



Obstruction theory to formality and homotopy equivalences

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The notion of formality



Formal topological spaces

R : commutative ground ring

Definition

A topological space X is **formal** if there exists a zig-zag of quasi-isomorphisms of dga algebras,

$$C_{\text{sing}}^{\bullet}(X; R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \dots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H_{\text{sing}}^{\bullet}(X; R) .$$

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→ Origins in rational homotopy theory (for $\mathbb{Q} \subset R$)

X formal \implies The cohomology ring $H_{\text{sing}}^{\bullet}(X, \mathbb{Q})$ completely determines the rational homotopy type of X .

Examples

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [Deligne, Griffiths, Morgan & Sullivan, 1975]

Formality of an algebraic structure

A : chain complex over R

\mathcal{P} : colored operad or properad

$\phi : \mathcal{P} \rightarrow \text{End}_A$: a dg \mathcal{P} -algebra structure

Definition

The dg \mathcal{P} -algebra (A, ϕ) is **formal** if

$$\exists (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \dots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \varphi_*) ,$$

where φ_* is the canonical \mathcal{P} -algebra structure on $H(A)$.

Examples

- X is formal = $(C_{\text{sing}}^\bullet(X; R), \cup)$ is formal as dga algebra
- $C(\mathcal{D}_k; \mathbb{R})$ is formal as an operad [Kontsevich, 1999]

Purity implies formality

(A, ϕ) : dg \mathcal{P} -algebra encoded by an operad \mathcal{P}

α : unit of infinite order in R

σ_α : the degree twisting by $\alpha =$ automorphism of $(H(A), \varphi_*)$
which acts via $\alpha^k \times$ on $H_k(A)$.

Theorem

If σ_α admits a chain-level lift, i.e. $\exists f \in \text{End}(A, \phi)$ s.t. $H(f) = \sigma_\alpha$, then (A, ϕ) is formal.

→ Deligne, Griffiths, Morgan, Sullivan [1975]

→ Sullivan [1977]

→ Guillén Santos, Navarro, Pascual, Roig [2005]

→ Drummond-Cole and Horel [2021]

Examples

- Petersen [2014], Boavida de Brito and Horel [2021]
The little disks operad \mathcal{D}_k & Grothendieck-Teichmüller group
- Riche, Soergel, Williamson [2014]
The extensions of parity sheaves on the flag variety.
- Drummond-Cole and Horel [2021]
 X : a complement of a hyperplane arrangement over \mathbb{C}
defined over a finite extension K of \mathbb{Q}_p .
 ℓ : a prime number different from p
→ $C^\bullet(X_{an}, \mathbb{Z}_\ell) \cong C_{et}^\bullet(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ [Artin].
→ A Frobenius action on $H_{et}^\bullet(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ is σ_q [Kim, 1994].

Questions

- Can we descend these results to other coefficient rings?
(e.g. $\mathbb{Z}(\ell), \dots$)
- Does the degree twisting criteria hold for other types of algebras? (e.g. Hopf algebras, involutive Lie bialgebras,...)
- Is the degree twisting the only homology automorphism satisfying this property?
- Which type of geometric structure can we detect by relaxing the formality condition ?

Kaledin classes & formality criteria

1. Higher structures

- Formality can be addressed as a deformation problem, using the operadic calculus.

2. Kaledin classes

- An obstruction theory to the formality over any ring

3. Formality criteria

- Formality descent with torsion coefficient, Automorphism lifts

4. Beyond formality

- Generalizing Kaledin classes to study homotopy equivalences



Higher structures



Homotopy retracts

Definition

(W, d_W) is a **homotopy retract** of (V, d_V) if there are maps

$$h \circlearrowleft (V, d_V) \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{i} \end{array} (W, d_W)$$

where $\text{id}_V - ip = d_V h + h d_V$ and i is a quasi-isomorphism .

Proposition

If R is a field, the cohomology of any cochain complex is a homotopy retract:

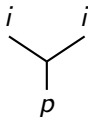
$$h \circlearrowleft (A, d_A) \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{i} \end{array} (H(A), 0) .$$

Transfer of algebraic structure

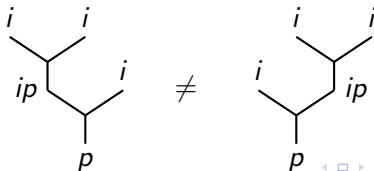
(A, d_A, ϕ) : a dga algebra and a homotopy retraction:

$$h \begin{array}{c} \curvearrowright \\ (A, d_A, \phi) \end{array} \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{i} \end{array} (H, d_H)$$

→ Transferred product: $\varphi_2 := p \circ \phi \circ i^{\otimes 2} : H^{\otimes 2} \rightarrow H$



Not associative in general!



