ is a $(0, q)$-form with values in $\mathcal{O}(\rho)$ and that $\bar{\partial}\xi = 0$. If $1 \leq q < n$, then there exists a solution to $\bar{\partial}v = \xi$. If $q = n$, then there exists a solution if $\rho \geq -n$.*

Proof of Theorem 9.1. According to Proposition 5.1, each f_j is a global holomorphic section of the line bundle $\mathcal{O}(d_\alpha)$ over $\mathbb{P}^n$. Defining $f := (f_1, f_2, \ldots, f_m)$, this will be a global holomorphic section of the direct sum bundle $E^* := \mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \oplus \ldots \oplus \mathcal{O}(d_m)$. Moreover, this section operates on $E = \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2) \oplus \ldots \oplus \mathcal{O}(-d_m)$, as argued in Proposition 4.2, and is, by assumption, nonvanishing at each point. In order to solve equation (9.1), we homogenize its right-hand side along with the polynomials, and look for existence of holomorphic sections in $\mathcal{O}(\mathbb{P}^n, E \otimes \mathcal{O}(\rho))$ satisfying

$$\delta_f q = z_0^\rho,$$

where z_0^ρ is a section of $\mathcal{O}(\rho)$. According to Proposition 8.1, there is a solution to the ∇-equation (8.5), and, according to Proposition 8.2, there exists a holomorphic solution if the corresponding $\bar{\partial}$-equations (8.4) are solvable.

Each $\bar{\partial}$-equation to solve is of the form $\bar{\partial} w_k = \xi_k$, where ξ_k is a $\bar{\partial}$-closed $(0, k-1)$-form with values in

$$\Lambda^k E \otimes \mathcal{O}(\rho) = \bigoplus_J \mathcal{O}(\rho - d_{j_1} - d_{j_2} - \ldots - d_{j_k}),$$

where the last equality follows from Proposition 4.3. The first $\bar{\partial}$-equation to solve is of the form $\bar{\partial} w_N = \xi_N$, where $N = \min(m, n+1)$. Hence, at worst, we would have to solve $\bar{\partial} w_{n+1} = \xi_{n+1}$. In this case the right-hand side is a $(0, n)$-form with values in

$$\bigoplus_J \mathcal{O}(\rho - d_{j_1} - d_{j_2} - \ldots - d_{j_{n+1}}).$$

Thus, Theorem 9.2 grants solvability if

$$\rho - d_{j_1} - d_{j_2} - \ldots - d_{j_{n+1}} \geq -n,$$

for all j. Therefore, choosing $\rho = d_1 + d_2 + \ldots + d_{n+1} - n$, the inequality is fulfilled for all possible selections of indices. Hence, this first $\bar{\partial}$-equation is solvable. The remaining $\bar{\partial}$-equations to solve are of degree less than n, and therefore solvable without further restraint.

Now, we are granted a global holomorphic section q of $E \otimes \mathcal{O}(\rho) = \mathcal{O}(\rho - d_1) \oplus \mathcal{O}(\rho - d_2) \oplus \ldots \oplus \mathcal{O}(\rho - d_m)$. But, the components of these sections are homogeneous polynomials in $\mathbb{C}^{n+1}$, according to Proposition 5.1. Dehomogenizing by putting $z_0 = 1$, we end up with polynomials Q_α in $\mathbb{C}^n$, such that equation (9.1) is satisfied. Moreover, $\deg(F_\alpha Q_\alpha) \leq \sum_1^{n+1} d_j - n$. □

Remark 1. It is a well-known fact that less then $n+1$ polynomials cannot satisfy the nonzero assumption in Theorem 9.1. One way to see this is to examin its proof. Assuming that they could satisfy the assumption, we see that the first $\bar{\partial}$-equation to be solved would be of degree less than n, thus, solvable regardless of the value of ρ. Thus, for $\rho = 0$, we could find a global holomorphic section q of E satisfying $fq \equiv 1$. But, according to Proposition 5.1, q is the zero section, which clearly contradicts $fq \equiv 1$. □

10. A Macaulay theorem on $\mathbb{P}^n \times \mathbb{P}^\nu$

In the preceding section, we saw how we without common zeroes at infinity of $\mathbb{P}^n$, could retrieved a good bound on the degree of the solutions to (9.1). Naturally, the question arises whether corresponding assumptions at infinity for other complex compactifications of $\mathbb{C}^n$ may result in similar bounds. Proposition 5.2 says that the $\mathcal{O}_z(\ell) \otimes \mathcal{O}_\zeta(\lambda)$ bundle allows a similar connection between sections and polynomials. Later, the existence of holomorphic sections comes down to solvability of $\bar{\partial}$-equations in the line bundle $\mathcal{O}_z(\ell) \otimes \mathcal{O}_\zeta(\lambda)$.

Given a polynomial F on $\mathbb{C}^n \times \mathbb{C}^\nu$, the homogenization

$$f := z_0^d \zeta_0^\delta F(z/z_0, \zeta/\zeta_0), \tag{10.1}$$

where $d = \deg_z F$ and $\delta = \deg_\zeta F$, results in a (d, δ)-homogeneous polynomial f, i.e., a global holomorphic section of $\mathcal{O}_z(d) \otimes \mathcal{O}_\zeta(\delta)$. Again mentioning cohomology, Proposition 5.2 says that $H^q(\mathbb{P}^n \times \mathbb{P}^\nu, \mathcal{O}_z(r) \otimes \mathcal{O}_\zeta(\rho)) = 0$ when $q = 0$ and either $r < 0$ or $\rho < 0$. For arbitrary q, the *Künneth formula* asserts that

$$H^q(\mathbb{P}^n \times \mathbb{P}^\nu, \mathcal{O}_z(r) \otimes \mathcal{O}_\zeta(\rho)) = \bigoplus_{q_1+q_2=q} H^{q_1}(\mathbb{P}^n, \mathcal{O}(r)) \otimes H^{q_2}(\mathbb{P}^\nu, \mathcal{O}(\rho)),$$

and hence vanishes if either $H^{q_1}(\mathbb{P}^n, \mathcal{O}(r))$ or $H^{q_2}(\mathbb{P}^\nu, \mathcal{O}(\rho))$ does, for every partition of q. See Griffiths–Harris [5], page 103–104 (the proof goes through even for r, $\rho \neq 0$). Alternatively, one can show the same thing with explicit integral formulas, see Götmark [6]. In terms of solvability of $\bar{\partial}$-equations, thus:

Theorem 10.1. *Let ξ be a $\bar{\partial}$-closed $(0, q)$-form, $q \geq 1$, with values in $\mathcal{O}_z(r) \otimes \mathcal{O}_\zeta(\rho)$ over $\mathbb{P}^n \times \mathbb{P}^\nu$. There exists a solution to the equation $\bar{\partial} v = \xi$ in the following cases:*

a) $q \neq n$, ν, $n + \nu$
b) $q = n$, and $r \geq -n$ (or $\rho < 0$)
c) $q = \nu$, and $\rho \geq -\nu$ (or $r < 0$)
d) $q = n + \nu$, and $r \geq -n$ or $\rho \geq -\nu$.

