GENERALIZED NEHARI MANIFOLD AND SEMILINEAR SCHRÖDINGER EQUATION WITH WEAK MONOTONICITY CONDITION ON THE NONLINEAR TERM

FRANCISCO ODAIR DE PAIVA, WOJCIECH KRYSZEWSKI, AND ANDRZEJ SZULKIN

Abstract. We study the Schrödinger equations $-\Delta u + V(x)u = f(x,u)$ in $\mathbb{R}^N$ and $-\Delta u - \lambda u = f(x,u)$ in a bounded domain $\Omega \subset \mathbb{R}^N$. We assume that f is superlinear but of subcritical growth and $u \mapsto f(x,u)/|u|$ is nondecreasing. In $\mathbb{R}^N$ we also assume that V and f are periodic in $x_1, \dots, x_N$. We show that these equations have a ground state and that there exist infinitely many solutions if f is odd in u. Our results generalize those in [11] where $u \mapsto f(x,u)/|u|$ was assumed to be strictly increasing. This seemingly small change forces us to go beyond methods of smooth analysis.

1. Introduction

We consider the semilinear Schrödinger equations

$$-\Delta u + V(x)u = f(x,u), \quad u \in H^1(\mathbb{R}^N) \tag{1.1}$$

and

$$-\Delta u - \lambda u = f(x,u), \quad u \in H^1_0(\Omega), \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $H^1(\mathbb{R}^N)$, $H^1_0(\Omega)$ are the usual Sobolev spaces. In both problems we make the following assumptions on f:

- (F_1) f is continuous and $|f(x,u)| \le C(1+|u|^{p-1})$ for some $C > 0$ and $p \in (2, 2^*)$, where $2^* := 2N/(N-2)$ if $N \ge 3$ and $2^* := +\infty$ if $N = 1$ or 2,
- (F_2) $f(x,u) = o(u)$ uniformly in x as $u \to 0$,
- (F_3) $F(x,u)/u^2 \to \infty$ uniformly in x as $|u| \to \infty$, where $F(x,u) := \int_0^u f(x,s)\,ds$,
- (F_4) $u \mapsto f(x,u)/|u|$ is non-decreasing on $(-\infty, 0)$ and on $(0, \infty)$.

The assumptions (F_1)–(F_3) appear in [11] while a condition corresponding to (F_4) is a little stronger there:

- (F_4') $u \mapsto f(x,u)/|u|$ is strictly increasing on $(-\infty, 0)$ and on $(0, \infty)$.

2010 *Mathematics Subject Classification.* 35J20, 35J60, 58E30.

Key words and phrases. Generalized Nehari manifold, Schrödinger equation, strongly indefinite functional, Clarke's subdifferential.

The first author was supported by FAPESP under the grant 2015/10545-0. This work was done while he was visiting the Mathematics Department of Stockholm University. He would like to thank the members of the Department for hospitality and a stimulating scientific atmosphere.

The second author was partially supported by the Polish National Science Center under grant 2013/09/B/ST1/01963.

As we shall see, this slightly weaker hypothesis will force us to go beyond methods of smooth analysis, and introducing a non-smooth approach in this context is in fact our main purpose. In what follows we shall frequently refer to different results and arguments in [11, 12]. When such reference is made, it should be understood that no stronger conditions than (F_1)–(F_4) were needed there.

The main results of this paper are the following two theorems:

Theorem 1.1. *Suppose f satisfies (F_1)–(F_4), V and f are 1-periodic in $x_1, \dots, x_N$ and $0 \notin \sigma(-\Delta + V)$, where $\sigma(\cdot)$ denotes the spectrum in $L^2(\mathbb{R}^N)$. Then equation* (1.1) *has a ground state solution. If moreover f is odd in u, then equation* (1.1) *has infinitely many pairs of geometrically distinct solutions.*

Theorem 1.2. *(i) Suppose f satisfies (F_1)–(F_4) and $\lambda \neq \lambda_k$ for any k, where λ_k is the k-th eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Then equation* (1.2) *has a ground state solution. If moreover f is odd in u, then equation* (1.1) *has infinitely many pairs of geometrically distinct solutions $\pm u_k$ such that the $L^\infty(\Omega)$-norm of u_k tends to infinity with k.*
(ii) If $\lambda = \lambda_k$ for some k, then the above results remain valid under the additional assumption that $f(x, u) \neq 0$ unless $u = 0$.

Similar results, but under the stronger condition (F_4'), have been proved in [11].

As usual, a *ground state* is a solution which minimizes the functional corresponding to the problem over the set of all nontrivial ($u \neq 0$) solutions. Later in this section we shall define what we mean by geometrically distinct solutions.

Existence of a ground state solution under the assumptions of Theorem 1.1 has been shown by S. Liu in [7]; since this result is an easy consequence of our approach, we include it here anyway. See also [16] where a number of results on ground states for problems similar to (1.1) and (1.2) has been proved and [13] where (F_4) has been further weakened. Existence of ground states for *systems* of equations has been discussed in [8]. Concerning existence of infinitely many solutions we know of a result by Tang [14] where a condition different from (F_4) has been introduced for (1.2), and by Zhong and Zou [16] where (1.1) and (1.2) have been considered under the same hypotheses as in Theorems 1.1 and 1.2. However, they needed an additional assumption which is not easy to verify unless $u \mapsto f(x, u)/|u|$ is "most times" strictly increasing.

Consider equation (1.1) under the assumptions of Theorem 1.1. Let $E := H^1(\mathbb{R}^N)$. The functional corresponding to (1.1) is

$$\Phi(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx - \int_{\mathbb{R}^N} F(x, u)\, dx.$$

It is well known (see e.g. [15]) that $\Phi \in C^1(E, \mathbb{R})$ and critical points of Φ are solutions for (1.1). Let $E = E^+ \oplus E^-$ be the decomposition corresponding to the positive and the negative part of the spectrum of $-\Delta + V$. Since $0 \notin \sigma(-\Delta + V)$, there exists an equivalent inner product $\langle .\,,.\rangle$ in E such that

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx, \tag{1.3}$$

where $u^\pm \in E^\pm$.

For equation (1.2) under the assumptions of Theorem 1.2 we put $E = H_0^1(\Omega)$ and we have the spectral decomposition $E = E^+ \oplus E^0 \oplus E^-$, where E^0 is the nullspace of $-\Delta - \lambda$ in E and $0 \le \dim(E^0 \oplus E^-) < \infty$. Also here we can choose an equivalent inner product such that the corresponding functional Φ is of the form (1.3), with $\mathbb{R}^N$ replaced by Ω.