→ Consider $\varphi_3 : H^{\otimes 3} \rightarrow H$

$$\begin{array}{c} \diagup \\ | \\ \diagdown \end{array} := \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ | \\ h \quad i \\ \diagdown \quad \diagup \\ | \\ p \end{array} - \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ | \\ i \quad h \\ \diagdown \quad \diagup \\ | \\ p \end{array}$$

Homotopy associative algebras

Definition (Stasheff, 1963)

A_∞ -algebra: a cochain complex H with a collection of maps

$$\varphi_n : H^{\otimes n} \rightarrow H$$

of degree $2 - n$, for all $n \geq 2$, which satisfy the relations

$$\partial \left(\begin{array}{c} 1 \quad 2 \quad \dots \quad n \\ \diagdown \quad | \quad / \\ \text{---} \end{array} \right) = \sum_{\substack{k+l=n+1 \\ 1 \leq j \leq k}} \pm \begin{array}{c} 1 \quad \dots \quad l \\ \diagdown \quad | \quad / \\ \text{---} \\ \diagdown \quad | \quad / \\ 1 \quad \dots \quad j \quad \dots \quad k \end{array}$$

Examples

- Every dga algebra (A, ϕ) is an A_∞ -algebra with $\varphi_n = 0$ for all $n \geq 3$.
- $(H, d_H, \varphi_2, \varphi_3, \dots)$

Homotopy transfer theorem

Theorem (Kadeishvili, 1982)

Given a dga algebra (A, d_A, ϕ) and a homotopy retract

$$h \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} (A, d_A, \phi) \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{i} \end{array} (H, d_H)$$

there exists an A_∞ -algebra structure on H such that p (and i) extend to A_∞ -quasi-isomorphisms:

$$(A, d_A, \phi) \rightsquigarrow (H, d_H, \varphi_2, \varphi_3, \varphi_4, \dots)$$

Homotopy morphisms

$(A, d_A, \phi_2, \dots), (H, d_H, \varphi_2, \dots) : A_\infty$ -algebras

Definition

A_∞ -morphism $f : A \rightsquigarrow H$ is a collection of linear maps

$$f_n : A^{\otimes n} \longrightarrow H, \quad n \geq 1,$$

of degree $1 - n$, which satisfy the relations

$$\sum_{\substack{k \geq 1 \\ i_1 + \dots + i_k = n}} \pm \begin{array}{c} \vee \qquad \vee \\ f_{i_1} \dots f_{i_k} \\ | \\ \phi_k \end{array} = \sum_{\substack{k+l=n+1 \\ 1 \leq j \leq k}} \pm \begin{array}{c} \vee \qquad \vee \\ \qquad \qquad \varphi_l \\ | \qquad \qquad | \\ j \qquad \qquad \phi_1 \\ | \qquad \qquad | \\ \phi_k \end{array}$$

where $\varphi_1 = d_H$ and $\phi_1 = d_A$.

Homotopy quasi-isomorphisms

Definition

A_∞ -quasi-isomorphism $f : A \xrightarrow{\sim} H$ is an A_∞ -morphism where $f_1 : A \rightarrow H$ is a quasi-isomorphism .

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Proposition (R is a field)

quasi-isos of associative algebras

A_∞ -quasi-iso

$$\exists (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \dots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B, \phi') \iff \exists (A, \phi) \xrightarrow{\sim} (B, \phi')$$

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Corollary

A dga algebra (A, ϕ) is formal if and only if

$$\exists (A, \phi) \xrightarrow{\sim} (H(A), \varphi_*) .$$



Kaledin classes



Hochschild complex

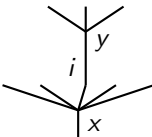
Transferred structure: $(H(A), \varphi_*, \varphi_3, \varphi_4, \dots)$

$$\varphi_n \in \text{Hom}(H(A)^{\otimes n}, H(A)), \quad |\varphi_n| = 2 - n$$

Hochschild cochain complex:

$$\mathfrak{g} := \prod_{n \geq 1} s^{-n+1} \text{Hom}(H(A)^{\otimes n}, H(A))$$

Lie bracket : $[x, y] := x \star y - (-1)^{|x||y|} y \star x$

$$x \star y := \sum_{i=1}^n (-1)^{(i-1)(m-1)}$$


for $x \in \text{Hom}(H(A)^{\otimes n}, H(A))$ and $y \in \text{Hom}(H(A)^{\otimes m}, H(A))$.