Theorem 10.2. *Let $F_1, F_2, \ldots, F_m$ be polynomials on $\mathbb{C}^{n+\nu}$ such that the homogenizations $f_1, f_2, \ldots, f_m$ lack common zeroes in $\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}^{\nu+1} \setminus \{0\}$. Then there exist polynomials $Q_1, Q_2, \ldots, Q_m$ satisfying (9.1), such that*

a) $\deg_z(F_\alpha Q_\alpha) \leq \sum_1^{n+\nu+1} d_{I_k} - n$ *and* $\deg_\zeta(F_\alpha Q_\alpha) \leq \sum_1^{\nu+1} \delta_{J_k} - \nu$.

Also, there exist polynomials satisfying (9.1), such that

b) $\deg_z(F_\alpha Q_\alpha) \leq \sum_1^{n+1} d_{I_k} - n$ *and* $\deg_\zeta(F_\alpha Q_\alpha) \leq \sum_1^{n+\nu+1} \delta_{J_k} - \nu$.

In either case, I and J are orderings of indices such that $d_{I_1} \geq d_{I_2} \geq \ldots \geq d_{I_m}$ and $\delta_{J_1} \geq \delta_{J_2} \geq \ldots \geq \delta_{J_m}$.

Proof. According to what was said above, each f_j is a global holomorphic section of $\mathcal{O}_z(d_j) \otimes \mathcal{O}_\zeta(\delta_j)$. It follows that $f := (f_1, f_2, \ldots, f_m)$ is a section of the direct sum bundle

$$E^* := \mathcal{O}_z(d_1) \otimes \mathcal{O}_\zeta(\delta_1) \oplus \ldots \oplus \mathcal{O}_z(d_m) \otimes \mathcal{O}_\zeta(\delta_m).$$

In view of Section 4, E^* is the dual bundle of

$$E = \mathcal{O}_z(-d_1) \otimes \mathcal{O}_\zeta(-\delta_1) \oplus \ldots \oplus \mathcal{O}_z(-d_m) \otimes \mathcal{O}_\zeta(-\delta_m).$$

Again, we homogenize the equation $\sum F_j Q_j \equiv 1$, and we want to conclude existence of holomorphic sections of $E \otimes \mathcal{O}_z(r) \otimes \mathcal{O}_\zeta(\rho)$ satisfying

$$\delta_f q = z_0^r \zeta_0^\rho, \tag{10.2}$$

where $z_0^r \zeta_0^\rho$ is a holomorphic section of $\mathcal{O}_z(r) \otimes \mathcal{O}_\zeta(\rho)$. By assumption, f is nonvanishing at each point, and hence, according to Proposition 8.1, there is a solution to the ∇-equation (8.5), and, according to Proposition 8.2 such holomorphic section exists if the corresponding $\bar\partial$-equations (8.4) are solvable.

Each of these $\bar\partial$-equations is of the form $\bar\partial w_k = \xi_k$, where ξ_k is a $\bar\partial$-closed $(0, k-1)$-form with values in

$$\Lambda^k E \otimes \mathcal{O}_z(r) \otimes \mathcal{O}_\zeta(\rho) = \bigoplus_{|J|=k} \mathcal{O}_z(r - d_{j_1} - \ldots - d_{j_k}) \otimes \mathcal{O}_\zeta(\rho - \delta_{j_1} - \ldots - \delta_{j_k}).$$

The first $\bar\partial$-equation to be solved is of the form $\bar\partial w_N = \xi_N$, where $N = \min(m, n+\nu+1)$. Given that $m \geq n+\nu+1$, the right-hand side turns out to be a $(0, n+\nu)$-form with values in

$$\bigoplus_{|J|=n+\nu+1} \mathcal{O}_z(r - d_{j_1} - \ldots - d_{j_{n+\nu+1}}) \otimes \mathcal{O}_\zeta(\rho - \delta_{j_1} - \ldots - \delta_{j_{n+\nu+1}}).$$

Thus, Theorem 10.1 grants solvability if either

$$r - d_{j_1} - d_{j_2} - \ldots - d_{j_{n+\nu+1}} \geq -n$$

or

$$\rho - \delta_{j_1} - \delta_{j_2} - \ldots - \delta_{j_{n+\nu+1}} \geq -\nu,$$

for each selection of indices J. Choosing either

$$r \geq d_{I_1} + d_{I_2} + \ldots + d_{I_{n+\nu+1}} - n$$

or

$$\rho \geq \delta_{J_1} + \delta_{J_2} + \ldots + \delta_{J_{n+\nu+1}} - \nu,$$

for orderings of indices such that $d_{I_1} \geq d_{I_2} \geq \ldots \geq d_{I_m}$ and $\delta_{J_1} \geq \delta_{J_2} \geq \ldots \geq \delta_{J_m}$, one of the inequalities will be fulfilled. In such case, this first $\bar\partial$-equation will be solvable.

The other $\bar\partial$-equations are solvable without further restraints, except for the cases when the right-hand side at hand is of bidegree $(0, n)$ or $(0, \nu)$. In these cases it takes its values in

$$\bigoplus_{|J|=n+1} \mathcal{O}_z(r - d_{j_1} - \ldots - d_{j_{n+1}}) \otimes \mathcal{O}_\zeta(\rho - \delta_{j_1} - \ldots - \delta_{j_{n+1}})$$

or

$$\bigoplus_{|J|=\nu+1} \mathcal{O}_z(r - d_{j_1} - \ldots - d_{j_{\nu+1}}) \otimes \mathcal{O}_\zeta(\rho - \delta_{j_1} - \ldots - \delta_{j_{\nu+1}}),$$

respectively, and, according to Theorem 10.1, we have solvability in these cases if

$$r \geq d_{I_1} + d_{I_2} + \ldots + d_{I_{n+1}} - n$$

and

$$\rho \geq \delta_{J_1} + \delta_{J_2} + \ldots + \delta_{J_{\nu+1}} - \nu.$$

Summarizing, we are granted the existence of a global holomorphic section of $E \otimes \mathcal{O}_z(r) \otimes \mathcal{O}_\zeta(\rho) = \mathcal{O}_z(r - d_1) \otimes \mathcal{O}_\zeta(\rho - \delta_1) \oplus \ldots \oplus \mathcal{O}_z(r - d_m) \otimes \mathcal{O}_\zeta(\rho - \delta_m)$ that solves (10.2). But, the components of such section are bihomogeneous polynomials on $\mathbb{C}^{n+1} \times \mathbb{C}^{\nu+1}$. Dehomogenizing by putting $z_0 = \zeta_0 = 1$, we end up with polynomials Q_α on $\mathbb{C}^n \times \mathbb{C}^\nu$, satisfying equation (9.1). Moreover, since it suffices to choose either

a) $r \geq \sum_1^{n+\nu+1} d_{I_k} - n$ and $\rho \geq \sum_1^{\nu+1} \delta_{J_k} - \nu$,
b) $r \geq \sum_1^{n+1} d_{I_k} - n$ and $\rho \geq \sum_1^{n+\nu+1} \delta_{J_k} - \nu$,

the degree estimates follow. □

Remark 2. One should note that the proof reveals something more than what has been stated. The critical cases for solvability of the $\bar{\partial}$-equations are when the $(0, q)$-forms are of degree n, ν and $n+\nu$, and take their values in

$$\mathcal{O}_z(r - d_{j_1} - \ldots - d_{j_{q+1}}) \otimes \mathcal{O}_\zeta(\rho - \delta_{j_1} - \ldots - \delta_{j_{q+1}}).$$

