Obstruction theory to formality and homotopy equivalences

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Topology seminar

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The notion of formality

Formal topological spaces

R : commutative ground ring

Definition

A topological space X is formal if there exists a zig-zag of quasi-isomorphisms of dga algebras,

$$C^\bullet_{\text{sing}}(X;R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^\bullet_{\text{sing}}(X;R)\ .$$

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$\rightarrow$ Origins in rational homotopy theory (for $\mathbb{Q} \subset R$)

X formal $\implies$ The cohomology ring $H^\bullet_{\mathrm{sing}}(X,\mathbb{Q})$ completely determines the rational homotopy type of X.

Examples

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [Deligne, Griffiths, Morgan & Sullivan, 1975]

Formality of an algebraic structure

A : chain complex over R

$\mathscr{P}$: colored operad or properad

$\phi : \mathscr{P} \to \mathrm{End}_A$: a dg $\mathscr{P}$-algebra structure

Definition

The dg $\mathscr{P}$-algebra (A, ϕ) is formal if

$$\exists\, (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \varphi_*) \ ,$$

where φ_* is the canonical $\mathscr{P}$-algebra structure on $H(A)$.

Examples

- X is formal $= (C^{\bullet}_{\mathrm{sing}}(X; R), \cup)$ is formal as dga algebra
- $C(\mathcal{D}_k; \mathbb{R})$ is formal as an operad [Kontsevich, 1999]

Examples

1. Algebraic geometry [Drummond-Cole–Horel, 2021, E., 2024]
   Étale cohomology of complements of hyperplane arrangements with coefficients in $\mathbb{Z}_{(\ell)}$
2. Representation theory [Riche–Soergel–Williamson, 2014]
   The extensions of parity sheaves on the flag variety
3. Riemannian geometry [Amann–Kapovich, 2012]
   Compact positive quaternionic–Kähler manifold
4. Mathematical physics [Kontsevich's quantization theorem, 1997]
   Hochschild complex of a polynomial algebra
5. Rational homotopy theory [Miller, 1979]
   De Rham algebra of highly connected manifolds
6. Contact geometry [Biswas–Fernández–Muñoz–Tralle, 2015]
   Sasakian manifold

Examples

1. Algebraic geometry [Drummond-Cole–Horel, 2021, E., 2024]
   Étale cohomology of complements of hyperplane arrangements
   $\implies$ Purity and Formality descent

2. Representation theory [Riche–Soergel–Williamson, 2014]
   Extensions of parity sheaves on the flag variety $\implies$ Purity

3. Mathematical physics [Kontsevich's quantization theorem, 1997]
   Hochschild complex of a polynomial algebra $\implies$ Intrinsic formality

4. Riemannian geometry [Amann–Kapovich, 2012]
   Compact positive quaternionic–Kähler manifold $\implies$ Domination

5. Rational homotopy theory [Miller, 1979]
   Highly connected manifolds $\implies$ Highly connected formality

6. Contact geometry [Biswas–Fernández–Muñoz–Tralle, 2015]
   Sasakian manifold $\implies$ Formality in fibrations

Purity implies formality

(A, ϕ) : dg $\mathscr{P}$-algebra encoded by an operad $\mathscr{P}$

α : unit of infinite order in R

σ_α : the degree twisting by α = automorphism of $(H(A), \varphi_1)$ which acts via $\alpha^k\times$ on $H^k(A)$.

Theorem

If σ_α admits a chain-level lift, i.e. $\exists\, f \in \mathrm{End}(A, \phi)$ s.t. $H(f) = \sigma_\alpha$, then (A, ϕ) is formal.

→ Deligne–Griffiths–Morgan–Sullivan [1975]

→ Sullivan [1977]

→ Guillén Santos–Navarro–Pascual–Roig [2005]

→ Cirici–Horel [2022]

Example

1. Algebraic geometry [Cirici-Horel, 2018, Drummond-Cole–Horel, 2021]

X : a complement of a hyperplane arrangement over $\mathbb{C}$ defined over a finite extension K of $\mathbb{Q}_p$.

ℓ : a prime number different from p.

$\rightarrow C^\bullet_{\mathrm{sing}}(X_{an}, \mathbb{Z}_\ell) \cong C^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ [Artin].