A formal deformation

Transferred structure:

$$(\varphi_*, \varphi_3, \varphi_4, \dots) \in \mathfrak{g} := \prod_{n \geq 1} s^{-n+1} \text{Hom}(H(A)^{\otimes n}, H(A))$$

A formal deformation:

$$\Phi := \varphi_* + \varphi_3 \hbar + \varphi_4 \hbar^2 + \dots \in \mathfrak{g}[[\hbar]] := \mathfrak{g} \widehat{\otimes} R[[\hbar]]$$

Proposition : $\text{ad}_\Phi := [\Phi, -]$ defines a differential on $\mathfrak{g}[[\hbar]]$

Twisted dg Lie algebra:

$$\mathfrak{g}[[\hbar]]^\Phi := (\mathfrak{g}[[\hbar]], [-, -], \text{ad}_\Phi)$$

Kaledin classes

$$\partial_{\hbar}\Phi := \varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \dots \in \mathfrak{g}[[\hbar]]$$

Lemma : $\partial_{\hbar}\Phi$ is a cycle in $\mathfrak{g}[[\hbar]]^{\Phi} := (\mathfrak{g}[[\hbar]], [-, -], \text{ad}_{\Phi})$,

$$\text{ad}_{\Phi}(\partial_{\hbar}\Phi) := [\Phi, \partial_{\hbar}\Phi] = 0 .$$

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Kaledin class:

$$K_{\Phi} := [\partial_{\hbar}\Phi] \in H^1\left(\mathfrak{g}[[\hbar]]^{\Phi}\right) .$$

n^{th} -truncated Kaledin class :

$$K_{\Phi}^n := [\varphi_3 + 2\varphi_4\hbar + \dots + (n-2)\varphi_n\hbar^{n-3}] \in H^1\left(\left(\mathfrak{g}[[\hbar]]/\hbar^{n-2}\right)^{\tilde{\Phi}}\right) .$$

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Theorem ([Kaledin, 2007], [Lunts, 2007])

$R : \mathbb{Q}$ -algebra

$(A, \phi) : dg$ associative algebra, $H(A)$ is a homotopy retract

- (A, ϕ) is gauge formal $\iff K_{\Phi} = 0$.
- (A, ϕ) is gauge n -formal $\iff K_{\Phi}^n = 0$.

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Theorem ([Kaledin, 2007], [Lunts, 2007], [Melani–Rubió, 2019])

R : \mathbb{Q} -algebra

: Binary Koszul operad

(A, ϕ) : dg \mathcal{P} -algebra that admits a transferred structure

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Theorem (E., 2024)

R : commutative ground ring

: (pr)operad colored in groupoids

n : integer such that $n!$ is invertible in R

(A, ϕ) : dg -algebra that admits a transferred structure

- (A, ϕ) is gauge formal $\iff K_{\Phi} = 0$.
- (A, ϕ) is gauge n -formal $\iff K_{\Phi}^n = 0$.



Formality criteria



Formality descent

(A, ϕ) : a dg \mathcal{P} -algebra that admits a transferred structure

$H_i(A)$: projective, finitely generated for all i .

S : faithfully flat commutative R -algebra.

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S : faithfully flat commutative R -algebra.

Proposition (E., 2024)

(A, ϕ) is gauge n -formal $\iff (A \otimes_R S, \phi \otimes 1)$ is gauge n -formal.

Proof.

$$H_{-1}(\mathfrak{g}_{H(A)}[[\hbar]]^\Phi) \otimes_{R[[\hbar]]} S[[\hbar]] \cong H_{-1}(\mathfrak{g}_{H(A \otimes_R S)}[[\hbar]]^{\Phi \otimes 1}) \quad \square$$

Examples

- $C(\mathcal{D}_k; \mathbb{R})$ is formal $\iff C(\mathcal{D}_k; \mathbb{Q})$ is formal [GSNPR, 2005]
- $\mathbb{Z}_{(\ell)} \subset \mathbb{Z}_\ell$

Complement of hyperplane arrangements

X : a **complement of a hyperplane arrangement** over \mathbb{C}
→ complement of a finite collection of affine hyperplanes in $\mathbb{A}_{\mathbb{C}}^n$.