At level $n + \nu$ it is sufficient to choose either

$$r \geq \sum_1^{n+\nu+1} d_{I_k} - n \text{ or } \rho \geq \sum_1^{n+\nu+1} \delta_{J_k} - \nu.$$

But incidently, it may occur that when high values on r are required, only low values of ρ are, and vice versa. Hence, it may occasionally not be neccesary for neither r nor ρ to satisfy the sufficient criteria, but rather be chosen so that either

$$r \geq \sum_1^{n+\nu+1} d_{j_k} - n \text{ or } \rho \geq \sum_1^{n+\nu+1} \delta_{j_k}$$

holds for each choice of $n + \nu + 1$ indices. □

Remark 3. It follows from the proof of Theorem 10.2, that for less than $n + \nu$ polynomials satisfying the assumption of the theorem, one can actually find polynomials satisfying (9.1), such that $\deg_z(F_\alpha Q_\alpha) \leq \sum_1^{n+1} d_{I_k} - n$ and $\deg_\zeta(F_\alpha Q_\alpha) \leq \sum_1^{\nu+1} \delta_{J_k} - \nu$. Though, as we shall see in Theorem 11.6, in such a case Macaulay's theorem can be applied on less of them, and will give a sharper bound. By the same argument used in Remark 1, the least number of polynomials satisfying the assumption of Theorem 10.2 is $\max(n, \nu) + 1$. □

11. Comparing the Macaulay theorems

Given polynomials $F_1, F_2, \dots, F_m$ on $\mathbb{C}^n$, we will in this section discuss the contribution made by a theorem of Macaulay type for the compactification $\mathbb{C}^{n+\nu} \hookrightarrow \mathbb{P}^n \times \mathbb{P}^\nu$. To this end, we will use the notation $d_j^* = \deg F_j$, $d_j = \deg_z F_j$ and $\delta_j = \deg_\zeta F_j$. Especially, we have the inequalities

$$d_j, \delta_j \leq d_j^* \leq d_j + \delta_j. \tag{11.1}$$

We will say that a variable z_j is of *polynomial degree* in a polynomial F_α if the coefficient corresponding to the term $z_j^{d_\alpha}$ of the polynomial F_α is nonzero.

We begin with two immediate observations:

Lemma 11.1. *For given homogeneous polynomials to lack common zeroes, each variable has to be of polynomial degree for some of the polynomials.*

Proof. If z_k is not of polynomial degree for any of the given polynomials, then $z_k = 1$, $z_j = 0$, $j \neq k$ is a common zero. □

Lemma 11.2. *If the given polynomials satisfy the assumption of Macaulay's theorem, then no sharper bound can be achieved by further addition of polynomials. The same is true for Theorem 10.2.*

11.1. **Cases when Macaulay's theorem is better.** In many cases, Macaulay's theorem gives a sharper bound than what can be achieved by a separated compactification $\mathbb{C}^{n+\nu} \hookrightarrow \mathbb{P}^n \times \mathbb{P}^\nu$. In view of inequality (11.1), it especially may happen if the separation of variables is made in a way so that high powers of z and ζ are included in different terms of the same polynomial.

Example 1. For the following polynomials on $\mathbb{C}^2$, the assumptions of Macaulay's theorem and Theorem 10.2 are both fullfiled:

$$z^d, \qquad z^d + \zeta^d, \qquad 1 + z\zeta^{d-1}.$$

Macaulay gives the bound $3d-2$ while Theorem 10.2 gives the bibounds $(2d, 2d-2)$ and $(2d-1, 2d-2)$. □

In special cases, one can draw the conclusion that Theorem 10.2 cannot generate a sharper bound. We shall state some cases when such conclusion can be made, by just comparing degrees. To this end, for the polynomials in question, we assume that the hypothesis of Macaulay's theorem is met.

Proposition 11.3. *Given three polynomials on $\mathbb{C}^2$, then Theorem 10.2 cannot give a sharper bound.*

Proof. Let F_1,F_2 and F_3 be polynomials on $\mathbb{C}_z \times \mathbb{C}_\zeta$. According to Lemma 11.1, both z and ζ must be of polynomial degree in some polynomial. Assume that z is so in, say, F_1, i.e., $d_1^* = d_1$. Then

$$\begin{aligned} d_1^* + d_2^* + d_3^* \leq & d_1 + (d_2 + \delta_2) + (d_3 + \delta_3) \\ & \leq d_1 + d_2 + d_3 + [\delta_1 + \delta_2 + \delta_3 - \min(\delta_j)] \,, \end{aligned}$$

and, since ζ is of polynomial degree for some f_j, the same argument yields

$$d_1^* + d_2^* + d_3^* \leq [d_1 + d_2 + d_3 - \min(d_j)] + \delta_1 + \delta_2 + \delta_3.$$

Hence, Macaulay's bound is sharper. □

The condition of only three polynomials is of importance. Adding further polynomials will not achieve a sharper bound, as stated in Lemma 11.2. But, as we shall see in Example 2 below, there are examples of more than three polynomials on $\mathbb{C}^2$ that lack common zeroes even at infinity, but not when any single one of them is removed. In such case, Theorem 10.2 can in fact yield a sharper bound. But, with an extra condition, included in the following generalization, the result holds even for more than three polynomials.

Proposition 11.4. *Given m polynomials on $\mathbb{C}^{n+\nu}$, ordered so that $d_1^* \geq d_2^* \geq \ldots \geq d_m^*$, and such that n, as well as the remaining ν, variables are of polynomial degree in different polynomials among the first $n+\nu+1$ ones, then Theorem 10.2, for this separation of variables, cannot give a sharper bound.*

Proof. The idea of the previous proof applies here as well. Necessarily, each $z_1, z_2, \ldots, z_n$ is of polynomial degree in, say, $F_1, F_2, \ldots, F_n$, i.e., $d_j^* = d_j$ for $j = 1, 2, \ldots, n$. Then

$$\sum_{j=1}^{n} d_j^* + \sum_{j=n+1}^{n+\nu+1} d_j^* \leq \sum_{j=1}^{n} d_j + \sum_{j=n+1}^{n+\nu+1} (d_j + \delta_j) \leq \sum_{k=1}^{n+\nu+1} d_{I_k} + \sum_{k=1}^{\nu+1} \delta_{J_k},$$

and, since each ζ_j is of polynomial degree for some, mutually different, F_j among the first $n + \nu + 1$, the same argument yields

$$\sum_{j=1}^{\nu} d_j^* + \sum_{j=\nu+1}^{n+\nu+1} d_j^* \leq \sum_{k=1}^{n+1} d_{I_k} + \sum_{k=1}^{n+\nu+1} \delta_{J_k},$$

where I and J are orderings such that $d_{I_1} \geq d_{I_2} \geq \ldots \geq d_{I_m}$ and $\delta_{J_1} \geq \delta_{J_2} \geq \ldots \geq \delta_{J_m}$. Hence, Macaulay's bound is sharper. □

Remark 4. Note that even makeing allowance for what was pointed out in Remark 2, Theorem 10.2 cannot yield a sharper bound. This is due to the focus on the $n + \nu + 1$ polynomials of highest total degree, and among them we have no freedom of choice to avoid counting the corresponding degrees in z or ζ. More precisely, we are led to solve the $\bar{\partial}$-equation of degree $n+\nu$ with values in

$$\mathcal{O}_z(r - d_1 - \ldots - d_{n+\nu+1}) \otimes \mathcal{O}_\zeta(\rho - \delta_1 - \ldots - \delta_{n+\nu+1}).$$