The following set introduced by Pankov [9] is called the *generalized Nehari manifold* or the *Nehari-Pankov manifold*:

$$\mathcal{M} := \big\{u \in E \setminus (E^0 \oplus E^-) : \Phi'(u)u = 0 \text{ and } \Phi'(u)v = 0 \text{ for all } v \in E^0 \oplus E^-\big\} \tag{1.4}$$

(E^0 is necessarily trivial in Theorem 1.1). (F_4) implies $f(x,u)u \ge 0$, and the assumptions of Theorem 1.2 imply that if $\dim E^0 > 0$, then $f(x,u)u > 0$ for $u \ne 0$. Hence $\mathcal{M}$ contains all nontrivial critical points of Φ. Note that if $E^0 \oplus E^- = \{0\}$, then $\mathcal{M}$ is the usual Nehari manifold [12]. Since this case is considerably easier to handle, we assume in what follows that $\sigma(-\Delta + V) \cap (-\infty, 0) \ne \emptyset$ in Theorem 1.1 and $\lambda \ge \lambda_1$ in Theorem 1.2. As in [11], for $u \notin E^0 \oplus E^-$ we define

$$\begin{aligned} E(u) &:= E^0 \oplus E^- \oplus \mathbb{R}u = E^0 \oplus E^- \oplus \mathbb{R}u^+ \\ \text{and}\quad \widehat{E}(u) &:= E^0 \oplus E^- \oplus \mathbb{R}^+u = E^0 \oplus E^- \oplus \mathbb{R}^+u^+, \end{aligned} \tag{1.5}$$

where $\mathbb{R}^+ = [0,\infty)$. It has been shown there that if (F_4) is replaced by (F_4'), then $\widehat{E}(u)$ intersects $\mathcal{M}$ at a unique point which is the unique global maximum of $\Phi|_{\widehat{E}(u)}$. It has been shown in [16] by an explicit example that if (F_4) but not (F_4') holds, then (in the framework of Theorem 1.2) $\widehat{E}(u)$ and $\mathcal{M}$ may intersect on a finite line segment. In the next section we shall show that $\widehat{E}(u) \cap \mathcal{M} \ne \emptyset$ and if $w \in \widehat{E}(u) \cap \mathcal{M}$, then there exist $\sigma_w > 0$, $\tau_w \ge \sigma_w$ such that $\widehat{E}(u) \cap \mathcal{M} = [\sigma_w, \tau_w]w$. In other words, $\widehat{E}(u) \cap \mathcal{M}$ is either a point or a finite line segment. We also show that a point $\widetilde{w} \in [\sigma_w, \tau_w]w$ is critical for Φ if and only if the whole segment $[\sigma_w, \tau_w]w$ consists of critical points.

In Theorem 1.1 the functional Φ is invariant with respect to the action of $\mathbb{Z}^N$ given by the translations $k \mapsto u(\cdot - k)$, $k \in \mathbb{Z}^N$. Hence if u is a solution of (1.1), then so is $u(\cdot - k)$. This and the preceding paragraph justify the following definition: Two solutions u_1 and u_2 are called *geometrically distinct* if $u_2 \ne u_1(\cdot - k)$ for any $k \in \mathbb{Z}^N$ and $u_2 \notin [\sigma_{u_1}, \tau_{u_1}]u_1$. In Theorem 1.2 there is no $\mathbb{Z}^N$-invariance but we still want to identify solutions in $\widehat{E}(u) \cap \mathcal{M}$. So u_1, u_2 are *geometrically distinct* if $u_2 \notin [\sigma_{u_1}, \tau_{u_1}]u_1$.

2. Preliminaries

In this section we assume that the hypotheses of Theorem 1.1 or 1.2 are satisfied. In particular, (F_1)–(F_4) hold. To simplify notation, Ω will stand for $\mathbb{R}^N$ or for a bounded domain in $\mathbb{R}^N$.

Lemma 2.1. *If $f(x,u) \neq 0$, then $F(x,u) < \frac{1}{2}f(x,u)u$.*

Proof. Suppose $u > 0$. Since $f(x,t)/t \to 0$ as $t \to 0$ and $f(x,u)/u > 0$,

$$F(x,u) = \int_0^u \frac{f(x,t)}{t}\, t\, dt < \frac{f(x,u)}{u} \int_0^u t\, dt = \frac{1}{2} f(x,u)u.$$

For $u < 0$ the proof is similar. □

The following result will be crucial for studying the structure of the set $\widehat{E}(u) \cap \mathcal{M}$.

Proposition 2.2. *Let $x \in \Omega$ be fixed and let $u, s, v \in \mathbb{R}$ be such that $s \geq 0$ and $f(x,u) \neq 0$. Then:*
(i)

$$g(s,v) := f(x,u)\left[\tfrac{1}{2}\left(s^2 - 1\right)u + sv\right] + F(x,u) - F(x, su+v) \leq 0 \tag{2.1}$$

for all x.
(ii) There exist $s_u \in (0,1]$, $t_u \geq 1$ such that $g(s,v) = 0$ if and only if $s \in [s_u, t_u]$ and $v = 0$ ($s_u = t_u$ not excluded). Moreover, for such s we have $f(x,su) = sf(x,u)$.

Part (i) of this proposition has been shown in [7] and it extends a similar result in [11] where (F_4') has been assumed (however, our s corresponds to $s+1$ in [7, 11]). Here we provide a different argument which will be needed in order to show part (ii).

Proof. Obviously, $g(1,0) = 0$. We shall show that $g(s,v) \to -\infty$ as $s + |v| \to \infty$. Put $z = z(s) := su + v$. Using Lemma 2.1, we obtain

$$\begin{aligned} g(s,v) &= f(x,u)\left[\tfrac{1}{2}\left(s^2-1\right)u + sv\right] + F(x,u) - F(x,z) \\ &< f(x,u)\left[\tfrac{1}{2}\left(s^2-1\right)u + s(z - su)\right] + \frac{1}{2}f(x,u)u - F(x,z) \\ &= -\tfrac{1}{2}s^2 f(x,u)u + sf(x,u)z - Az^2 + (Az^2 - F(x,z)). \end{aligned}$$

Since the quadratic form (in s and z) above is negative definite if $A > 0$ is a constant large enough and since $Az^2 - F(x,z)$ is bounded above according to (F_3), $g(s,v) \to -\infty$ as $s + |v| \to \infty$ as claimed.