K : a finite extension of \mathbb{Q}_p

q : order of the residue field of the ring of integers of K

ℓ : a prime number different from p

s : order of q in $\mathbb{F}_{\ell}^{\times}$

Proposition (Dummond-Cole – Horel, 2021)

If X is defined over K , i.e. $\exists K \hookrightarrow \mathbb{C}$ and $\exists \mathcal{X}$ a complement of a hyperplane arrangement over K s.t. $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^{\bullet}(X_{an}, \mathbb{Z}_{\ell})$ is gauge $(s - 1)$ -formal.

Formality descent $\implies C^{\bullet}(X_{an}, \mathbb{Z}_{(\ell)})$ is gauge $(s - 1)$ -formal.

Triviality of fibrations

Theorem (E., 2024)

X : a simply connected topological space

F : a nilpotent space of finite \mathbb{Q} -type.

A fibration $\xi : E \rightarrow X$ with fiber $F_{\mathbb{Q}}$ is trivial up to homotopy iff $\xi \otimes \mathbb{R}$ is trivial up to homotopy.

Example

The Fadell–Neuwirth fibration :

$$\xi : \text{Conf}_{n-1}(\mathbb{R}^d) \longrightarrow \text{Conf}_n(\mathbb{S}^d) \longrightarrow \mathbb{S}^d .$$

If d is odd, $\xi \otimes \mathbb{R}$ is trivial up to homotopy [Haya Enriquez, 2022]

$\implies \xi$ is trivial up to homotopy.

Intrinsic formality

A graded \mathcal{P} -algebra (H, φ_*) is **intrinsically formal** if every \mathcal{P} -algebra (A, ϕ) such that $(H(A), \varphi_*) = (H, \varphi_*)$ is itself gauge formal.

$$\mathfrak{g}^{\varphi_*} : (\mathfrak{g}, [-, -], d + [\varphi_*, -])$$

Proposition (E., 2023)

$$H^1(\mathfrak{g}^{\varphi_*}) = 0 \implies (H, \varphi_*) \text{ intrinsically formal.}$$

Previous works: [Hinich, 2003]

Properadic coformality of spheres

Example (Kontsevich, Takeda, Vlassopoulos, 2021)

$A = C_*(\Omega S^n; R)$ has an n -pre-Calabi-Yau structure

$$\phi = m_{(1)} + m_{(2)} + m_{(3)} + \dots$$

where $m_{(\ell)}$ is a cyclically anti-symmetric collection of maps

$$m_{(\ell)}^{k_1, \dots, k_\ell} : sA^{k_1} \otimes \dots \otimes sA^{k_\ell} \rightarrow A^\ell .$$

In particular, $m_{(1)}^0 = 0$, $m_{(1)}$ is an A_∞ -algebra, $m_{(2)}$ is a noncommutative Poisson bivector up to a $[m_{(1)}, -]$ -exact term,

$$m_{(2)} \star m_{(2)} = [m_{(1)}, m_{(3)}] .$$

= a polyvector field + integrability condition with respect to a noncommutative analogue of the Schouten-Nijenhuis bracket.

Properadic coformality of spheres

The n -pre-Calabi-Yau algebra structure on $A = C_*(\Omega S^n; R)$ has vanishing copairing:

$$m_{(2)}^{0,0} = 0$$

→ always the case where A is connective and $n \geq 1$.

Theorem (E., Takeda, 2025)

If R is a \mathbb{Q} -algebra, (H, φ_) is intrinsically formal as an n -pre-Calabi-Yau algebra structure with vanishing copairing.*

Automorphism lifts

R : a field

(A, ϕ) : a dg \mathcal{P} -algebra that admits a transferred structure s.t.
 $H(A)$ is finite dimensional.

Corollary (E., 2024)

Suppose that there exists $u \in \text{Aut}(H(A), \varphi_0)$ such that for all $k < n$, and all p -tuples (k_1, \dots, k_p) ,

$$\text{Spec}(u_{k_1+\dots+k_p+k}) \cap \text{Spec}(u_{k_1} \otimes \dots \otimes u_{k_p}) = \emptyset ,$$

where $u_i := u|_{H_i(A)}$. If u admits a lift at the level of chains then (A, ϕ) is gauge n -formal.