This imposes a necessary condition on either r or ρ. Assume it to be on r, i.e., $r \geq \sum_1^{n+\nu+1} d_j - n$. Also, the $\bar{\partial}$-equation of degree ν with values in

$$\mathcal{O}_z(r - d_{n+1} - \ldots - d_{n+\nu+1}) \otimes \mathcal{O}_\zeta(\rho - \delta_{n+1} - \ldots - \delta_{n+\nu+1}),$$

has to be solved. But

$$r - \sum_{n+1}^{n+\nu+1} d_j \geq \sum_1^n d_j - n \geq 0,$$

since $d_j > 0$ for $j = 1, 2, \ldots, n$. Hence, we need $\rho \geq \sum_{n+1}^{n+\nu+1} \delta_j - \nu$. The inequalities of the proof follow all the same. □

Corollary 11.5. *Given m polynomials on $\mathbb{C}^n$, ordered so that $d_1^* \geq d_2^* \geq \ldots \geq d_m^*$, and such that each variable is of polynomial degree in different polynomials among the first $n+1$, then Theorem 10.2 cannot give a sharper bound.*

Proof. For each partition of the n variables, apply Proposition 11.4. □

11.2. **Cases when Macaulay's theorem is not better.** There are cases when Theorem 10.2 do yield a sharper bound than Macaulay's theorem. As mentioned above, one way to construct such examples is, for $n+2$ points in $\mathbb{C}^n$, to find $n+2$ polynomials that zeroes on all points but one each, and that still meet the assumptions of Theorem 10.2 and Macaulay. Note that they will have a common zero if any polynomial is removed.

Example 2. We choose the four points $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ on $\mathbb{C}^2$, and the polynomials

$$z^d\zeta - z^d - \zeta + 1, \qquad \zeta^{k+1} - z^k\zeta, \qquad z^{d+1} + z^d\zeta, \qquad z^d\zeta,$$

that each zero on all points but the corresponding one, i.e., no one can be excluded. The assumptions of both Theorem 10.2 and Macaulay hold, and the respective bounds are $(2d, k+2)$ and $3d+1$. Hence, a sharper bound is achived by Theorem 10.2 as soon as $k < d-1$. □

An additional remark should be made. Another aspect in which Theorem 10.2 is better, is that it does give the more specific bibound (r, ρ), i.e., the bound r on z and ρ on ζ. Moreover, not necessarily is any factor of the polynomials of degree r in z and ρ in ζ. A priori, this changes nothing about the conclusion we can draw of the total bound.

11.3. **Cases when Macaulay's theorem cannot be applied.** A main contribution of Theorem 10.2 is in cases when Macaulay's theorem cannot be applied. Since different compactifications of $\mathbb{C}^n$ separates the infinity in different ways, such cases are possible. As mentioned in Lemma 11.1, for Macaulay's theorem to be applicable, a necessary contition is for each variable to exist of polynomial degree. By the same lemma, for Theorem 10.2 to be applicable, it is merely necessary that each variable exists of polynomial degree relative the ones in its separation, i.e., ζ is of polynomial degree in $\zeta + z_1 z_2$, considering z as constant.

Example 3. We consider the following polynomials on $\mathbb{C}^4$:

$$z_1^d, \qquad z_1^2 + \zeta, \qquad z_1 + z_2, \qquad z_3^3 + 1, \qquad \zeta + z_2^2 + z_3^2.$$

Since ζ is not of polynomial degree in any polynomial, Macaulay's theorem cannot be applied. Though, separating ζ, it is of polynomial degree, considering z as constant. Theorem 10.2 may, and does, still apply. □

In this manner, one can construct further examples of polynomials to which Macaulay's theorem cannot be applied.

Example 4. Let $F_1, F_2, \dots, F_{n+1}$ on $\mathbb{C}^n$ and $G_1, G_2, \dots, G_{\nu+1}$ on $\mathbb{C}^\nu$ be two sets of nonconstant polynomials, both whose homogenizations lack nontrivial common zeroes, respectively. Then will the separated homogenization of the $n + \nu + 1$ polynomials

$$F_1(z), \quad \dots, \quad F_n(z), \quad F_{n+1}(z)G_1(\zeta), \quad \dots, \quad F_{n+1}(z)G_{\nu+1}(\zeta),$$

on $\mathbb{C}^{n+\nu}$ lack nontrivial common zeroes. That, since if at some point each $F_1, \dots, F_n$ zeroes, then F_{n+1} will not, as well as some G_j. Hence, Theorem 10.2 can be applied, but Macaulay's theorem cannot, since no ζ_j is of polynomial degree. □

Next we shall justify what was posed in Remark 3, following the proof of Theorem 10.2.

Theorem 11.6. *Let $m \leq n + \nu$, and suppose that there are m nonconstant polynomials in $\mathbb{C}^n_z \times \mathbb{C}^\nu_\zeta$ such that the assumption of Theorem 10.2 holds. Then, either $n+1$ of them are independent of ζ, or $\nu + 1$ of them are independent of z. In either case, they meet the assumption of Macaulay's theorem, and the bound is sharper than that of Theorem 10.2 on all m polynomials.*

Before the proof, without any details, we introduce some notation from algebraic geometry.We have projections

$$\pi_1 : \mathbb{P}^n \times \mathbb{P}^\nu \to \mathbb{P}^n \text{ and } \pi_2 : \mathbb{P}^n \times \mathbb{P}^\nu \to \mathbb{P}^\nu.$$

A hypersurface $d \in H^2(\mathbb{P}^n) = \mathbb{Z}$ can be denoted by $d \cdot e$, for some generator e and $d \in \mathbb{Z}$. Via the pull-backs $E = \pi_1^* e_1$ and $F = \pi_2^* e_2$, a hypersurface $H \in H^2(\mathbb{P}^n \times \mathbb{P}^\nu)$ is homologous to $d_1E + d_2F$. Since E and F are of codimension 1, $\dim E^\alpha \geq n - \alpha$ and $\dim F^\alpha \geq \nu - \alpha$. Given a (d, δ)-homogeneous polynomial f, the zero set $f(z, \zeta) = 0$ is a hypersurface of $\mathbb{P}^n \times \mathbb{P}^\nu$. Thus, we can denote it by $dE + \delta F$. For a further survey on these concepts, see any introductory book on algebraic geometry, e.g., [5].