$\rightarrow$ A Frobenius action on $H^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ is σ_q [Kim, 1994].

Questions

- Can we descend these results to other coefficient rings? (e.g. $\mathbb{Z}_{(\ell)}$)
- Does the degree twisting criteria hold for other types of algebras? (e.g. Hopf algebras, Lie bialgebras,...)
- Is the degree twisting the only homology automorphism satisfying this property?
- Can we incorporate all the aformentionned criteria into a single theory and generalize them to any coefficient ring / other types of algebras?

Outline

1. Higher structures
   - Formality can be addressed as a deformation problem

2. Obstruction theory
   - Kaledin classes
   - Obstruction sequences to homotopy equivalences

3. Formality criteria
   - Purity
   - Formality descent
   - Intrinsic formality
   - Formality under domination
   - Highly connected formality
   - Formality in fibrations

Higher structures

Homotopy retracts

Definition

(W, d_W) is a homotopy retract of (V, d_V) if there are maps

$$h \circlearrowright (V, d_V) \underset{i}{\overset{p}{\rightleftarrows}} (W, d_W)$$

where $\mathrm{id}_V - ip = d_V h + h d_V$ and i is a quasi-isomorphism .

Proposition

If R is a field, the cohomology of any cochain complex is a homotopy retract:

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H(A), 0) \; .$$

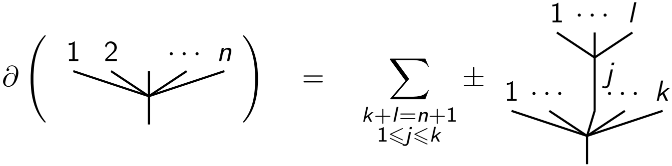
Homotopy associative algebras

Definition (Stasheff, 1963)

A_∞-algebra: a cochain complex H with a collection of maps

$$\varphi_n : H^{\otimes n} \to H$$

of degree $2-n$, for all $n \geqslant 2$, which satisfy the relations



Examples

- Every dga algebra (A, ϕ) is an A_∞-algebra with $\varphi_n = 0$ for all $n \geqslant 3$.

Homotopy transfer theorem

Theorem (Kadeishvili, 1982)

Given a dga algebra (A, d_A, ϕ) *and a homotopy retract*

$$h \circlearrowright (A, d_A, \phi) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

there exists an A_∞*-algebra structure on* H *such that* p *(and* i*) extend to* A_∞*-quasi-isomorphisms:*

$$(A, d_A, \phi) \rightsquigarrow (H, d_H, \varphi_2, \varphi_3, \varphi_4, \dots)$$

Homotopy morphisms

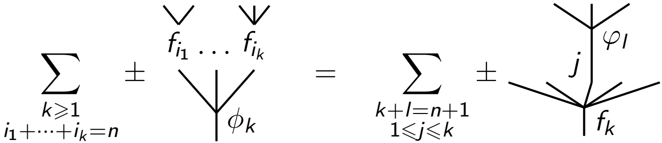
$(A, d_A, \phi_2, \ldots), \ (H, d_H, \varphi_2, \ldots) : \ A_\infty$-algebras

Definition

A_∞-morphism $f : A \rightsquigarrow H$ is a collection of linear maps

$$f_n : A^{\otimes n} \longrightarrow H, \quad n \geqslant 1 \ ,$$

of degree $1 - n$, which satisfy the relations



where $\varphi_1 = d_H$ and $\phi_1 = d_A$.

Homotopy quasi-isomorphisms

Definition

A_∞-quasi-isomorphism $f : A \xrightarrow{\sim} H$ is an A_∞-morphism where $f_1 : A \to H$ is a quasi-isomorphism .