Frobenius & Weil numbers

K : a finite extension of \mathbb{Q}_p

q : order of the residue field of the ring of integers \mathcal{O}_K

ℓ : a prime number different from p

X : a smooth proper K -scheme

Definition

$\alpha \in \overline{\mathbb{Q}}_\ell$ is a **Weil number of weight n** if

$$\forall \iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}, \quad |\iota(\alpha)| = q^{n/2} .$$

Theorem (Deligne, 1974)

For all n , the eigenvalues of a Frobenius action on $H_{\text{et}}^n(X_{\overline{K}}, \mathbb{Q}_\ell)$ are Weil numbers of weight n .

Theorem

Let X be a smooth and proper scheme over \mathbb{C} . The algebra $C^\bullet(X_{\text{an}}, \mathbb{Q})$ is formal.

Proof.

- There exists a smooth and proper model \mathcal{X} over \mathcal{O}_K .

$$C^\bullet(X_{\text{an}}, \mathbb{Q}_\ell) \cong C_{\text{et}}^\bullet(\mathcal{X}_{\overline{K}}, \mathbb{Q}_\ell)$$

- Let u be the Frobenius action on $H_{\text{et}}^\bullet(\mathcal{X}_{\overline{K}}, \mathbb{Q}_\ell)$.
- For all $k \geq 1$, (k_1, \dots, k_p) and $s := k_1 + \dots + k_p$,

$$\begin{array}{ccc} \text{Spec}(u_{s+k}) & \cap & \text{Spec}(u_{k_1} \otimes \dots \otimes u_{k_p}) = \emptyset . \\ \downarrow \psi & & \downarrow \psi \\ \alpha & & \beta \\ |\iota(\alpha)| = q^{\frac{s+k}{2}} & > & |\iota(\beta)| = q^{\frac{s}{2}} \end{array}$$



Previous work: [Deligne, 1980]

Let Sch_K be the category of smooth and proper schemes over K of *good reduction*, i.e. for which there exists a smooth and proper model over \mathcal{O}_K .

Theorem (E., 2024)

Let \mathbb{V} be a groupoid and let \mathcal{P} be a \mathbb{V} -colored operad in sets. Let X be a \mathcal{P} -algebra in Sch_K . The dg \mathcal{P} -algebra $C_\bullet(X_{\text{an}}, \mathbb{Q})$ is formal.

Example (Guillén Santos, Navarro, Pascual, & Roig, 2005)

$\overline{\mathcal{M}}$ the cyclic operad of moduli spaces of stable algebraic curves
 $C_\bullet(\overline{\mathcal{M}}_{\text{an}}; \mathbb{Q})$ is formal



Beyond formality



Homotopy equivalences between algebraic structure

Definition

The dg \mathcal{P} -algebras (A, ϕ) and (B, ψ) are

- homotopy equivalent

$$\exists (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \dots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B, \psi)$$

- gauge homotopy equivalent if $\exists (A, \phi) \overset{\sim}{\rightsquigarrow} (B, \psi)$.

Example

(A, ϕ) is formal if it is homotopy equivalent to $(H(A), \varphi_*)$

Question

- Can we generalize Kaledin classes to study homotopy equivalences ?

Obstruction sequences to homotopy equivalences

Let (A, φ) and (B, ψ) be two \mathcal{P} -algebras with $H(A) \cong H(B)$.

→ **obstruction sequence** $(\vartheta_k)_{1 \leq k \leq n}$ which is either

- an infinite sequence of vanishing classes, when $n = \infty$;
- a finite sequence of trivial classes that ends on $\vartheta_n \neq 0$.

The index $n \in \llbracket 1, \infty \rrbracket$ of the last class only depends on ϕ and ψ .

Theorem (E., 2025)

The algebras (A, φ) and (B, ψ) are gauge homotopy equivalent modulo $\mathcal{F}^k \mathfrak{g}$ for all k if and only if $n = \infty$.

Minimal model on highly connected variety

Theorem (E., 2025)

Let \mathbb{K} be a field. Let M^d be a compact k -connected oriented C^∞ -manifold where d is smaller than $(\ell + 1)k + 2$. For every prime number p ,

$$C_{\text{sing}}^*(M, \mathbb{K})$$

is homotopy equivalent to an A_∞ -algebra $(H_{\text{sing}}^*(M, \mathbb{K}), \varphi)$, with $\varphi_n = 0$ for $n \geq \ell$.



Thank you for your attention!