Proof. Repeat any polynomial until $m = n + \nu$. Let $f_1, f_2, \dots, f_{n+\nu}$ be the (d_j, δ_j)-homogenizations on $\mathbb{C}^{n+1} \times \mathbb{C}^{\nu+1}$, and let $d_jE + \delta_jF$ denote their zero sets. By means of algebraic geometry, their common zero set is the intersection

$$\begin{aligned}\prod_{j=1}^{n+\nu}(d_jE + \delta_jF) &= \sum_{k=0}^{n+\nu}\left[\sum_{|J|=n+\nu} d_{J_1}d_{J_2}\dots d_{J_k}\delta_{J_{k+1}}\dots\delta_{J_{n+\nu}}\right]E^kF^{n+\nu-k} \\ &= \left[\sum_{|J|=n+\nu} d_{J_1}d_{J_2}\dots d_{J_n}\delta_{J_{n+1}}\delta_{J_{n+2}}\dots\delta_{J_{n+\nu}}\right]E^nF^\nu.\end{aligned}$$

Since

$$\dim E^nF^\nu \geq (1, 1),$$

a necessary condition for the common zeroes assumption of Theorem 10.2 to hold is that

$$\sum_{|J|=n+\nu} d_{J_1}d_{J_2}\dots d_{J_n}\delta_{J_{n+1}}\delta_{J_{n+2}}\dots\delta_{J_{n+\nu}} = 0. \tag{11.2}$$

Suppose that condition (11.2) holds. Assume that for n of the polynomials f_j, say $f_1, f_2, \dots, f_n$, we have $d_j \neq 0$. Condition (11.2) asserts that $\delta_j = 0$, for at least one of $f_{n+1}, f_{n+2}, \dots, f_{n+\nu}$. Replace and rename this polynomial with the first among $f_1, f_2, \dots, f_n$ such that $\delta_j \neq 0$. Repeat this step until we have an ordering of the polynomials such that $d_j \neq 0$, $\delta_j = 0$ for $f_1, f_2, \dots, f_n$. For condition (11.2) to hold, δ_j has to equal zero for at least one of the remaining polynomials, and we are left with $n + 1$ polynomials on $\mathbb{C}^{n+1}$.

If there were no n polynomials such that $d_j \neq 0$, then the non-constant condition implies that there are $\nu + 1$ polynomials satisfying $d_j = 0$, $\delta_j \neq 0$, i.e., $\nu + 1$ polynomials on $\mathbb{C}^{\nu+1}$.

Suppose that these $n + 1$ polynomials on $\mathbb{C}^{n+1}$ do not satisfy the common zeroes assumption of Macaulay's theorem, i.e., they have a common zero $\hat{z} \in \mathbb{C}^{n+1}\setminus\{0\}$. Then, according to Remark 1 following the proof of Macaulay's theorem, the $\nu - 1$ remaining polynomials on $\mathbb{C}^{n+1} \times \mathbb{C}^{\nu+1}$ must have a common zero on $\{\hat{z}\} \times \mathbb{C}^{\nu+1}\setminus\{0\}$. But this contradicts that the common zeroes assumption of Theorem 10.2 was met.

Note that the bound generated by Macaulay on these $n + 1$ polynomials will be counted for with Theorem 10.2 applied on the original $n + \nu$ polynomials. ∎

12. Max Nöther theorems with an additional bound of Macaulay type

Given polynomials $F_1, F_2, \dots, F_m$ in $\mathbb{C}^n$, in previous sections we have seen how we, via some homogenization, could transfer the problem of finding polynomials $Q_1, Q_2, \dots, Q_m$ such that

$$\sum_j F_j Q_j = \Phi, \tag{12.1}$$

to the question of existence of holomorphic sections. In order for the Koszul complex method to be fully operationable, we had the pointwise nonvanishing-condition on the homogenization $f = (f_1, f_2, \dots, f_m)$.

If we were to allow f to be zero at certain points, the ideal (F) will not be trivial, and we have the additional condition that Φ has to belong to (F). Moreover, Proposition 8.1 does not assure a solution to the ∇-equation. In this case, the discussed method still reduces existence of holomorphic sections to solvability of $\bar{\partial}$-equations, provided that it is possible to solve the ∇-equation.

12.1. **Compactifying $\mathbb{C}^n$ by $\mathbb{P}^n$.** Via homogenization (9.2), f is a global holomorphic section of $E^* = \mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \oplus \dots \oplus \mathcal{O}(d_m)$, operating on $E = \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2) \oplus \dots \oplus \mathcal{O}(-d_m)$. Hence, equation (12.1) is transformed to

$$\delta_f q = z_0^\rho \varphi, \tag{12.2}$$

over $\mathcal{O}(\mathbb{P}^n, E \otimes \mathcal{O}(\rho + \deg \Phi))$. The solvability of the arising $\bar{\partial}$-equations is treated in Theorem 9.2. The additional difficulty turns out to be to find a solution to $\nabla v = z_0^\rho \varphi$. For this purpose, we have the following identity:

Lemma 12.1. *Let E and F be two vector bundles over X, $v \in \mathcal{E}(X, F)$, $u \in \mathcal{E}_{0,q}(X, \Lambda^k E \otimes L)$ and $f \in \mathcal{O}(X, E^*)$. If $q + k$ is odd, then*

$$\nabla(u \wedge v) = \nabla u \wedge v - u \wedge \nabla v.$$

Especially, the equality holds for u, $v \in \bigoplus_{k=1}^N \mathcal{E}_{0,k-1}(X, \Lambda^k E \otimes L)$.

Proof. From definition of the δ_f and $\bar{\partial}$-operator, we have

$$\begin{aligned}\nabla(u \wedge v) =& (\delta_f - \bar{\partial})(u \wedge v) = \delta_f(u \wedge v) - \bar{\partial}(u \wedge v) \\ =& \delta_f u \wedge v + (-1)^{q+k} u \wedge \delta_f v - (\bar{\partial} u \wedge v + (-1)^{q+k} u \wedge \bar{\partial} v) \\ =& (\delta_f u - \bar{\partial} u) \wedge v - u \wedge (\delta_f v - \bar{\partial} v) = \nabla u \wedge v - u \wedge \nabla v.\end{aligned}$$

The result follows by linearity. □

We shall present a result, due to Max Nöther, see [12] or [5], allowing certain common zeroes. The classical theorem is stated for n polynomials on $\mathbb{C}^n$. But we shall see that our method disregards this restriction, in a natural way. Moreover, if we strenghten the assumption to no zeroes at all, then $1 \in (F)$ and Macaulay's theorem falls out as a special case.

Theorem 12.2. *Let $F_1, F_2, \dots, F_m$ be polynomials on $\mathbb{C}^n$ such that the homogenizations $f_1, f_2, \dots, f_m$ lack nontrivial common zeroes for $z_0 = 0$, and let Φ be a polynomial in the ideal (F). Then there exist polynomials $Q_1, Q_2, \dots, Q_m$ satisfying (12.1) such that*

$$\deg F_\alpha Q_\alpha \leq \max(\deg \Phi, \sum_{j=1}^{n+1} d_j - n)$$

if $m \geq n+1$, and $\deg F_\alpha Q_\alpha \leq \deg \Phi$ otherwise.

Remark 5. Before proving this theorem, we note that the zero set $\mathcal{Z}$ of f is closed, and by assumption, does not intersect the infinity. Hence, $\mathcal{Z}$ has to be compact in $\mathbb{C}^n$. □

Proof. Since certain common zeroes of the F_j's are allowed, f may vanish at certain points. Hence, we have to establish solvability of the ∇-equation (8.5).

Denote the zero set of f by $\mathcal{Z}$. We do know that $\nabla u \equiv 1$ is solvable over a subbundle of E, outside $\mathcal{Z}$, thus, makes $v = u \otimes z_0^\rho \varphi$ a solution outside $\mathcal{Z}$.