It follows that g has a maximum ≥ 0 on the set $\{(s,v) : s \geq 0\}$. As

$$g(0,v) = -\tfrac{1}{2}f(x,u)u + F(x,u) - F(x,v) < -F(x,v) \leq 0$$

(by Lemma 2.1), the maximum is attained at some (s,v) with $s > 0$. Then

$$g_v'(s,v) = sf(x,u) - f(x, su+v) = 0 \tag{2.2}$$

and

$$g_s'(s,v) = (su+v)f(x,u) - uf(x, su+v) = 0. \tag{2.3}$$

Using (2.2) in (2.3) we obtain $vf(x,u) = 0$. Hence $v = 0$ and

$$g_s'(s,0) = su^2 \left(\frac{f(x,u)}{u} - \frac{f(x,su)}{su} \right) = 0.$$

By (F_4), there must exist s_u, t_u such that $s_u \in (0,1]$, $t_u \geq 1$ and $g'_s(s,0) = 0$ if and only if $s \in [s_u, t_u]$. For such s we have $g(s,0) = g(1,0) = 0$ and $f(x, su) = sf(x,u)$. □

Corollary 2.3. *Suppose $u \in \mathcal{M}$ and let $s \geq 0$, $v \in E^0 \oplus E^-$. Then*

$$\int_\Omega \left(f(x,u)\left[\tfrac{1}{2}\left(s^2-1\right)u + sv\right] + F(x,u) - F(x, su+v)\right) dx \leq 0$$

and there exist $0 < s_u \leq 1 \leq t_u$ such that equality holds if and only if $s \in [s_u, t_u]$, $v = 0$. Moreover, for such s and almost all $x \in \Omega$, $f(x, su) = sf(x,u)$.

Proof. If $u \in \mathcal{M}$, then $f(x, u(x)) \neq 0$ for x on a set of positive measure. According to Proposition 2.2, inequality (2.1) holds for such x and there exist $s_{u(x)} \in (0,1]$, $t_{u(x)} \geq 1$ such that the left-hand side of (2.1) is zero if and only if $s \in [s_{u(x)}, t_{u(x)}]$ and $v(x) = 0$. Moreover, for such s, $f(x, su(x)) = sf(x, u(x))$. Now one takes $s_u := \operatorname{ess\,sup}\{s_{u(x)} : f(x,u(x)) \neq 0\}$ and $t_u := \operatorname{ess\,inf}\{t_{u(x)} : f(x,u(x)) \neq 0\}$.

Note that if $f(x,u(x)) = 0$, then $F(x,u(x)) = \int_0^{u(x)} f(x,t)\,dt = 0$ because $f(x,t) = 0$ for t between 0 and $u(x)$ according to (F_4). Hence the integrand above is ≤ 0 also in this case. □

Proposition 2.4. *(i) If $u \in E \setminus (E^0 \oplus E^-)$, then $\widehat{E}(u) \cap \mathcal{M} \neq \emptyset$.*
(ii) If $w \in \widehat{E}(u) \cap \mathcal{M}$, then there exist $0 < s_w \leq 1 \leq t_w$ such that $\widehat{E}(u) \cap \mathcal{M} = [s_w, t_w]w$. Moreover, $\Phi(sw) = \Phi(w)$, $\Phi'(sw) = s\Phi'(w)$ for all $s \in [s_w, t_w]$ and $\Phi(z) < \Phi(w)$ for all other $z \in \widehat{E}(u)$.
(iii) $\mathcal{M}$ is bounded away from $E^0 \oplus E^-$, closed and $c := \inf_{w \in \mathcal{M}} \Phi(w) > 0$. Moreover, $\Phi|_{\mathcal{M}}$ is coercive, i.e., $\Phi(u) \to \infty$ as $u \in \mathcal{M}$ and $\|u\| \to \infty$.

Note that an immediate consequence is that if w is a critical point of Φ, then the whole line segment $[s_w, t_w]w$ consists of critical points.

Proof. (i) The conclusion can be found in [11, Lemma 2.6 and Theorem 3.1], see also [12, Proposition 39]. The proof is by showing that $\Phi(z) \leq 0$ for $z \in \widehat{E}(u)$ and $\|z\|$ large enough, and then weak upper semicontinuity of $\Phi|_{\widehat{E}(u)}$ implies that there exists a positive maximum.

(ii) For each $z \in \widehat{E}(u)$ we have $z = sw + v$, where $s \geq 0$ and $v = v^0 + v^- \in E^0 \oplus E^-$. It has been shown in the course of the proof of [11, Proposition 2.3] and [12, Proposition 39] that

$$\Phi(z) - \Phi(w) = \Phi(sw+v) - \Phi(w) = -\frac{1}{2}\|v^-\|^2 + \int_\Omega \left(f(x,w)\left[\tfrac{1}{2}\left(s^2-1\right)w + sv\right] + F(x,w) - F(x, sw+v)\right) dx$$

(again, keep in mind that our s corresponds to $s+1$ in [11, 12]). Hence according to Corollary 2.3, $\Phi(z) \leq \Phi(w)$ for all $z \in \widehat{E}(u)$ and $\Phi(z) = \Phi(w)$ if and only if $z \in [s_w, t_w]w$. That $\Phi(sw) = \Phi(w)$ for $s \in [s_w, t_w]$ is clear and since $\Phi(sw) = \max_{\widehat{E}(u)} \Phi(z)$, it is also clear that $\widehat{E}(u) \cap \mathcal{M} = [s_w, t_w]w$ and $\Phi(z) < \Phi(w)$ for other z. The equality $\Phi'(sw) = s\Phi'(w)$ follows immediately from the fact that $f(x, sw) = sf(x, w)$.