Proposition (R is a field)

quasi-isos of associative algebras $\qquad$ A_∞*-quasi-iso*

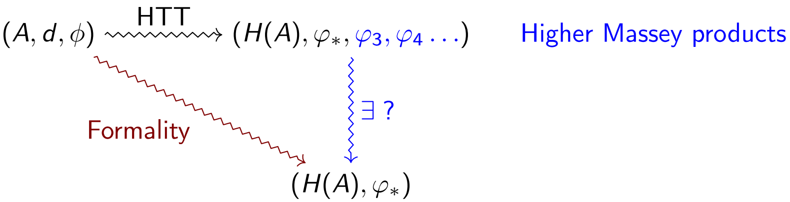
$$\exists\,(A,\phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B,\phi') \iff \exists\,(A,\phi) \overset{\sim}{\rightsquigarrow} (B,\phi')$$

Corollary

A dga algebra (A,ϕ) is formal if and only if

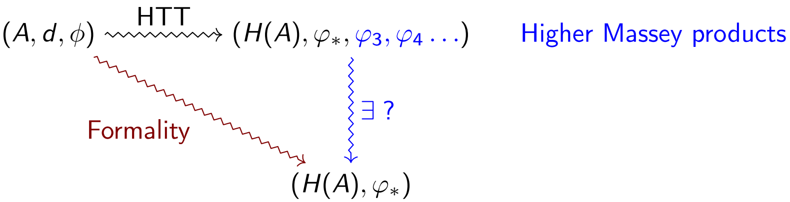
$$\exists\,(A,\phi) \overset{\sim}{\rightsquigarrow} (H(A),\varphi_*)\ .$$

An equivalent characterization of formality



$\Longrightarrow$ If the higher Massey products vanish, then (A, d, ϕ) is formal.

An equivalent characterization of formality



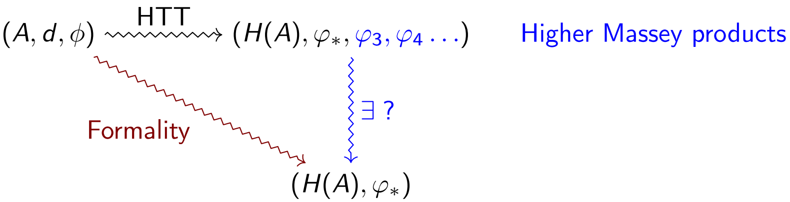
$\implies$ If the higher Massey products vanish, then (A, d, ϕ) is formal.

Definition

A dga (A, ϕ) is

- gauge formal if $\exists$ $(A, \phi) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_*)$.
- gauge n-formal if $\exists$ $(A, \phi) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_*, 0, \ldots, 0, \psi_{n+1}, \ldots)$.
- of n-depth if $\exists$ $(A, \phi) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_*, \psi_3, \ldots, \psi_n, 0, \ldots)$.

An equivalent characterization of formality



$\Longrightarrow$ If the higher Massey products vanish, then (A, d, ϕ) is formal.

Definition

A dg $\Omega\mathscr{C}$-algebra (A, d, ϕ) over a reduced conilpotent coproperad $\mathscr{C}$ is

- gauge formal if $\exists\ (A, \phi) \ \stackrel{\sim}{\rightsquigarrow}\ (H(A), \varphi_*)$.
- gauge *n*-formal if $\exists\ (A, \phi) \ \stackrel{\sim}{\rightsquigarrow}\ (H(A), \varphi_*, 0, \ldots, 0, \psi_{n+1}, \ldots)$.
- of *n*-depth if $\exists\ (A, \phi) \ \stackrel{\sim}{\rightsquigarrow}\ (H(A), \varphi_*, \psi_3, \ldots, \psi_n, 0, \ldots)$.

Obstruction classes

Kaledin classes

The convolution dg Lie algebra associated to $H(A)$:

$$\mathfrak{g} := \left(\mathrm{Hom}\left(\overline{\mathscr{C}}, \mathrm{End}_{H(A)}\right), [-,-], d\right)$$

Kaledin classes:

$$K_\Phi := \left[\varphi_2 + 2\varphi_3\hbar + 3\varphi_4\hbar^2 + \cdots\right] \in H^1\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right)$$

$$K_\Phi^n := \left[\varphi_2 + 2\varphi_3\hbar + \cdots + n\varphi_{n+1}\hbar^{n-1}\right] \in H^1\left(\left(\mathfrak{g}[\![\hbar]\!]/\hbar^n\right)^{\widetilde{\Phi}}\right) .$$

Theorem (E., 2024)

Let (A,ϕ) be an $\Omega\mathscr{C}$-algebra that admits a transferred structure.