Since $\Phi \in (F)$ and the zero set of F is compact, we can also find a local solution to $\nabla \psi = z_0^\rho \varphi$ in a neigbourhood of $\mathcal{Z}$. We introduce a cutoff function χ with support in this neighbourhood, and such that $\chi \equiv 1$ in a smaller neighbourhood of $\mathcal{Z}$. We combine these solutions as $\chi\psi + (1-\chi)v$, defined in all of $\mathbb{P}^n$. Applying ∇, we have

$$\begin{aligned}\nabla(\chi\psi + (1-\chi)v) =& \chi z_0^\rho \varphi - \bar\partial\chi \wedge \psi + (1-\chi) z_0^\rho \varphi - \bar\partial(1-\chi) \wedge v \\ =& z_0^\rho \varphi + \bar\partial\chi \wedge (v - \psi).\end{aligned}$$

By Lemma 12.1,

$$\nabla(u \wedge (v-\psi)) = \nabla u \wedge (v - \psi) - u \wedge \nabla(v-\psi) = v - \psi,$$

where defined, and since $\nabla \bar\partial \chi = 0$, the same lemma gives

$$\nabla(\bar\partial\chi \wedge u \wedge (v-\psi)) = -\bar\partial\chi \wedge \nabla(u \wedge (v-\psi)) = -\bar\partial\chi \wedge (v-\psi),$$

for the globally defined $\bar\partial\chi \wedge u \wedge (v-\psi)$. Hence,

$$\chi\psi + (1-\chi)v + \bar\partial\chi \wedge u \wedge (v-\psi)$$

is a global solution to (8.5).

Proposition 8.2 assures the existence of holomorphic sections satisfying (12.2) if the $\bar\partial$-equations, corresponding to the ∇-solution, are solvable. As in the proof of the Macaulay theorem, for $m \geq n+1$, Theorem 9.2 grants a solution if

$$\rho + \deg \Phi - \sum_{j=1}^{n+1} d_j \geq -n.$$

Choosing

$$\rho = \min\{k \in \mathbb{N} | \, k + \deg \Phi \geq \sum_{j=1}^{n+1} d_j - n\},$$

there exists, accordingly, a global holomorphic section of

$$E \otimes \mathcal{O}(\rho + \deg \Phi) = \mathcal{O}(\rho + \deg \Phi - d_1) \oplus \ldots \oplus \mathcal{O}(\rho + \deg \Phi - d_m),$$

i.e., homogeneous polynomials in $\mathbb{C}^{n+1}$. Dehomogenizing, we are left with polynomials $Q_1, Q_2, \ldots, Q_m$, such that

$$\deg F_\alpha Q_\beta \leq \rho + \deg \Phi.$$

If $\deg \Phi \geq \sum_{j=1}^{n+1} d_j - n$, then $\rho = 0$ and $\deg \Phi$ is the bound. Otherwise, the bound is $\sum_{j=1}^{n+1} d_j - n$. If $m \leq n$, then we can choose $\rho = 0$, and the bound is $\deg \Phi$. □

12.2. **Compactifying $\mathbb{C}^{n+\nu}$ by $\mathbb{P}^n \times \mathbb{P}^\nu$.** Via homogenization (10.1), f is a global holomorphic section of

$$E^* = \mathcal{O}_z(d_1) \otimes \mathcal{O}_\zeta(\delta_1) \oplus \ldots \oplus \mathcal{O}_z(d_m) \otimes \mathcal{O}_\zeta(\delta_m),$$

operating on

$$E = \mathcal{O}_z(-d_1) \otimes \mathcal{O}_\zeta(-\delta_1) \oplus \ldots \oplus \mathcal{O}_z(-d_m) \otimes \mathcal{O}_\zeta(-\delta_m).$$

Hence, equation (12.1) is transformed to

$$\delta_f q = z_0^r \zeta_0^\rho \varphi, \tag{12.3}$$

over $\mathcal{O}(\mathbb{P}^n \times \mathbb{P}^\nu, E \otimes \mathcal{O}_z(r + \deg_z \Phi) \otimes \mathcal{O}_\zeta(\rho + \deg_\zeta \Phi))$. For this setting, we can solve $\nabla v = z_0^r \zeta_0^\rho \varphi$ in a similar way as above. The solvability of the arising $\bar{\partial}$-equations is treated in Theorem 10.1.

Example 5. Consider the following polynomials in $\mathbb{C}^2$,

$$z + \zeta^2, \qquad\qquad z^2\zeta^2 + z^2.$$

Neither contains a constant term, i.e., z_0 cannot be of polynomial degree. Homogenizing through (9.2) gives

$$z_0 z + \zeta^2, \qquad\qquad z^2\zeta^2 + z_0^2 z^2,$$

and we see that $(0, 1, 0)$ is a common zero. Hence, Theorem 12.2 cannot be applied. Though, homogenized through (10.1), they become

$$z\zeta_0^2 + z_0\zeta^2, \qquad\qquad z^2\zeta^2 + z^2\zeta_0^2.$$

And we see that for $z_0 = 0$ or $\zeta_0 = 0$, there are no common zeroes. Hence, a result on $\mathbb{P}^n \times \mathbb{P}^\nu$ in lines with the generalization of Max Nöther's, would be applicable in cases when the prior one is not. □

Theorem 12.3. *Let $F_1, F_2, \ldots, F_m$ be polynomials on $\mathbb{C}^n_z \times \mathbb{C}^\nu_\zeta$ such that the homogenizations $f_1, f_2, \ldots, f_m$ lack nontrivial common zeroes for $z_0 = 0$ and for $\zeta_0 = 0$, and let Φ be a polynomial in the ideal (F). Then there exist polynomials $Q_1, Q_2, \ldots, Q_m$ satisfying (12.1) such that*

a) $\deg_z(F_\alpha Q_\alpha) \le \max(\deg_z \Phi, \sum_1^{n+\nu+1} d_{I_k} - n)$ *and*
$\deg_\zeta(F_\alpha Q_\alpha) \le \max(\deg_\zeta \Phi, \sum_1^{\nu+1} \delta_{J_k} - \nu)$.

Also, there exist polynomials satisfying (12.1) such that

b) $\deg_z(F_\alpha Q_\alpha) \le \max(\deg_z \Phi, \sum_1^{n+1} d_{I_k} - n)$ *and*
$\deg_\zeta(F_\alpha Q_\alpha) \le \max(\deg_\zeta \Phi, \sum_1^{n+\nu+1} \delta_{J_k} - \nu)$,

or if $m \le n + \nu$,

c) $\deg_z(F_\alpha Q_\alpha) \le \max(\deg_z \Phi, \sum_1^{n+1} d_{I_k} - n)$ *and*
$\deg_\zeta(F_\alpha Q_\alpha) \le \max(\deg_\zeta \Phi, \sum_1^{\nu+1} \delta_{J_k} - \nu)$.

In either case, I and J are orderings of indices such that $d_{I_1} \ge d_{I_2} \ge \ldots \ge d_{I_m}$ and $\delta_{J_1} \ge \delta_{J_2} \ge \ldots \ge \delta_{J_m}$.

Proof. Since certain common zeroes of the F_j's are allowed, f may vanish at certain points. We have to establish solvability of the ∇-equation (8.5) before we can apply our method.

Denote the zero set of f by $\mathcal{Z}$. We do know that $\nabla u \equiv 1$ is solvable over a subbundle of E, outside $\mathcal{Z}$, thus, makes $v = u \otimes z_0^r \zeta_0^\rho \varphi$ a solution outside $\mathcal{Z}$.

