Operadic calculus and formality criteria

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The notion of formality

Formal topological spaces

R : commutative ground ring

Definition

A topological space X is formal if there exists a zig-zag of quasi-isomorphisms of dga algebras,

$$C^\bullet_{\mathrm{sing}}(X;R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^\bullet(X;R)\ .$$

→ Origins in rational homotopy theory (for $R \subset \mathbb{Q}$)

X formal $\implies$ $H^\bullet(X,\mathbb{Q})$ completely determines the rational homotopy type of the space.

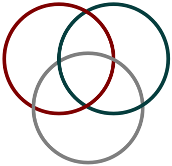
Examples

→ Formal spaces

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [DGMS, 1975]

→ Nonformal spaces

- The complement of the Borromean rings



Formality of an algebraic structure

A : cochain complex over R

(A, ν) : differential graded algebraic structure over A, e.g.

- $\rightarrow$ a dg associative algebra,
- $\rightarrow$ a dg Lie algebra,
- $\rightarrow$ a dg operad,
- $\rightarrow$ $\cdots$

Definition

The dg algebra (A, ν) is formal if

$$\exists\, (A, \nu) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \overline{\nu}) \ .$$

Examples

- X is formal $\Longleftrightarrow (C^{\bullet}_{\mathrm{sing}}(X; R), \cup)$ is formal as dga algebra
- $C^{\bullet}_{\mathrm{sing}}(\mathcal{D}_k; \mathbb{R})$ is formal as an operad, where $\mathcal{D}_k$ is the little k-discs operad [Kontsevich, 1999]

Higher structures & operadic calculus

Homotopy retracts

Definition

(W, d_W) is a homotopy retract of (V, d_V) if there are maps of cochain complexes

$$h \circlearrowright (V, d_V) \underset{i}{\overset{p}{\rightleftarrows}} (W, d_W)$$

where $\mathrm{id}_V - ip = d_V h + h d_V$ and i is a quasi-isomorphism

Proposition

If R is a field, any chain complex admits its cohomology as a homotopy retract

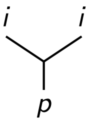
$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H(A), 0)$$

Transfer of algebraic structure

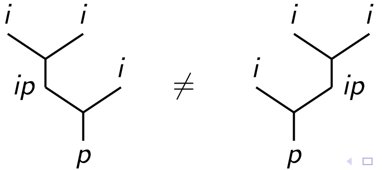
→ (A, d, ν) a dga algebra and a homotopy retract:

$$h \circlearrowright (A, d_A, \nu) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

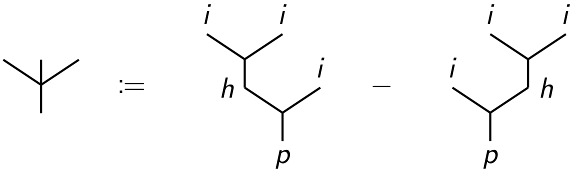
→ Transferred product: $\mu_2 := p \circ \nu \circ i^{\otimes 2} : H^{\otimes 2} \to H$



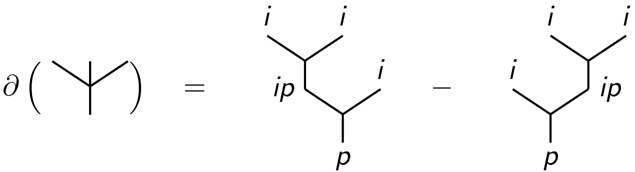
Not associative in general!



→ Consider $\mu_3 : H^{\otimes 3} \to H$

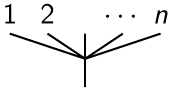


→ In $\mathrm{Hom}(H^{\otimes 3}, H)$:

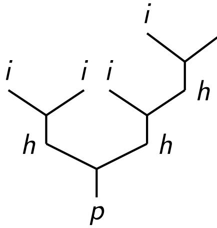


→ μ_2 is associative up to the homotopy μ_3.

→ $\mu_n : H^{\otimes n} \to H$, for all $n \geq 2$



$:= \sum_{\mathrm{PBT}_n} \pm$



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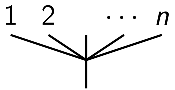
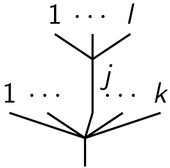


$\Big) = \sum_{\substack{k+l=n+1 \\ 1 \leq j \leq k}} \pm$



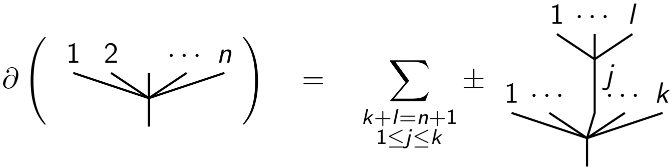
Homotopy associative algebras

Definition

A_∞-algebra: a chain complex H with a collection of maps

$$\mu_n : H^{\otimes n} \to H$$

of degree $n-2$, for all $n \geq 2$, which satisfy the relations



Examples

- Every dga algebra (A, ν) is an A_∞-algebra with $\mu_n = 0$ for all $n \geq 3$.
- $(H, d_H, \mu_2, \mu_3, \dots)$

Homotopy transfer theorem

Theorem (Kadeishvili, 1982)