- *If R is a $\mathbb{Q}$-algebra, (A,ϕ) is gauge formal $\iff K_\Phi = 0$.*
- *If $n!$ is invertible in R, (A,ϕ) is gauge n-formal $\iff K_\Phi^n = 0$.*

Previous works: [Kaledin, 2007], [Lunts, 2007], [Melani–Rubió, 2019]

Homotopy equivalences between algebraic structure

Definition

The dg $\Omega\mathscr{C}$-algebras (A,ϕ) and (B,ψ) are

- homotopy equivalent

$$\exists\,(A,\phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B,\psi)$$

- gauge homotopy equivalent if $\exists\,(A,\phi) \overset{\sim}{\rightsquigarrow} (B,\psi)$.
- gauge n-homotopy equivalent if (A,ϕ) is gauge homotopy equivalent to an $\Omega\mathscr{C}$-algebra (B,φ) such that

$$\varphi - \psi \in \mathcal{F}^{n+1}\mathfrak{g}\ .$$

Example

(A,ϕ) is formal $\iff$ it is homotopy equivalent to $(H(A),\varphi_1)$

Obstruction sequences to homotopy equivalences

Let (A, ϕ) and (B, ψ) be two $\Omega\mathscr{C}$-algebras admitting transferred structures and such that $H(A) \cong H(B)$.

$\rightarrow$ obstruction sequence $(\vartheta_k)_{1 \leqslant k \leqslant m}$ which is either

- an infinite sequence of vanishing classes, when $m = \infty$;
- a finite sequence of trivial classes that ends on $\vartheta_m \neq 0$.

Proposition

The index $m \in [\![1, \infty]\!]$ only depends on φ and ψ: this is their homotopy equivalence degree.

Theorem (E., 2025)

Let (A, ϕ) and (B, ψ) be two $\Omega\mathscr{C}$-algebras admitting transferred structures and such that $H(A) \cong H(B)$.

1. *If R is a $\mathbb{Q}$-algebra, the algebras are gauge k-homotopy equivalent for all k if and only if $m = \infty$.*
2. *If $m!$ is invertible in R, the algebras are gauge $(m-1)$-homotopy equivalent but not gauge m-homotopy equivalent if and only if $m \in \mathbb{N}$.*

Formality criteria

Formality descent

(A, ϕ) : an dg $\Omega\mathscr{C}$-algebra that admits a transferred structure

$H^i(A)$: projective, finitely generated for all i.

S : faithfully flat commutative R-algebra.

Theorem (E., 2024)

1. (A, ϕ) *gauge* n*-formal* $\Longleftrightarrow (A \otimes_R S, \phi \otimes 1)$ *gauge* n*-formal.*
2. (A, ϕ) *is of depth* $n \Longleftrightarrow (A \otimes_R S, \phi \otimes 1)$ *if of depth* n*.*

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Examples

- D_k : little k-disks operads [Guillén Santos–Navarro–Pascual–Roig]
  $C(\mathcal{D}_k;\mathbb{R})$ is formal $\Longleftrightarrow$ $C(\mathcal{D}_k;\mathbb{Q})$ is formal
- $\mathbb{Z}_{(\ell)} \subset \mathbb{Z}_\ell$

Formality descent

X : a complement of an hyperplane arrangement over $\mathbb{C}$
$\rightarrow$ complement of a finite collection of affine hyperplanes in $\mathbb{A}^n_{\mathbb{C}}$.

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

ℓ : a prime number different from p

s : order of q in $\mathbb{F}_\ell^\times$

Theorem (Cirici–Horel, 2018, Dummond-Cole–Horel, 2021)
If X is defined over K then $C^\bullet(X_{an}, \mathbb{Z}_\ell)$ is gauge $(s-1)$-formal.

Formality descent

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Theorem (E., 2024)

If X is defined over K then $C^\bullet(X_{an}, \mathbb{Z}_{(\ell)})$ is gauge $(s-1)$-formal.

Proof.

Formality descent □

Automorphism lifts

R : a field

(A, ϕ) : a dg $\Omega\mathscr{C}$-algebra that admits a transferred structure

$H^i(A)$: projective, finitely generated for all i.
$H(A)$ is finite dimensional.