Since $\Phi \in (F)$ and the zero set of F is compact, we can also find a local solution to $\nabla \psi = z_0^r \zeta_0^\rho \varphi$ in a neigbourhood of $\mathcal{Z}$. We introduce a cutoff function χ with support in this neighbourhood, and such that $\chi \equiv 1$ in a smaller neighbourhood of $\mathcal{Z}$. We combine these solutions as $\chi\psi + (1-\chi)v$, defined on all of $\mathbb{P}^n \times \mathbb{P}^\nu$. Applying ∇, we have

$$\begin{aligned}\nabla(\chi\psi + (1-\chi)v) =& \chi z_0^r \zeta_0^\rho \varphi - \bar\partial\chi \wedge \psi + (1-\chi) z_0^r \zeta_0^\rho \varphi - \bar\partial(1-\chi) \wedge v \\ =& z_0^r \zeta_0^\rho \varphi + \bar\partial\chi \wedge (v - \psi).\end{aligned}$$

By Lemma 12.1,

$$\nabla(u \wedge (v-\psi)) = \nabla u \wedge (v-\psi) - u \wedge \nabla(v-\psi) = v - \psi,$$

where defined, and since $\nabla\bar\partial\chi = 0$, the same lemma gives

$$\nabla(\bar\partial\chi \wedge u \wedge (v-\psi)) = -\bar\partial\chi \wedge \nabla(u \wedge (v-\psi)) = -\bar\partial\chi \wedge (v-\psi),$$

for the globally defined $\bar\partial\chi \wedge u \wedge (v-\psi)$. Hence,

$$\chi\psi + (1-\chi)v + \bar\partial\chi \wedge u \wedge (v-\psi)$$

is a global solution to the ∇-equation (8.5).

Proposition 8.2 assures existence of holomorphic sections satisfying (12.3) if the $\bar\partial$-equations, corresponding to the ∇-solution, are solvable. As in the proof of Theorem 10.2, for $m \ge n+\nu+1$, Theorem 10.1 grants a solution if either

$$r + \deg_z \Phi - \sum_{k=1}^{n+\nu+1} d_{I_k} \geq -n$$

and

$$\rho + \deg_\zeta \Phi - \sum_{k=1}^{\nu+1} \delta_{J_k} \geq -\nu,$$

or

$$r + \deg_z \Phi - \sum_{k=1}^{n+1} d_{I_k} \geq -n$$

and

$$\rho + \deg_\zeta \Phi - \sum_{k=1}^{n+\nu+1} \delta_{J_k} \geq -\nu.$$

Choosing r and ρ as small as possible, in either case, then there exists, accordingly, a global holomorphic section of

$$\begin{aligned} E \otimes \mathcal{O}_z(r + \deg_z \Phi) \otimes \mathcal{O}_\zeta(\rho + \deg_\zeta \Phi) \\ = \mathcal{O}_z(r + \deg_z \Phi - d_1) \otimes \mathcal{O}_\zeta(\rho + \deg_\zeta \Phi - \delta_1) \oplus \dots \\ \oplus \mathcal{O}_z(r + \deg_z \Phi - d_m) \otimes \mathcal{O}_\zeta(\rho + \deg_\zeta \Phi - \delta_m), \end{aligned}$$

i.e., bihomogeneous polynomials in $\mathbb{C}^{n+\nu+1}$. Dehomogenizing, we are left with polynomials $Q_1, Q_2, \dots, Q_m$, such that either

a) $\deg_z(F_\alpha Q_\alpha) \leq \max(\deg_z \Phi, \sum_1^{n+\nu+1} d_{I_k} - n)$ and $\deg_\zeta(F_\alpha Q_\alpha) \leq \max(\deg_\zeta \Phi, \sum_1^{\nu+1} \delta_{J_k} - \nu)$,

b) $\deg_z(F_\alpha Q_\alpha) \leq \max(\deg_z \Phi, \sum_1^{n+1} d_{I_k} - n)$ and $\deg_\zeta(F_\alpha Q_\alpha) \leq \max(\deg_\zeta \Phi, \sum_1^{n+\nu+1} \delta_{J_k} - \nu)$,

or if $m \leq n + \nu$,

c) $\deg_z(F_\alpha Q_\alpha) \leq \max(\deg_z \Phi, \sum_1^{n+1} d_{I_k} - n)$ and $\deg_\zeta(F_\alpha Q_\alpha) \leq \max(\deg_\zeta \Phi, \sum_1^{\nu+1} \delta_{J_k} - \nu)$,

where I and J orderings of indices such that $d_{I_1} \geq d_{I_2} \geq \dots \geq d_{I_m}$ and $\delta_{J_1} \geq \delta_{J_2} \geq \dots \geq \delta_{J_m}$. □

Remark 6. As for the proof of Theorem 10.2, note that the proof reveals something more than what has been stated. At the solution of the $\bar{\partial}$-equations, it may occasionally be sufficient with r and ρ such that

$$r + \deg_z \Phi \geq \sum_{k=1}^{n+\nu+1} d_{j_k} - n \;\text{ or }\; \rho + \deg_\zeta \Phi \geq \sum_{k=1}^{n+\nu+1} \delta_{j_k} - \nu$$

holds, for each set of $n + \nu + 1$ indices, but neither

$$r + \deg_z \Phi \geq \sum_1^{n+\nu+1} d_{I_k} - n \;\text{ nor }\; \rho + \deg_\zeta \Phi \geq \sum_1^{n+\nu+1} \delta_{J_k} - \nu.$$

□

13. An exhaustive Max Nöther theorem with Macaulay bound on products of projective space

We may generalize even further. Via the compactification

$$\mathbb{C}^n \hookrightarrow \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_\mu},$$

where $n = n_1 + n_2 + \ldots + n_\mu$, we construct the tensor bundle $\mathcal{O}_1(\ell_1) \otimes \mathcal{O}_2(\ell_2) \otimes \ldots \otimes \mathcal{O}_\mu(\ell_\mu)$ of the pull-back bundles $\mathcal{O}_j(\ell_j)$ over $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_\mu}$. By induction, we have:

a) if $\ell_j < 0$ for any $j = 1, \ldots, \mu$, then
$\mathcal{O}(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_\mu}, \mathcal{O}_1(\ell_1) \otimes \ldots \otimes \mathcal{O}_\mu(\ell_\mu)) = \{0\}$.

b) if $\ell_j \geq 0$ for all $j = 1, \ldots, \mu$, then
$\mathcal{O}(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_\mu}, \mathcal{O}_1(\ell_1) \otimes \ldots \otimes \mathcal{O}_\mu(\ell_\mu))$
$= \{(\ell_1, \ldots, \ell_\mu)\text{-homogeneous polynomials on } \mathbb{C}^{n_1+1} \times \ldots \times \mathbb{C}^{n_\mu+1}\}$.

This tells us that $H^q(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_\mu}, \mathcal{O}_1(\rho_1) \otimes \ldots \otimes \mathcal{O}_\mu(\rho_\mu)) = 0$ when $q = 0$ and any $\rho_j < 0$. By induction, Künneth's formula asserts that

$$H^q(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_\mu}, \mathcal{O}_1(\rho_1) \otimes \ldots \otimes \mathcal{O}_\mu(\rho_\mu)) = \bigoplus_{q=q_1+\ldots+q_\mu} \bigotimes_{j=1}^{\mu} H^{q_j}(\mathbb{P}^{n_j}, \mathcal{O}_j(\rho_j)).$$

In terms of solvability of $\bar{\partial}$-equations, thus:

Theorem 13.1. *Let ξ be a $\bar{\partial}$-closed $(0,q)$-form, $q \geq 1$, with values in $\mathcal{O}_1(\rho_1) \otimes \mathcal{O}_2(\rho_2) \otimes \ldots \otimes \mathcal{O}_\mu(\rho_\mu)$ over $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_\mu}$. There exists a solution to the equation $\bar{\partial} v = \xi$ in the following cases:*

a) if for each index set $J \subset \{1, 2, \ldots, \mu\}$, we have $\sum_J n_{j_\ell} \neq q$.

b) if for each index set $J \subset \{1, 2, \ldots, \mu\}$ such that $\sum_J n_{j_\ell} = q$, there exists $j \in J$ such that $\rho_j \geq -n_j$, or $j \notin J$ such that $\rho_j < 0$.

