



Kaledin classes & formality criteria

Coline Emprin

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Laboratoire Paul Painlevé – Université de Lille

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Kaledin classes & formality criteria

1. The notion of formality

2. Higher structures

- The formality can be addressed as a deformation problem, using the operadic calculus.

3. Kaledin obstruction classes

- Construction of an obstruction theory to formality over any coefficient ring.

4. Formality criteria

- Formality descent with torsion coefficients
- Intrinsic formality criterium
- Degree twisting & automorphisms lifts



The notion of formality



Formal topological spaces

R : commutative ground ring

Definition

A topological space X is **formal** if there exists a zig-zag of quasi-isomorphisms of differential graded associative algebras,

$$C_{\text{sing}}^{\bullet}(X; R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H_{\text{sing}}^{\bullet}(X; R) .$$

→ Origins in rational homotopy theory (for $\mathbb{Q} \subset R$)

X formal $\implies H^{\bullet}(X, \mathbb{Q})$ completely determines the rational homotopy type of X .

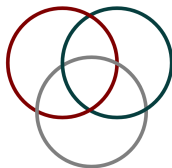
Examples

→ Formal spaces

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [DGMS, 1975]

→ Nonformal spaces

- The complement of the Borromean rings



Formality of an algebraic structure

A : cochain complex over R

(A, ϕ) : differential graded algebraic structure over A , e.g.

- a dg associative algebra,
- a dg Lie algebra,
- ...

Definition

The dg algebra (A, ϕ) is **formal** if

$$\exists (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \varphi_*) ,$$

where φ_* denotes the induced structure on $H(A)$.

Examples

- X formal = $(C_{\text{sing}}^\bullet(X; R), \cup)$ is formal as dga algebra
- $C(\mathcal{D}_k; \mathbb{R})$ is formal as an operad [Kontsevich, 1999]



Higher structures



Homotopy retracts

Definition

(W, d_W) is a **homotopy retract** of (V, d_V) if there are maps

$$h \circlearrowleft (V, d_V) \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} (W, d_W)$$

where $\text{id}_V - ip = d_V h + h d_V$ and i is a quasi-isomorphism .

Proposition

If R is a field, the cohomology of any cochain complex is a homotopy retract:

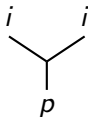
$$h \circlearrowleft (A, d_A) \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} (H(A), 0) .$$

Transfer of algebraic structure

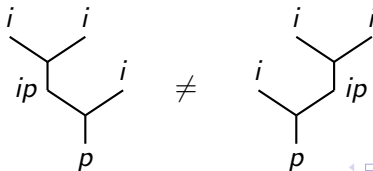
(A, d_A, ϕ) : a dga algebra and a homotopy retraction:

$$h \circlearrowleft (A, d_A, \phi) \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} (H, d_H)$$

→ Transferred product: $\varphi_2 := p \circ \phi \circ i^{\otimes 2} : H^{\otimes 2} \rightarrow H$



Not associative in general!



→ Consider $\varphi_3 : H^{\otimes 3} \rightarrow H$

$$\text{Y-junction} := \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ h \quad \quad i \\ \diagup \quad \diagdown \\ p \end{array} - \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ i \quad \quad h \\ \diagup \quad \diagdown \\ p \end{array}$$

→ In $\text{Hom}(H^{\otimes 3}, H)$:

$$\partial \left(\text{Y-junction} \right) = \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ ip \quad \quad i \\ \diagup \quad \diagdown \\ p \end{array} - \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ i \quad \quad ip \\ \diagup \quad \diagdown \\ p \end{array}$$

→ φ_2 is associative up to the homotopy φ_3 .

$\rightarrow \varphi_n : H^{\otimes n} \rightarrow H$, for all $n \geq 2$

$$\begin{array}{c} 1 \quad 2 \quad \dots \quad n \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \end{array} := \sum_{\text{PBT}_n} \pm \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ i \quad i \quad i \quad h \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ h \quad h \quad h \quad p \end{array}$$

$$\partial \left(\begin{array}{c} 1 \quad 2 \quad \dots \quad n \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \end{array} \right) = \sum_{\substack{k+l=n+1 \\ 1 \leq j \leq k}} \pm \begin{array}{c} 1 \quad \dots \quad l \\ \diagdown \quad \diagup \\ j \quad \dots \quad k \\ \diagdown \quad \diagup \end{array}$$

Homotopy associative algebras

Definition (Stasheff, 1963)