Theorem (E., 2024)

Suppose that there exists $u \in \mathrm{Aut}(H(A), \varphi_1)$ such that for all $k < n$, and all p-tuples $(k_1, \ldots, k_p)$,

$$\mathrm{Spec}(u_{k_1+\cdots+k_p+k}) \cap \mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p}) = \varnothing \ ,$$

where $u_i := u_{|H^i(A)}$. If u admits a lift at the level of chains then (A, ϕ) is gauge n-formal.

Frobenius & Weil numbers

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers $\mathcal{O}_K$

ℓ : a prime number different from p

X : a smooth proper K-scheme

Definition

$\alpha \in \overline{\mathbb{Q}}_\ell$ is a Weil number of weight n if

$$\forall\, \iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}, \quad |\iota(\alpha)| = q^{n/2} \ .$$

Theorem (Deligne, 1974)

For all n, the eigenvalues of a Frobenius action on $H^n_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ are Weil numbers of weight n.

Let Sch_K be the category of smooth and proper schemes over K of *good reduction*, i.e. for which there exists a smooth and proper model over $\mathcal{O}_K$.

Theorem (E., 2024)

Let $\mathbb{V}$ be a groupoid and let $\mathscr{P}$ be a $\mathbb{V}$-colored operad in sets. Let X be a $\mathscr{P}$-algebra in Sch_K. The dg $\mathscr{P}$-algebra $C_\bullet(X_{\mathrm{an}}, \mathbb{Q})$ is formal.

Example (Guillén Santos–Navarro–Pascual–Roig, 2005)

Let $\overline{\mathcal{M}}$ the cyclic operad of moduli spaces of stable algebraic curves of genus zero. The cyclic operad $C_\bullet(\overline{\mathcal{M}}_{an}; \mathbb{Q})$ is formal.

Intrinsic formality

A graded $\Omega\mathscr{C}$-algebra (H, φ_*) is intrinsically formal if every dg $\Omega\mathscr{C}$-algebra (A, ϕ) such that $(H(A), \varphi_1) \cong (H, \varphi_*)$ is gauge formal.

The convolution dg Lie algebra associated to $H(A)$:

$$\mathfrak{g}_{H(A)} := \left(\mathrm{Hom}\left(\overline{\mathscr{C}}, \mathrm{End}_{H(A)}\right), [-,-], d\right)$$

Theorem (E., 2024)

$\mathfrak{g}^{\varphi_*}$: $(\mathfrak{g}_{H(A)}, [-,-], d + [\varphi_*, -])$

$$H^1(\mathfrak{g}^{\varphi_*}) = 0 \implies (H, \varphi_*) \textit{ intrinsically formal.}$$

Previous works: [Hinich, 2003] → Tamarkin's proof of Kontsevich formality

Intrinsic formality

Example (Kontsevich–Takeda–Vlassopoulos, 2021)

$A = C_*(\Omega S^n; R)$ has an pre-Calabi-Yau (or V_∞-algebra) structure

$$\phi = \underbrace{m_{(1)}}_{A_\infty\text{-alg}} + \underbrace{m_{(2)}}_{\substack{\text{Poisson bivector} \\ \text{up to homotopy}}} + m_{(3)} + \cdots$$

where $m_{(\ell)}$ is a cyclically anti-symmetric collection of maps

$$m_{(\ell)}^{k_1,\dots,k_\ell} : sA^{k_1} \otimes \cdots \otimes sA^{k_\ell} \to A^\ell \ .$$

$\implies$ encodes Poincaré duality.

Intrinsic formality

The pre-CY algebra on $C_*(\Omega S^n; R)$ has vanishing copairing:

$$m_{(2)}^{0,0} = 0$$

$\to$ always the case where A is connective and $n \geqslant 1$.

Theorem (E.-Takeda, 2025)

If R is a $\mathbb{Q}$-algebra, $(H(A), \varphi_1)$ is intrinsically formal as an n-pre-CY algebra structure with vanishing copairing.