Though its complexity, in the special case of $n_j = 1$, i.e., for the compatification $\mathbb{C}^n \hookrightarrow (\mathbb{P}^1)^n = \mathbb{P}^1 \times \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$, then in particular:

Corollary 13.2. *Let ξ be a $\bar{\partial}$-closed $(0,q)$-form, $q \geq 1$, with values in $\mathcal{O}_1(\rho_1) \otimes \mathcal{O}_2(\rho_2) \otimes \ldots \otimes \mathcal{O}_n(\rho_n)$ over $(\mathbb{P}^1)^n$. There exists a solution to the equation $\bar{\partial} v = \xi$ if $\rho_j \geq -1$ for all j, but at most $q-1$.*

For $\mathbb{C}^n \hookrightarrow \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_\mu}$, we have the homogenization

$$f = F\left(\frac{z_1}{z_{1,0}}, \frac{z_2}{z_{2,0}}, \ldots, \frac{z_\mu}{z_{\mu,0}}\right) \prod_{j=1}^{\mu} z_{j,0}^{d_j},$$

where $d_j = \deg_j F$.

Theorem 13.3. *Let $F_1, F_2, \ldots, F_m$ be polynomials on $\mathbb{C}^n$ such that the homogenizations $f_1, f_2, \ldots, f_m$ lack nontrivial common zeroes for any $z_{j,0} = 0$, and let Φ be a polynomial in the ideal (F). Then there exist polynomials $Q_1, Q_2, \ldots, Q_m$ satisfying (12.1) such that*

a) $\deg_j(F_\alpha Q_\alpha) \leq \max(\deg_j \Phi, \sum_{\ell=1}^{j+1} d_{j,I_\ell} - 1)$, for each j,

or if $m \leq n$,

b) $\deg_j(F_\alpha Q_\alpha) \leq \max(\deg_j \Phi, \sum_{\ell=1}^{j+1} d_{j,I_\ell} - 1)$*, for each* $j < m-1$ *and* $\deg_j(F_\alpha Q_\alpha) \leq \max(\deg_j \Phi, \sum_{\ell=1}^{m-1} d_{j,I_\ell} - 1)$ *for* $j \geq m-1$*.*

In either case, and for each j*,* I *is an ordering of indices such that* $d_{j,I_1} \geq d_{j,I_2} \geq \ldots \geq d_{j,I_m}$*.*

Proof. It follows directly from Theorem 13.4 below, combined with Corollary 13.2. □

Example 6. None of the variables z, ζ or ω exist of polynomial degree for

$$z\zeta\omega, \qquad z^2\zeta + \omega^3, \qquad \zeta\omega + 1, \qquad z\omega + 1.$$

Hence, Macaulay's theorem cannot be used. Separating z, then ζ still does not, and separating ζ, then z still does not exists of polynomial degree. Using Theorem 10.2, the only possibility left is for the separation $\mathbb{P}^2_{z,\zeta} \times \mathbb{P}^1_\omega$. Homogenized, the polynomials take the form

$$z\zeta\omega, \qquad z^2\zeta\omega_0^3 + z_0^3\omega^3, \qquad \zeta\omega + z_0\omega_0, \qquad z\omega + z_0\omega_0,$$

and we see that $z = \omega_0 = 1$, $z_0 = \zeta = \omega = 0$ is a common zero. Though, on the compactification $(\mathbb{P}^1)^3$, Theorem 13.3 can be be used and gives six tribounds which at best sums up to 8. □

Unlike what was laid down in Theorem 10.2 and 12.3, on the degree estimating results of Max Nöther and Macaulay type deduced through the discussed procedure for line bundles over projective space, an exhaustive result with respect to what has been said about $\bar{\partial}$-cohomology groups is:

Theorem 13.4. *Let* $F_1, F_2, \ldots, F_m$ *be polynomials on* $\mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \ldots \times \mathbb{C}^{n_\mu}$ *such that the homogenizations* $f_1, f_2, \ldots, f_m$ *lack nontrivial common zeroes for any* $z_{j,0} = 0$*, and let* Φ *be a polynomial in the ideal* (F)*. Then there exist polynomials* $Q_1, Q_2, \ldots, Q_m$ *satisfying (12.1) such that*

$$\deg_j F_\alpha Q_\alpha \leq \rho_j + \deg_j \Phi,$$

where $\rho_1, \rho_2, \ldots, \rho_\mu$ *are any nonnegative integers such that:*

$$\forall \begin{cases} I \subset \{1, 2, \ldots, m\} \\ J \subset \{1, 2, \ldots, \mu\} \\ k \in \{1, 2, \ldots, N-1\} \end{cases} : \begin{cases} |I| = k+1 \\ \sum_J n_{j_\ell} = k \end{cases}$$

$$: \exists \begin{cases} j \in J : \rho_j + \deg_j \Phi \geq \sum_{\ell=1}^{k+1} d_{j,I_\ell} - n_j \\ \quad or \\ j \notin J : \rho_j + \deg_j \Phi < \sum_{\ell=1}^{k+1} d_{j,I_\ell} \end{cases},$$

where $N = \min(m, n+1)$*,* J *is a set of indices for the separations,* I *is a set of indices for the polynomials and* k *is the degree as a* $(0,q)$*-form of the* $\bar{\partial}$*-equation imposing corresponding condition.*

Proof. Via homogenization, $f = (f_1, f_2, \dots, f_m)$ is a global holomorphic section of

$$E^* = \bigoplus_{i=1}^{m} \bigotimes_{j=1}^{\mu} \mathcal{O}_j(d_{j,i})$$

operating on

$$E = \bigoplus_{i=1}^{m} \bigotimes_{j=1}^{\mu} \mathcal{O}_j(-d_{j,i}).$$

Via homogenization, we seek a global holomorphic section of

$$E \otimes \bigotimes_{j=1}^{\mu} \mathcal{O}_j(\rho_j + \deg_j \Phi),$$

satisfying

$$\delta_f q = \varphi \prod_{j=1}^{\mu} z_{j,0}^{\rho_j}.$$

The very same procedure applied in the proof of Theorem 12.3 takes the existence of global holomorphic sections down to the solvability of $\bar{\partial}$-equations. The right-hand sides of those equations are $(0,k)$-forms that take values in

$$\Lambda^{k+1} E \otimes \bigotimes_{j=1}^{\mu} \mathcal{O}_j(\rho_j + \deg_j \Phi) = \bigoplus_{|I|=k+1} \bigotimes_{j=1}^{\mu} \mathcal{O}_j(\rho_j + \deg_j \Phi - \sum_{\ell=1}^{k+1} d_{j,I_\ell}).$$

Under the hypothesis of the theorem, the existence of holomorphic sections follows by Theorem 13.1. □

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