(iii) That $c > 0$ has been shown in [11, Lemma 2.4] and is an immediate consequence of the fact that $\Phi(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$ as $u \to 0$, $u \in E^+$. Since $\Phi|_{E^0 \oplus E^-} \le 0$, $\mathcal{M}$ is bounded away from $E^0 \oplus E^-$ and hence closed. Finally, according to Proposition 2.7 and the proof of Theorem 3.1 in [11], $\Phi|_{\mathcal{M}}$ is coercive. □

Remark 2.5. If f satisfies (F_1)–(F_4) and is of the form $f(x, u) = a(x)h(u)$, where $h(u) \neq 0$ for $u \neq 0$, then $s_w = t_w = 1$ in Proposition 2.4, i.e. $\widehat{E}(u)$ intersects $\mathcal{M}$ at a unique point. Assuming the contrary, suppose $t_w > 1$ and $w > 0$ on a set of positive measure (other cases are treated similarly). So meas$\{x : w(x) > d\}$ is positive for some $d > 0$. We claim that $h(t)/t$ is constant for $0 < t < d$. Otherwise there exist $s \in (1, t_w]$, t_0 and $\varepsilon > 0$ such that $\varepsilon < t_0 < d - \varepsilon$ and

$$\frac{h(t)}{t} < \frac{h(st)}{st} \quad \text{for all } t \in (t_0 - \varepsilon, t_0 + \varepsilon).$$

Since the sets $\{x : w(x) > t_0 + \varepsilon\}$ and $\{x : w(x) < t_0 - \varepsilon\}$ have positive measure, so does the set $\{x : w(x) \in (t_0 - \varepsilon, t_0 + \varepsilon)\}$, see [1]. But this contradicts the last statement of Corollary 2.3. Hence $h(t)/t$ is constant for $0 < t < d$ and $h(t)/t \to 0$ as $t \to 0$. So $h(t) = 0$ on $(0, d)$ which is impossible.

According to Proposition 2.4, for each $u \in E^+ \setminus \{0\}$ there exist w and $0 < \sigma_w \le \tau_w$ such that

$$m(u) := [\sigma_w, \tau_w]w = \widehat{E}(u) \cap \mathcal{M} \subset E.$$

This is a multivalued map from $E^+ \setminus \{0\}$ to E. However, the map $\widehat{\Psi} : E^+ \setminus \{0\} \to \mathbb{R}$ given by

$$\widehat{\Psi}(u) := \Phi(m(u)) = \max_{z \in \widehat{E}(u)} \Phi(z)$$

is single-valued because Φ is constant on $\widehat{E}(u) \cap \mathcal{M}$. In fact more is true:

Proposition 2.6. *The map $\widehat{\Psi}$ is locally Lipschitz continuous.*

Proof. If $u_0 \in E^+ \setminus \{0\}$, then there exist a neighbourhood $U \subset E^+ \setminus \{0\}$ of u_0 and $R > 0$ such that $\Phi(w) \le 0$ for all $u \in U$ and $w \in \widehat{E}(u)$, $\|w\| \ge R$. For otherwise we can find sequences (u_n), (w_n) such that $u_n \to u_0$, $w_n \in \widehat{E}(u_n)$, $\Phi(w_n) > 0$ and $\|w_n\| \to \infty$. But $u_0, u_1, u_2, \dots$ is a compact set, hence according to [11, Lemma 2.5], $\Phi(w) \le 0$ for some R and all $w \in \widehat{E}(u_j)$, $j = 0, 1, 2, \dots$, $\|w\| \ge R$, which is a contradiction.

Let U, R be as above and $s_1u_1 + v_1 \in m(u_1)$, $s_2u_2 + v_2 \in m(u_2)$, where $u_1, u_2 \in U$ and $v_1, v_2 \in E^0 \oplus E^-$. Then $\|m(u_1)\|, \|m(u_2)\| \le R$. By the maximality property of $m(u)$ and the mean value theorem,

$$\begin{aligned}\widehat{\Psi}(u_1) - \widehat{\Psi}(u_2) &= \Phi(s_1u_1 + v_1) - \Phi(s_2u_2 + v_2) \le \Phi(s_1u_1 + v_1) - \Phi(s_1u_2 + v_1) \\ &\le s_1 \sup_{t \in [0,1]} \|\Phi'(s_1(tu_1 + (1-t)u_2) + v_1)\| \, \|u_1 - u_2\| \le C\|u_1 - u_2\|,\end{aligned}$$

where the constant C depends on R but not on the particular choice of points in $m(u_1)$, $m(u_2)$. Similarly, $\widehat{\Psi}(u_2) - \widehat{\Psi}(u_1) \le C\|u_1 - u_2\|$ and the conclusion follows. □

Remark 2.7. It has been shown in [11] that if (F_4') holds instead of (F_4), then $\widehat{\Psi} \in C^1(E^+ \setminus \{0\}, \mathbb{R})$. An easy inspection of the arguments in [11] or [12] shows that if for each $u \in E^+ \setminus \{0\}$ there exists a unique positive maximum of $\Phi|_{\widehat{E}(u)}$, then $\widehat{\Psi}$ is still of class C^1. Hence in particular, if f is as in Remark 2.5, then the conclusions of Theorems 1.1 and 1.2 hold with the same proofs as in [11].

However, under our assumptions we can in general only assert that $\widehat{\Psi}$ is locally Lipschitz continuous (because $u \mapsto m(u)$ may not be single-valued). Therefore, instead of the derivative of $\widehat{\Psi}$ we shall use Clarke's subdifferential [4]. The study of minimax methods for differential equations whose associated functional is merely locally Lipschitz continuous has been initiated by Chang in [3]. We recall some notions and facts taken from [3, 4]. They may also be found conveniently collected in Section 7.1 of [2]. The *generalized directional derivative* of $\widehat{\Psi}$ at u in the direction v is defined by

$$\widehat{\Psi}^\circ(u; v) := \limsup_{\substack{h \to 0 \\ t \downarrow 0}} \frac{\widehat{\Psi}(u + h + tv) - \widehat{\Psi}(u + h)}{t}.$$

The function $v \mapsto \widehat{\Psi}^\circ(u; v)$ is convex and its subdifferential $\partial\widehat{\Psi}(u)$ is called the *generalized gradient* (or *Clarke's subdifferential*) of $\widehat{\Psi}$ at u, that is,

$$\partial\widehat{\Psi}(u) := \{w \in E^+ : \widehat{\Psi}^\circ(u; v) \ge \langle w, v\rangle \text{ for all } v \in E^+\}. \tag{2.4}$$