A_∞ -algebra: a cochain complex H with a collection of maps

$$\varphi_n : H^{\otimes n} \rightarrow H$$

of degree $2 - n$, for all $n \geq 2$, which satisfy the relations

$$\partial \left(\begin{array}{c} 1 \quad 2 \quad \dots \quad n \\ \diagdown \quad | \quad \diagup \\ \text{ } \end{array} \right) = \sum_{\substack{k+l=n+1 \\ 1 \leq j \leq k}} \pm \begin{array}{c} 1 \quad \dots \quad l \\ \diagdown \quad | \quad \diagup \\ \text{ } \\ \diagdown \quad | \quad \diagup \\ 1 \quad \dots \quad j \quad \dots \quad k \\ \text{ } \end{array}$$

Examples

- Every dga algebra (A, ϕ) is an A_∞ -algebra with $\varphi_n = 0$ for all $n \geq 3$.
- $(H, d_H, \varphi_2, \varphi_3, \dots)$

Homotopy transfer theorem

Theorem (Kadeishvili, 1982)

Given a dga algebra (A, d_A, ϕ) and a homotopy retract

$$h \circlearrowleft (A, d_A, \phi) \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} (H, d_H)$$

there exists an A_∞ -algebra structure on H such that p (and i) extend to A_∞ -quasi-isomorphisms:

$$(A, d_A, \phi) \rightsquigarrow (H, d_H, \varphi_2, \varphi_3, \varphi_4, \dots)$$

Homotopy morphisms

$(A, d_A, \phi_2, \dots), (H, d_H, \varphi_2, \dots) : A_\infty$ -algebras

Definition

A_∞ -morphism $f : A \rightsquigarrow H$ is a collection of linear maps

$$f_n : A^{\otimes n} \longrightarrow H, \quad n \geq 1,$$

of degree $1 - n$, which satisfy the relations

$$\sum_{\substack{k \geq 1 \\ i_1 + \dots + i_k = n}} \pm \begin{array}{c} \vee \quad \vee \\ f_{i_1} \dots f_{i_k} \\ | \\ \phi_k \end{array} = \sum_{\substack{k+l=n+1 \\ 1 \leq j \leq k}} \pm \begin{array}{c} \vee \quad \vee \\ \quad \quad \varphi_l \\ | \quad j \\ f_k \end{array}$$

where $\varphi_1 = d_H$ and $\phi_1 = d_A$.

Homotopy quasi-isomorphisms

Definition

A_∞ -quasi-isomorphism $f : A \xrightarrow{\sim} H$ is an A_∞ -morphism where $f_1 : A \rightarrow H$ is a quasi-isomorphism .

Proposition (R is a field)

quasi-isos of associative algebras

A_∞ -quasi-iso

$$\exists (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B, \phi') \iff \exists (A, \phi) \xrightarrow{\sim} (B, \phi')$$

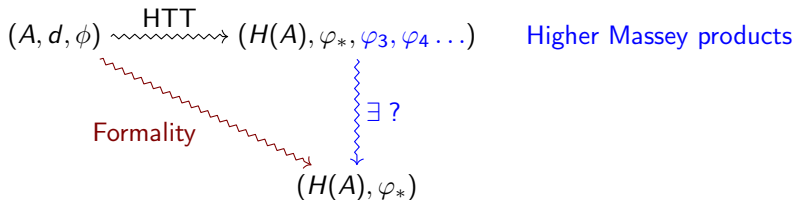
Corollary

A dga algebra (A, ϕ) is formal if and only if

$$\exists (A, \phi) \xrightarrow{\sim} (H(A), \varphi_*) .$$

An equivalent characterization of formality

(A, d, ϕ) a dga algebra such that $H(A)$ is a homotopy retract



$\implies$ If the higher Massey products vanish, then (A, d, ϕ) is formal.