Given a dga algebra (A, d_A, ν) and a homotopy retract

$$h \circlearrowright (A, d_A, \nu) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

there exists an A_∞-algebra structure on H such that p (and i) extend to A_∞-quasi-isomorphisms:

$$p_\infty : (A, d_A, \nu) \overset{\sim}{\rightsquigarrow} (H, d_H, \mu_2, \mu_3, \dots)$$

Homotopy morphisms

$(A, d_A, \nu_2, \dots), (H, d_H, \mu_2, \dots) : A_\infty$-algebras

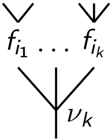
Definition

A_∞-morphism $f : A \rightsquigarrow H$ is a collection of linear maps

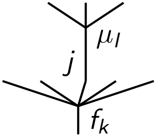
$$f_n : A^{\otimes n} \longrightarrow H, \quad n \geq 1 ,$$

of degree $n-1$, which satisfy the relations

$$\sum_{\substack{k\geq 1 \\ i_1+\cdots+i_k=n}} \pm$$



$$= \sum_{\substack{k+l=n+1 \\ 1\leq j\leq k}} \pm$$



where $\mu_1 = d_H$ and $\nu_1 = d_A$.

Homotopy quasi-isomorphisms

Definition

A_∞-quasi-isomorphism $f : A \overset{\sim}{\rightsquigarrow} H$ is an A_∞-morphism where $f_1 : A \to H$ is a quasi-isomorphism.

Proposition

A, H dga algebras

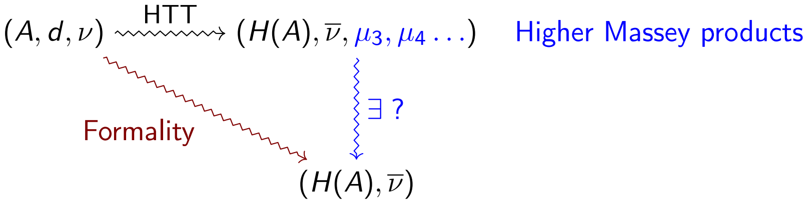
$$\exists\, A \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H \iff \exists\, A \overset{\sim}{\rightsquigarrow} H$$

Corollary

A dga algebra (A, d, ν) is formal $\iff \exists\, (A, d, \nu) \overset{\sim}{\rightsquigarrow} (H(A), 0, \overline{\nu})$.

An equivalent characterization of formality

(A, d, ν) a dga algebra such that $H(A)$ is a homotopy retract



$\Longrightarrow$ If the higher Massey products vanish, then (A, d, ν) is formal.

Definition

- (A, d, ν) is formal if $\exists\ (H(A), \overline{\nu}, \mu_3, \mu_4 \ldots) \overset{\sim}{\rightsquigarrow} (H(A), \overline{\nu})$.
- (A, d, ν) is n-formal if $\exists\ (H(A), \overline{\nu}, \mu_3, \mu_4 \ldots) \overset{\sim}{\rightsquigarrow} (H(A), \overline{\nu}, 0, \ldots, 0, \mu'_{n+1}, \ldots)$.

There is nothing special with the dga algebra structure!

Examples

- dg commutative algebras and C_∞-algebras
- dg Lie algebras and L_∞-algebras
- dg Frobenius algebras and $Frobenius_\infty$-algebras
- dg Lie bialgebras and $LieBi_\infty$-algebras
- ...
- dg $\mathcal{P}$-algebra and $\mathcal{P}_\infty$-algebras, for any Koszul (pr)operad $\mathcal{P}$.

$\Longrightarrow$ Operadic calculus provides a unified framework to deal with all types of algebraic structures

Formality criteria

The degree twisting

(A, d, ν) : a dg $\mathcal{P}$-algebra s.t. $H(A)$ is a homotopy retract

α : a unit in R.

σ_α : the degree twisting by α

$\rightarrow$ linear automorphism of $H(A)$ which acts via $\alpha^k \times$ on $H^k(A)$.

Theorem (Drummond-Cole – Horel, 2021)

Suppose that σ_α admits a lift, i.e. $\exists\, f \in \mathrm{End}(A, \nu)$ s.t. $H(f) = \sigma_\alpha$.

- $\forall k,\ \alpha^k - 1 \in R^\times \implies (A, d, \nu)$ *is formal.*
- $\forall k \leq n,\ \alpha^k - 1 \in R^\times \implies (A, d, \nu)$ *is n-formal.*

Theorem (Drummond-Cole – Horel, 2021)

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Heuristic :

→ Higher Massey products have to be compatible with the lift.

→ They intertwine multiplication by α^l with multiplication by α^k with $l \neq k$.

→ They have to vanish

Complement of subspace arrangements

X : a complement of a hyperplane arrangement over $\mathbb{C}$
$\rightarrow$ complement of a finite collection of affine hyperplanes in $\mathbb{A}^n_{\mathbb{C}}$.

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

l : a prime number different from p

h : order of q in $\mathbb{F}_l^\times$

Proposition

If X is defined over K, i.e. $\exists\ K \hookrightarrow \mathbb{C}$ and $\exists\ \mathcal{X}$ a complement of a hyperplane arrangement over K s.t. $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^\bullet(X_{an}, \mathbb{Z}_l)$ is $(h-1)$-formal.

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Heuristic :

- → $C^\bullet(X_{an}, \mathbb{Z}_l) \cong C^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_l)$.
- → The action of a Frobenius on $H_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_l)$ is σ_q, [Kim, 1994].

Thank you for your attention!