In [2] E is a Banach space and the generalized gradient is in the dual space E^*. Since here we work in a Hilbert space, we may assume via duality that $\partial\widehat{\Psi}(u)$ is a subset of E (or more precisely, of E^+). A point u is called a *critical point* of $\widehat{\Psi}$ if $0 \in \partial\widehat{\Psi}(u)$, i.e. $\widehat{\Psi}^0(u; v) \ge 0$ for all $v \in E^+$, and a sequence (u_n) is called a *Palais-Smale sequence* for $\widehat{\Psi}$ (PS-sequence for short) if $\widehat{\Psi}(u_n)$ is bounded and there exist $w_n \in \partial\widehat{\Psi}(u_n)$ such that $w_n \to 0$. The functional $\widehat{\Psi}$ satisfies the *PS-condition* if each PS-sequence has a convergent subsequence. Below we collect some notation which we shall need:

$$S^+ := \{u \in E^+ : \|u\| = 1\}, \quad T_uS^+ := \{v \in E^+ : \langle u, v\rangle = 0\}, \quad \Psi := \widehat{\Psi}|_{S^+},$$
$$\Psi^d := \{u \in S^+ : \Psi(u) \le d\}, \quad \Psi_c := \{u \in S^+ : \Psi(u) \ge c\}, \quad \Psi_c^d := \Psi_c \cap \Psi^d,$$
$$K := \{u \in S^+ : 0 \in \partial\widehat{\Psi}(u)\} \quad K_c := \Psi_c^c \cap K, \quad \partial\Psi(u) := \partial\widehat{\Psi}(u), \text{ where } u \in S^+.$$

Note that the symbol $\partial\Psi(u)$ stands for $\partial\widehat{\Psi}(u)$ when u is restricted to S^+. This is in consistence with the notation $\Psi = \widehat{\Psi}|_{S^+}$. As we shall see in the proof of the next proposition, $\widehat{\Psi}^\circ(u; su) = 0$ for all $s \in \mathbb{R}$. Hence $\partial\Psi(u) \subset T_uS^+$.

Proposition 2.8. *(i) $u \in S^+$ is a critical point of $\widehat{\Psi}$ if and only if $m(u)$ consists of critical points of Φ. The corresponding critical values coincide.*
(ii) $(u_n) \subset S^+$ is a PS-sequence for $\widehat{\Psi}$ if and only if there exist $w_n \in m(u_n)$ such that (w_n) is a PS-sequence for Φ.

Proof. (i) Let $u \in S^+$. We shall show that $\widehat{\Psi}^\circ(u;v) \geq 0$ for all $v \in E^+$ if and only if $m(u)$ consists of critical points. Note first that there exists an orthogonal decomposition $E = E(u) \oplus T_u S^+$, and by the maximizing property of $m(u)$, $\Phi'(w)v = 0$ for all $w \in m(u)$ and $v \in E(u)$. Let $s \in \mathbb{R}$ be fixed. Since $\widehat{\Psi}(u) = \widehat{\Psi}(\sigma u)$ for all $\sigma > 0$ and $\widehat{\Psi}$ is locally Lipschitz continuous,

$$|\widehat{\Psi}(u+h+t(su)) - \widehat{\Psi}(u+h)| = |\widehat{\Psi}((1+ts)u+h) - \widehat{\Psi}((1+ts)(u+h))| \leq Ct|s|\|h\|$$

for $\|h\|$ and $t > 0$ small. Hence $\widehat{\Psi}^\circ(u;su) = 0$ for all $s \in \mathbb{R}$. So we only need to consider $v \in T_u S^+$.

Let $s_u u + z_u$, where $s_u > 0$ and $z_u \in E^0 \oplus E^-$, denote an (arbitrarily chosen) element of $m(u)$. Then, using the maximizing property of $m(u)$ and the mean value theorem,

$$\begin{aligned}\widehat{\Psi}(u+h+tv) - \widehat{\Psi}(u+h) &= \Phi(s_{u+h+tv}(u+h+tv) + z_{u+h+tv}) - \Phi(s_{u+h}(u+h) + z_{u+h}) \\ &\leq \Phi(s_{u+h+tv}(u+h+tv) + z_{u+h+tv}) - \Phi(s_{u+h+tv}(u+h) + z_{u+h+tv}) \\ &= ts_{u+h+tv}\Phi'(s_{u+h+tv}(u+h+\theta tv) + z_{u+h+tv})v\end{aligned}$$

for some $\theta \in (0,1)$. Dividing by t and letting $h \to 0$ and $t \downarrow 0$ via subsequences we obtain

$$\widehat{\Psi}^\circ(u;v) \leq s^*\Phi'(s^*u + z^*)v, \tag{2.5}$$

where $s_n := s_{u+h_n+t_nv} \to s^* > 0$ and $z_n := z_{u+h_n+t_nv} \rightharpoonup z^*$. This follows because $\mathcal{M}$ is bounded away from 0 and $\Phi|_{\mathcal{M}}$ coercive, hence s_n and z_n must be bounded. We claim that $s^*u + z^* \in \mathcal{M}$. Indeed, taking subsequences once more, writing $z_n = z_n^0 + z_n^- \in E^0 \oplus E^-$ and using Fatou's lemma,

$$\begin{aligned}\widehat{\Psi}(u) &= \lim_{n\to\infty} \widehat{\Psi}(u+h_n+t_nv) = \lim_{n\to\infty} \Phi(s_n(u+h_n+t_nv) + z_n) \\ &= \lim_{n\to\infty}\left(\frac{1}{2}\|s_n(u+h_n+t_nv)\|^2 - \frac{1}{2}\|z_n^-\|^2 - \int_\Omega F(x, s_n(u+h_n+t_nv)+z_n)\,dx\right) \\ &\leq \frac{1}{2}\|s^*u\|^2 - \frac{1}{2}\|(z^*)^-\|^2 - \int_\Omega F(x, s^*u+z^*)\,dx \leq \widehat{\Psi}(u).\end{aligned}$$

This implies that $\|z_n\| \to \|z^*\|$ (recall $\dim E^0 < \infty$), hence $z_n \to z^*$ and $s_n(u+h_n+t_nv) + z_n \to s^*u + z^*$. As $\mathcal{M}$ is closed, the claim follows. Since $\widehat{E}(u) \cap \mathcal{M}$ may be a line segment, it is not sure that s^* and z^* are the same for different v. However, if s_1^*, s_2^* and z_1^*, z_2^* correspond to v_1 and v_2, then by Proposition 2.4, $s_1^*u + z_1^* = \tau(s_2^*u + z_2^*)$ and $\Phi'(s_1^*u + z_1^*)v_2 = \tau\Phi'(s_2^*u + z_2^*)v_2$ for some $\tau > 0$. Taking this into account, we see from (2.5) that if $y \in \partial\Psi(u)$, then

$$\langle y, v\rangle \leq \widehat{\Psi}^\circ(u;v) \leq \tau(v)\Phi'(s^*u + z^*)v, \tag{2.6}$$

where τ is bounded and bounded away from 0 (by constants independent of v). It follows immediately that u is a critical point of Ψ if and only if $m(u)$ consists of critical points of Φ.