Definition

- (A, d, ϕ) is **gauge formal** if $\exists (H(A), \varphi_*, \varphi_3, \varphi_4 \dots) \rightsquigarrow (H(A), \varphi_*)$.
- (A, d, ϕ) is **gauge n -formal** if

$$\exists (H(A), \varphi_*, \varphi_3, \varphi_4 \dots) \rightsquigarrow (H(A), \varphi_*, 0, \dots, 0, \varphi'_{n+1}, \dots) .$$



Kaledin classes



Hochschild complex

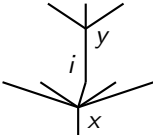
Transferred structure: $(H(A), \varphi_*, \varphi_3, \varphi_4, \dots)$

$$\varphi_n \in \text{Hom}(H(A)^{\otimes n}, H(A)), \quad |\varphi_n| = 2 - n$$

Hochschild cochain complex:

$$\mathfrak{g} := \prod_{n \geq 1} s^{-n+1} \text{Hom}(H(A)^{\otimes n}, H(A))$$

Lie bracket : $[x, y] := x \star y - (-1)^{|x||y|} y \star x$

$$x \star y := \sum_{i=1}^n (-1)^{(i-1)(m-1)} \text{diagram}$$


for $x \in \text{Hom}(H(A)^{\otimes n}, H(A))$ and $y \in \text{Hom}(H(A)^{\otimes m}, H(A))$.

A formal deformation

Transferred structure:

$$(\varphi_*, \varphi_3, \varphi_4, \dots) \in \mathfrak{g} := \prod_{n \geq 1} s^{-n+1} \operatorname{Hom}(H(A)^{\otimes n}, H(A))$$

A formal deformation:

$$\Phi := \varphi_* + \varphi_3 \hbar + \varphi_4 \hbar^2 + \dots \in \mathfrak{g}[[\hbar]] := \mathfrak{g} \hat{\otimes} R[[\hbar]]$$

Proposition : $\operatorname{ad}_\Phi := [\Phi, -]$ defines a differential on $\mathfrak{g}[[\hbar]]$

Twisted dg Lie algebra:

$$\mathfrak{g}[[\hbar]]^\Phi := (\mathfrak{g}[[\hbar]], [-, -], \operatorname{ad}_\Phi)$$

Kaledin classes

$$\partial_{\hbar}\Phi := \varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \cdots \in \mathfrak{g}[[\hbar]]$$

Lemma : $\partial_{\hbar}\Phi$ is a cycle in $\mathfrak{g}[[\hbar]]^{\Phi} := (\mathfrak{g}[[\hbar]], [-, -], \text{ad}_{\Phi})$,

$$\text{ad}_{\Phi}(\partial_{\hbar}\Phi) := [\Phi, \partial_{\hbar}\Phi] = 0 .$$

Kaledin class:

$$K_{\Phi} := [\partial_{\hbar}\Phi] \in H^1\left(\mathfrak{g}[[\hbar]]^{\Phi}\right) .$$

n^{th} -truncated Kaledin class :

$$K_{\Phi}^n := [\varphi_3 + 2\varphi_4\hbar + \cdots + (n-2)\varphi_n\hbar^{n-3}] \in H^1\left((\mathfrak{g}[[\hbar]]/\hbar^{n-2})^{\tilde{\Phi}}\right) .$$

Kaledin classes

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Theorem ([Kaledin, 2007], [Lunts, 2007])

$R : \mathbb{Q}$ -algebra

$(A, \phi) : dg \text{ associative algebra, } H(A) \text{ is a homotopy retract}$

- (A, ϕ) is gauge formal $\iff K_{\Phi} = 0$.
- (A, ϕ) is gauge n -formal $\iff K_{\Phi}^n = 0$.

Kaledin class:

$$K_{\Phi} := [\varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \cdots] \in H^1(\mathfrak{g}[[\hbar]]^{\Phi})$$

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Theorem ([Kaledin, 2007], [Lunts, 2007], [Melani–Rubió, 2019])

R : $\mathbb{Q}$ -algebra

$\mathcal{P}$: Koszul operad

(A, ϕ) : dg $\mathcal{P}$ -algebra such that $H(A)$ is a homotopy retract

- (A, ϕ) is gauge formal $\iff K_{\Phi} = 0$.
- (A, ϕ) is gauge n -formal $\iff K_{\Phi}^n = 0$.

Kaledin class:

$$K_{\Phi} := [\varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \cdots] \in H^1(\mathfrak{g}[[\hbar]]^{\Phi})$$

n^{th} -truncated Kaledin class :

$$K_{\Phi}^n := [\varphi_3 + 2\varphi_4\hbar + \cdots + (n-2)\varphi_n\hbar^{n-3}] \in H^1\left((\mathfrak{g}[[\hbar]]/\hbar^{n-2})^{\tilde{\Phi}}\right)$$

Theorem (E., 2023)

R : commutative ring

$\mathcal{P}$: (Pr)operad, possibly coloured in groupoids

(A, ϕ) : dg $\mathcal{P}$ -algebra such that $H(A)$ is a homotopy retract

- (A, ϕ) is gauge formal $\iff K_{\Phi} = 0$.
- (A, ϕ) is gauge n -formal $\iff K_{\Phi}^n = 0$.