Formality under domination

Theorem (Milivojević–Stelzig–Zoller, 2023)

X : *closed orientable manifold (or rational Poincaré duality space)*

Y : *manifold (or space)*

f : *a map* $Y \to X$ *that hits a fundamental class*

If Y *is formal then* X *is formal.*

Formality under domination

Theorem (Milivojević–Stelzig–Zoller, 2023)

X : closed orientable manifold (or rational Poincaré duality space)

Y : manifold (or space)

f : a map $Y \to X$ that hits a fundamental class

If Y is formal then X is formal.

Examples

1. Equidimensional manifolds with a degree-zero map.
2. f is a finite ramified covering map.
3. f is an orientable fibration $F \to M \to N$ s.t. $H(M) \twoheadrightarrow H(F)$.

   $\implies$ With their twistor fibration, compact positive quaternion Kähler manifolds are formal [Amann–Kapovich, 2012].

Formality under domination

Theorem (Milivojević–Stelzig–Zoller, 2023)

$f : (A, \phi) \xrightarrow{\sim} (B, \psi)$ *an* ∞*-morphism between* A_∞*-algebras.*

- *R is a characteristic zero field,*
- *f has a homotopy A-bimodule retract, i.e.* $\exists\, r : B \to A$ *of* A_∞*-A-bimodules such that* $H(r_{0,0} \circ f_1) = \mathrm{Id}$*.*

If (B, ψ) *is formal then* (A, ϕ) *is formal.*

Example

Formality descent $f : (A, \phi) \hookrightarrow (A \otimes_R S, \psi \otimes 1)$

Formality under domination

Theorem (E, 2025)

$f : (A, \phi) \overset{\sim}{\rightsquigarrow} (B, \psi)$ an ∞-morphism between $\Omega\mathscr{C}$-algebras that admit transferred structures

- *R is a ring*
- *f has a homotopy A-module retract.*

If (B, ψ) is gauge n-formal or of depth n then (A, ϕ) is as well.

Example

Formality descent $f : (A, \phi) \hookrightarrow (A \otimes_R S, \psi \otimes 1)$

Highly connected formality

Theorem (E., 2025)

R : a field

M^d : a compact k-connected C^∞-manifold $d \leqslant \ell k + 2$

The singular cochains $C^\bullet_{\text{sing}}(M, R)$ are of depth ℓ.

Theorem (E., 2025)

q : a prime number

K : a separately closed field in which q is invertible

X^d : a k-connected irreducible, proper, smooth variety over K
$d \leqslant \ell k + 2$

The étale cochaines $C^\bullet_{\text{et}}(X, \mathbb{F}_q)$ are of depth ℓ.

Highly connected formality

Theorem (E., 2025)

R : a field

M^d : a compact k-connected C^∞-manifold $d \leqslant \ell k + 2$

The singular cochains $C^\bullet_{\mathrm{sing}}(M, R)$ are of depth ℓ.

Theorem (E., 2025)

q : a prime number such that

K : a separately closed field in which q is invertible

X^d : a k-connected irreducible, proper, smooth variety over K
$d \leqslant \ell k + 2$

The étale cochaines $C^\bullet_{\mathrm{et}}(X, \mathbb{F}_q)$ are of depth ℓ.

Formality in fibrations

Theorem (Biswas–Fernández–Muñoz–Tralle, 2015)

(M, g) : a Sasakian manifold, i.e. Riemannian manifold such that $M \times \mathbb{R}$ equipped with the cone metric $t^2 g + dt^2$ is Kähler.

Then $\Omega^*(M)$ is of depth 3. In particular, there exists a simply connected K-contact non-Sasakian manifold.

Proof.

A cdga with $A^i = 0$ for $i \notin [0, 2n]$

$\omega \in A^2$: nondegenerate with hard Lefschetz property

If A is formal, then $A \otimes \Lambda(y)$ with $d(y) = \omega$ is of depth 3. □

Formality in fibrations

Theorem (Zhou, 2023)

M : a formal manifold

$\omega \in \Omega^*(M)$ an even-dimensional integral differential form

X : a sphere bundle over M with Euler class ω

1. *If ω is exact then X is formal.*
2. *If ω is non-exact then $\Omega^*(X)$ is of depth 3.*

Proof.
If (A, ϕ) is formal and $\omega = d\theta$ with ω even, then $A[\theta] = A \oplus \theta A$ is of depth 3. □

Thank you for your attention!