(ii) The proof is very similar here. We take $y_n \in \partial\Psi(u_n)$ and $w_n \in m(u_n)$. Since $\Phi|_{\mathcal{M}}$ is coercive, boundedness of $\Phi(m(u_n))$ implies that $(m(u_n))$ is bounded. As in (2.6), we see that

$$\langle y_n, v\rangle \le \widehat{\Psi}^\circ(u_n; v) \le \tau_n(v)\Phi'(w_n)v, \tag{2.7}$$

where τ_n is bounded and bounded away from 0 because so is $m(u_n)$. So the conclusion follows. □

Note that if $(w_n) \subset (m(u_n))$ is a PS-sequence for Φ, then so is any sequence $(w_n') \subset (m(u_n))$.

Finally for this section we construct a pseudo-gradient vector field $H : S^+ \setminus K \to TS^+$ for Ψ. For $u \in S^+$, let

$$\partial^-\Psi(u) := \left\{ p \in \partial\Psi(u) : \|p\| = \min_{a\in\partial\Psi(u)} \|a\| \right\} \quad \text{and} \quad \mu(u) := \inf_{a\in S^+} \{\|\partial^-\Psi(a)\| + \|u-a\|\}.$$

Since $\partial\Psi(u)$ is closed and convex, p as above exists and is unique, cf. [2, 3]. Hence

$$K = \{u \in S^+ : \partial^-\Psi(u) = 0\}.$$

The map $u \mapsto \|\partial^-\Psi(u)\|$ is lower semicontinuous [2, Proposition 7.1.1(vi)] but not continuous in general. The reason for introducing the function μ is that it regularizes $\|\partial^-\Psi(u)\|$. The idea comes from [5] where a similar function has been defined.

Lemma 2.9. *The function μ is continuous and $u \in K$ if and only if $\mu(u) = 0$.*

Proof. Let $u, v, a \in S^+$. Then

$$\mu(u) \le \|\partial^-\Psi(a)\| + \|u-a\| \le \|\partial^-\Psi(a)\| + \|v-a\| + \|u-v\|,$$

and taking the infimum over a on the right-hand side we obtain $\mu(u) \le \mu(v) + \|u-v\|$. Reversing the roles of u and v we see that $|\mu(u) - \mu(v)| \le \|u-v\|$. Hence μ is (Lipschitz) continuous.

Since $0 \le \mu(u) \le \|\partial^-\Psi(u)\|$, it is clear that $\mu(u) = 0$ if $u \in K$. Suppose $\mu(u) = 0$. Then there exist a_n such that $\partial^-\Psi(a_n) \to 0$ and $a_n \to u$, so $u \in K$ by the lower semicontinuity of $u \mapsto \|\partial^-\Psi(u)\|$. □

Proposition 2.10. *There exists a locally Lipschitz continuous function $H : S^+\setminus K \to TS^+$ such that $\|H(u)\| \le 1$ and $\inf\{\langle p, H(u)\rangle : p \in \partial\Psi(u)\} > \frac{1}{2}\mu(u)$ for all $u \in S^+ \setminus K$. If Φ is even, then H may be chosen to be odd.*

Proof. Let $u \in S^+ \setminus K$ and put $v_u := \partial^-\Psi(u)/\|\partial^-\Psi(u)\|$. Consider the map

$$\chi : w \mapsto \inf_{p\in\partial\Psi(w)} \langle p,\, v_u - \langle v_u, w\rangle w\rangle - \frac{1}{2}\mu(w), \quad w \in S^+ \setminus K$$

(note that $v_u - \langle v_u, w\rangle w \in T_w(S^+)$). Since $\partial\Psi(u)$ is convex, $\inf_{p\in\partial\Psi(u)}\langle p, v_u\rangle \ge \|\partial^-\Psi(u)\| \ge \mu(u)$ and therefore $\chi(u) \ge \frac{1}{2}\mu(u) > 0$. Moreover, since

$$\inf_{p\in\partial\Psi(w)} \langle p,\, v_u - \langle v_u, w\rangle w\rangle = -\sup_{p\in\partial\Psi(w)} \langle p,\, \langle v_u, w\rangle w - v_u\rangle = -\widehat{\Psi}^0(w;\, \langle v_u, w\rangle w - v_u)$$

(see Proposition 7.1.1(vii) and property (c) on p. 168 in [2]) and $\widehat{\Psi}^0$ is upper semicontinuous in both arguments [2, Proposition 7.1.1(vii)], χ is lower semicontinuous. Hence there exists a neighbourhood U_u of u such that $\chi(w) > 0$ for all $w \in U$.

The remaining part of the proof is standard. Take a locally finite open refinement $(U_{u_i})_{i\in I}$ of the open cover $(U_u)_{u\in S^+\setminus K}$ and a subordinated locally Lipschitz continuous partition of unity $\{\lambda_i\}_{i\in I}$. Define

$$H(u) := \sum_{i\in I} \lambda_i(u)v_{u_i}, \quad u \in S^+ \setminus K.$$

It is easy to see that H satisfies the required conclusions.

If Φ is even, then so is Ψ and we may replace $H(u)$ with $\frac{1}{2}(H(u) - H(-u))$. □

3. Proofs of Theorems 1.1 and 1.2

Since the arguments are very similar to those appearing in [11, 12], we shall describe them rather briefly and concentrate on pointing out the main differences.