Formality criteria



Formality descent

(A, ϕ) : a dg $\mathcal{P}$ -algebra s.t. $H(A)$ is a homotopy retract

$H^i(A)$: projective, finitely generated for all i .

S : faithfully flat commutative R -algebra.

Proposition (E., 2023)

(A, ϕ) is gauge n -formal $\iff (A \otimes_R S, \phi \otimes 1)$ is gauge n -formal.

Proof.

$$\begin{array}{ccc}
 H^1(\mathfrak{g}_{H(A)}[[\hbar]]^\Phi) \otimes_{R[[\hbar]]} S[[\hbar]] & \cong & H^1(\mathfrak{g}_{H(A \otimes_R S)}[[\hbar]]^{\Phi \otimes 1}) \\
 \downarrow & & \downarrow \\
 K_\Phi \otimes 1 = 0 & \iff & K_{\Phi \otimes 1} = 0
 \end{array}$$

□

Examples

- $C(\mathcal{D}_k; \mathbb{R})$ is formal $\iff C(\mathcal{D}_k; \mathbb{Q})$ is formal [GSNPR, 2005]
- $\mathbb{Z}_{(\ell)} \subset \mathbb{Z}_\ell$

Intrinsic formality

A graded $\mathcal{P}$ -algebra (H, φ_*) is **intrinsically formal** if every $\mathcal{P}$ -algebra (A, ϕ) such that $(H(A), \varphi_*) = (H, \varphi_*)$ is itself gauge formal.

$$\mathfrak{g}^{\varphi_*} : (\mathfrak{g}, [-, -], d + [\varphi_*, -])$$

Proposition (E., 2023)

$$H^1(\mathfrak{g}^{\varphi_*}) = 0 \implies (H, \varphi_*) \text{ intrinsically formal.}$$

Previous works: [Hinich, 2003]

Tamarkin's proof of Kontsevich formality

k : a characteristic zero field

A : a polynomial algebra over k

Theorem (Hinich, 2003)

The shifted cohomological Hochschild complex $C(A; A)[1]$ is formal as a dg Lie algebra.

Proof.

- $\text{Lie}[1] \subset \text{Gerst}$;
- $(HH^\bullet(A), \varphi_*)$ has a Gerst-algebra structure;
- $C(A; A)$ has a Gerst_∞ -algebra structure inducing φ_* ;
- $(HH^\bullet(A), \varphi_*)$ is intrinsically formal as a Gerst-algebra;
 $\rightarrow H^1(\mathfrak{g}^{\varphi_*}) = 0$, where $\mathfrak{g} = \text{Hom}(\overline{\text{Gerst}}, \text{End}_{HH^\bullet(A)})$.

The degree twisting

(A, ϕ) : a dg $\mathcal{P}$ -algebra s.t. $H(A)$ is a homotopy retract

α : a unit in R .

σ_α : the **degree twisting** by α

→ linear automorphism of $H(A)$ which acts via $\alpha^k \times$ on $H^k(A)$.

Theorem (Drummond-Cole – Horel, 2021)

Suppose that σ_α admits a lift, i.e. $\exists f \in \text{End}(A, \phi)$ s.t. $H(f) = \sigma_\alpha$.

- $\forall k, \alpha^k - 1 \in R^\times \implies (A, \phi)$ is gauge formal.
- $\forall k \leq n, \alpha^k - 1 \in R^\times \implies (A, \phi)$ is gauge n -formal.

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- $\forall k \leq n, \alpha^k - 1 \in R^\times \implies (A, \phi)$ is gauge n -formal.

Heuristic :

- Higher Massey products have to be compatible with the lift.
- They intertwine multiplication by α^l with multiplication by α^k with $l \neq k$.
- They have to vanish

Previous works: [DGMS, 1975], [Sullivan, 1977], [GSNPR, 2005]

Complement of subspace arrangements

X : a **complement of a hyperplane arrangement** over $\mathbb{C}$
→ complement of a finite collection of affine hyperplanes in $\mathbb{A}_{\mathbb{C}}^n$.