We start with Theorem 1.1. First we want to show that there exists a minimizer for Ψ on S^+. It follows from the results of Section 2 that

$$c := \inf_{w\in\mathcal{M}} \Phi(w) = \inf_{u\in S^+} \Psi(u) > 0.$$

According to Ekeland's variational principle [6], there exists a sequence $(u_n) \subset S^+$ such that $\Psi(u_n) \to c$ and

$$\Psi(w) \geq \Psi(u_n) - \frac{1}{n}\|w - u_n\| \quad \text{for all } w \in S^+. \tag{3.1}$$

For a given $v \in T_{u_n}S^+$, let $z_n(t) := (u_n + tv)/\|u_n + tv\|$. Since $\|u_n + tv\| - 1 = O(t^2)$ as $t \to 0$ and $\widehat{\Psi}(u_n + tv) = \Psi(z_n(t))$, it follows from (3.1) that

$$\widehat{\Psi}^\circ(u_n; v) \geq \limsup_{t\downarrow 0} \frac{\widehat{\Psi}(u_n + tv) - \widehat{\Psi}(u_n)}{t} = \limsup_{t\downarrow 0} \frac{\Psi(z_n(t)) - \Psi(u_n)}{t} \geq -\frac{1}{n}\|v\|.$$

Since $m(u_n)$ is bounded by coercivity of $\Phi|_{\mathcal{M}}$, the second inequality in (2.7) implies that

$$-\frac{1}{n}\|v\| \leq \widehat{\Psi}^\circ(u_n; v) \leq \tau_n(v)\Phi'(w_n)v,$$

where $w_n \in m(u_n) \subset \mathcal{M}$ and τ_n is bounded and bounded away from 0. So recalling $\Phi'(w_n)v = 0$ for all $v \in E(w_n)$, it follows that (w_n) is a bounded PS-sequence for Φ. Now we may proceed exactly as in the proof of Theorem 1.1 in [11], pp. 3811–3812 (or in the proof of Theorem 40 in [12]). More precisely, one shows invoking Lions' lemma [15, Lemma 1.21] in a rather standard way that there exists a sequence $(y_n) \subset \mathbb{R}^N$ such that

$$\int_{|x-y_n|<1} w_n^2\, dx \geq \varepsilon \quad \text{for } n \text{ large enough and some } \varepsilon > 0,$$

and since Φ and $\mathcal{M}$ are invariant by translations $u(\cdot) \mapsto u(\cdot - k)$, $k \in \mathbb{Z}^N$, we may assume (y_n) is bounded. So passing to a subsequence, $w_n \rightharpoonup w \neq 0$. This w is a solution and an additional argument shows it is a ground state, see [11] or [12] for more details.

Suppose now f is odd in u and note that Ψ is even and invariant by translations by elements of $\mathbb{Z}^N$. To prove that there exist infinitely many geometrically distinct solutions we assume the contrary. Since to each $[\sigma_w, \tau_w]w \subset \mathcal{M}$ there corresponds a unique point $u \in S^+$, K consists of finitely many orbits $\mathcal{O}(u) := \{u(\cdot - k) : u \in K,\ k \in \mathbb{Z}^N\}$. We choose a subset $\mathcal{F} \subset K$ such that $\mathcal{F} = -\mathcal{F}$ and each orbit has a unique representative in $\mathcal{F}$. Now an easy inspection shows that Lemmas 2.11 and 2.13 in [11] hold, i.e. the mapping $\check{m} : u \mapsto u^+/\|u^+\|$ from $\mathcal{M}$ to S^+ is Lipschitz continuous and $\kappa := \inf\{v - w\| : v, v \in K,\ v \neq w\} > 0$.

Proposition 3.1 (Lemma 2.14 in [11])**.** *Let $d \geq c$. If $(v_n^1), (v_n^2) \subset \Psi^d$ are two PS-sequences for Ψ, then either $\|v_n^1 - v_n^2\| \to 0$ as $n \to \infty$ or $\limsup_{n\to\infty} \|v_n^1 - v_n^2\| \geq \rho(d) > 0$, where ρ depends on d but not on the particular choice of PS-sequences in Ψ^d.*

The argument is exactly the same as in [11], taking into account that by Proposition 2.8, to $(v_n^j) \subset \Psi^d$ there correspond PS-sequences (u_n^j) with $u_n^j \in m(v_n^j)$, $j = 1, 2$. Once u_n^j have been chosen, one follows the lines of [11].

Let H be the vector field constructed in Proposition 2.10 and consider the flow given by

$$\frac{d}{dt}\eta(t, w) = -H(\eta(t, w)), \quad \eta(0, w) = w,$$

defined on the set

$$\mathcal{G} := \{(t, w) : w \in S^+ \setminus K,\ T^-(w) < t < T^+(w)\},$$

where $(T^-(w), T^+(w))$ is the maximal existence time for the trajectory passing through w at $t = 0$.

Proposition 3.2 (cf. Lemma 2.15 in [11])**.** *For each $w \in S^+ \setminus K$ the limit $\lim_{t\to T^+(w)} \eta(t, w)$ exists and is a critical point of Ψ.*

Proof. We adapt the argument in [11].

If $T^+(w) < \infty$, then for $0 \leq s < t < T^+(w)$ we have

$$\|\eta(t, w) - \eta(s, w)\| \leq \int_s^t \|H(\eta(\tau, w))\|\, d\tau \leq t - s,$$

hence $\lim_{t\to T^+(w)} \eta(t, w)$ exists and must be a critical point (or the flow can be continued for $t > T^+(w)$).

Let $T^+(w) = \infty$. It suffices to show that for each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that $\|\eta(t_\varepsilon, u) - \eta(t, u)\| < \varepsilon$ for all $t \geq t_\varepsilon$. Assuming the contrary, we find $\varepsilon \in (0, \rho(d)/2)$ and $t_n \to \infty$ such that $\|\eta(t_n, w) - \eta(t_{n+1}, w)\| = \varepsilon$ for all n. Choose the smallest $t_n^1 \in (t_n, t_{n+1})$ such that $\|\eta(t_n, w) - \eta(t_n^1, w)\| = \varepsilon/3$. Recall from Lemma 2.9 that μ is continuous and set