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

ℓ : a prime number different from p

s : order of q in $\mathbb{F}_{\ell}^{\times}$

Proposition (Cirici – Horel, 2022)

If X is defined over K , i.e. $\exists K \hookrightarrow \mathbb{C}$ and $\exists \mathcal{X}$ a complement of a hyperplane arrangement over K s.t. $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^{\bullet}(X_{an}, \mathbb{Z}_{\ell})$ is gauge $(s - 1)$ -formal.

Proposition (Cirici – Horel, 2022)

If X is defined over K , i.e. $\exists K \hookrightarrow \mathbb{C}$ and $\exists \mathcal{X}$ a complement of a hyperplane arrangement over K s.t. $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^\bullet(X_{an}, \mathbb{Z}_\ell)$ is gauge $(s-1)$ -formal.

Heuristic :

- $C^\bullet(X_{an}, \mathbb{Z}_\ell) \cong C_{et}^\bullet(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ [Artin]
- The action of a Frobenius on $H_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ is σ_q , [Kim, 1994].

Formality descent $\implies C^\bullet(X_{an}, \mathbb{Z}_{(\ell)})$ is gauge $(s-1)$ -formal.

Automorphism lifts

(A, ϕ) : a dg $\mathcal{P}$ -algebra s.t. $H(A)$ is a homotopy retract

Theorem (E., 2023)

Suppose that $\exists u \in \text{Aut}(H(A), \varphi_*)$ admitting a cochain lift. Let

$$\text{Ad}_u : \text{End}_{H(A)} \rightarrow \text{End}_{H(A)}$$

s.t. $\forall p \in \mathbb{N}$, and $\forall \psi \in \text{End}_{H(A)}(p) = \text{Hom}(H(A)^{\otimes p}, H(A))$

$$\text{Ad}_u(p)(\psi) = u \circ \psi \circ (u^{-1})^{\otimes p}.$$

1. If $\text{Ad}_u - \text{id}$ is invertible, then (A, ϕ) is gauge formal.
2. If $\text{Ad}_u - \text{id}$ is invertible on the elements of degree k for all $k < n$, then (A, ϕ) is gauge n -formal.

Automorphism lifts

R : a characteristic zero field

(A, ϕ) : a dg $\mathcal{P}$ -algebra s.t. $H(A)$ is a homotopy retract and finite dimensional.

Corollary

Suppose that there exists $u \in \text{Aut}(H(A), \varphi_)$ such that for all $k < n$, and all p -tuples $(k_1, \dots, k_p)$,*

$$\text{Spec}(u_{k_1+\dots+k_p+k}) \cap \text{Spec}(u_{k_1} \otimes \dots \otimes u_{k_p}) = \emptyset ,$$

where $u_i := u|_{H_i(A)}$. If u admits a lift at the level of chains then (A, ϕ) is gauge n -formal.

Frobenius & Weil numbers

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

ℓ : a prime number different from p

X : a smooth proper K -scheme

Definition

$\alpha \in \overline{\mathbb{Q}}_\ell$ is a **Weil number of weight n** if

$$\forall \iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}, \quad |\iota(\alpha)| = q^{n/2}.$$

Theorem (Deligne, 1974)

For all n , the eigenvalues of a Frobenius action on $H_{\text{et}}^n(X_{\overline{K}}, \mathbb{Q}_\ell)$ are Weil numbers of weight n .

Corollary

For every smooth proper K -scheme X , $C^\bullet(X_{an}, \mathbb{Q}_\ell)$ is formal.

Proof.

- $C^\bullet(X_{an}, \mathbb{Q}_\ell) \xrightarrow{\sim} C_{et}^\bullet(X_{\overline{K}}, \mathbb{Q}_\ell)$
- Let u be the Frobenius action on $H_{et}^\bullet(X_{\overline{K}}, \mathbb{Q}_\ell)$ and fix $\iota : \overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$.
- For all $k \geq 1$, $(k_1, \dots, k_p)$ and $s := k_1 + \dots + k_p$,

$$\begin{array}{ccc}
 \text{Spec}(u_{s+k}) & \cap & \text{Spec}(u_{k_1} \otimes \dots \otimes u_{k_p}) = \emptyset . \\
 \color{red}{\Psi} & & \color{blue}{\Psi} \\
 \color{red}{\alpha} & & \color{blue}{\beta} \\
 |\iota(\alpha)| = q^{\frac{s+k}{2}} & > & |\iota(\beta)| = q^{\frac{s}{2}}
 \end{array}$$



Previous works: [Deligne, 1980], [GSNPR, 2005]



Thank you for your attention!