$\kappa_n := \min_{s\in[t_n,t_n^1]} \mu(\eta(s,w))$. Then, using Proposition 2.10 and [2, Proposition 7.1.1(viii)],

$$\begin{aligned}\frac{\varepsilon}{3} &= \|\eta(t_n^1,w)-\eta(t_n,w)\| \le \int_{t_n}^{t_n^1} \|H(\eta(s,w))\|\,ds \le t_n^1 - t_n \\ &\le \frac{2}{\kappa_n}\int_{t_n}^{t_n^1} \inf_{p\in\partial\Psi(\eta(s,u))} \langle p, H(\eta(s,w))\rangle\,ds = -\frac{2}{\kappa_n}\int_{t_n}^{t_n^1} \sup_{p\in\partial\Psi(\eta(s,u))} \langle p, -H(\eta(s,w))\rangle\,ds \\ &\le -\frac{2}{\kappa_n}\int_{t_n}^{t_n^1} \frac{d}{ds}\Psi(\eta(s,w))\,ds = \frac{2}{\kappa_n}(\Psi(\eta(t_n,w)) - \Psi(\eta(t_n^1,w))).\end{aligned}$$

Since Ψ is bounded below, $\Psi(\eta(t_n,w)) - \Psi(\eta(t_n^1,w)) \to 0$, hence $\kappa_n \to 0$ and we may find $s_n^1 \in [t_n,t_n^1]$ such that if $z_n^1 := \eta(s_n^1,w)$, then $\mu(z_n^1) \to 0$. By the definition of μ there exist w_n^1 such that $w_n^1 - z_n^1 \to 0$ and $\partial^-\Psi(w_n^1) \to 0$. So $\limsup_{n\to\infty} \|w_n^1 - \eta(t_n,w)\| \le \varepsilon/3$. Similarly, there exists a largest $t_n^2 \in (t_n^1,t_{n+1})$ with $\|\eta(t_{n+1},w) - \eta(t_n^2,w)\| = \varepsilon/3$ and we find w_n^2 with $\partial^-\Psi(w_n^2) \to 0$ and $\limsup_{n\to\infty} \|w_n^2 - \eta(t_{n+1},w)\| \le \varepsilon/3$. It follows that $\varepsilon/3 \le \limsup_{n\to\infty} \|w_n^1 - w_n^2\| \le 2\varepsilon < \rho(d)$, a contradiction to Proposition 3.1. $\square$

Proposition 3.3 (cf. Lemma 2.16 in [11]). *Let $d \ge c$. Then for each $\delta > 0$ there exists $\varepsilon > 0$ such that $\Psi_{d-\varepsilon}^{d+\varepsilon} \cap K = K_d$ and $\lim_{t\to T^+(w)} \Psi(\eta(t,w)) < d-\varepsilon$ for all $w \in \Psi^{d+\varepsilon}\setminus U_\delta(K_d)$, where $U_\delta(K_d)$ is the open δ-neighbourhood of K_d.*

The proof requires changes which, in view of the arguments of Proposition 3.2, are rather obvious (in particular, $\nabla\Psi$ in the definition of τ in [11] should be replaced by μ).

With all these prerequisites, existence of infinitely many solutions is obtained by repeating the arguments on pp. 3817–3818 in [11]. Let

$$c_k := \inf\{d \in \mathbb{R} : \gamma(\Psi^d) \ge k\}, \quad k = 1,2,\dots,$$

where γ denotes Krasnoselskii's genus [10]. Using the flow η and Proposition 3.3 one shows $K_{c_k} \ne 0$ and $c_k < c_{k+1}$ for all k. This contradicts our assumption that there are finitely many geometrically distinct solutions.

Now we turn our attention to Theorem 1.2. Here there is no $\mathbb{Z}^N$-symmetry but instead there is a compact embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $q \in [1,2^*)$. Using this, one sees as in the proof of Theorem 3.1 in [11] or Theorem 37 in [12] that the ground state exists. The minimizing PS-sequence is extracted by using Ekeland's variational principle in the same way as at the beginning of this section. To obtain infinitely many solutions for odd f one first shows as in [11, Theorem 3.2] (or in [12, Section 4.2]) that Ψ satisfies the PS-condition. Now a standard minimax argument as in [11, Theorem 3.2] can be employed. Note that with the aid of the vector field H and suitable cutoff functions one can construct a deformation in the usual way as e.g. in [10] (see also [3]). We leave the details to the reader.

References

[1] A. Chabi and A. Haraux, *Un théorème de valeurs intermédiaires dans les espaces de Sobolev et applications*, Ann. Fac. Sci. Toulouse 7 (1985), 87–100.

[2] J. Chabrowski, *Variational Methods for Potential Operator Equations*, de Gruyter, Berlin, 1997.

[3] K. C. Chang, *Variational methods for non-differentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. 80 (1981), 102–129.

[4] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Les publications CRM, Université de Montréal, 1989.

[5] A. Ćwiszewski and W. Kryszewski, *Equilibria of set-valued maps: a variational approach*, Nonl. Anal. 48 (2002), 707–746.

[6] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. 47 (1974), 324–353.

[7] S. Liu, *On superlinear Schrödinger equations with periodic potential*, Calc. Var. PDE 45 (2012), 1–9.

[8] J. Mederski, *Ground states of a system of nonlinear Schrödinger equations with periodic potentials*, Comm. PDE, DOI: 10.1080/03605302.2016.1209520.

[9] A. Pankov, *Periodic nonlinear Schrödinger equation with application to photonic crystals*, Milan J. Math. 73 (2005), 259–287.

[10] M. Struwe, *Variational Methods*, Springer, New York, 1990.

[11] A. Szulkin and T. Weth, *Ground state solutions for some indefinite variational problems*, J. Func. Anal. 257 (2009), 3802–3822.

[12] A. Szulkin and T. Weth, *The method of Nehari manifold*, Handbook of Nonconvex Analysis and Applications, 597–632, Int. Press, Somerville, MA, 2010.

[13] X. H. Tang, *New super-quadratic conditions on ground state solutions for superlinear Schrödinger equation*, Adv. Nonl. Stud. 14 (2014), 361–373.

[14] X. H. Tang, *Non-Nehari manifold method for superlinear Schrödinger equation*, Taiwanese J. Math. 18 (2014), 1957–1979.

[15] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[16] X. Zhong and W. Zou, *Ground state and multiple solutions via generalized Nehari manifold*, Nonlinear Anal. 102 (2014), 251–263.

Universidade Federal de São Carlos, Departamento de Matemática, 13565-905 São Carlos, Brazil

E-mail address: `odair@dm.ufscar.br`

Faculty of Mathematics and Computer Sciences, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

E-mail address: `wkrysz@mat.umk.pl`

Department of Mathematics, Stockholm University, 106 91 Stockholm, Sweden

E-mail address: `andrzejs@math.su.se`