A concentration phenomenon for semilinear elliptic equations

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For a domain $\Omega \subset \mathbb{R}^N$ we consider the equation $-\Delta u+V(x)u = Q_n(x)|u|^{p-2}u$ with zero Dirichlet boundary conditions and $p \in (2, 2^*)$. Here $V \geq 0$ and Q_n are bounded functions that are positive in a region contained in Ω and negative outside, and such that the sets $\{Q_n > 0\}$ shrink to a point $x_0 \in \Omega$ as $n \to \infty$. We show that if u_n is a nontrivial solution corresponding to Q_n, then the sequence (u_n) concentrates at x_0 with respect to the H^1 and certain L^q-norms. We also show that if the sets $\{Q_n > 0\}$ shrink to two points and u_n are ground state solutions, then they concentrate at one of these points.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a domain and consider the problem

$$-\Delta u + V(x)u = Q(x)|u|^{p-2}u, \quad u \in H_0^1(\Omega), \tag{1.1}$$

where $H_0^1(\Omega)$ is the usual Sobolev space. Suppose $V, Q \in L^\infty(\Omega)$, $V \geq 0$ and $2 < p < 2^*$, where $2^* := 2N/(N-2)$ if $N \geq 3$, $2^* := \infty$ if $N = 1$ or 2. If Ω is unbounded, assume in addition that 0 is not in the spectrum of $-\Delta + V$ (i.e., $\sigma(-\Delta+V) \subset (0,\infty)$; this condition is automatically satisfied for bounded Ω). Multiplying (1.1) by u and integrating over Ω it follows immediately that $u = 0$ is the only solution if $Q \leq 0$. On the other hand, if $Q > 0$ on a bounded set of positive measure, then it is easy to see that there exists a solution

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$u \neq 0$ to (1.1). This will be shown in the next section and is in principle well known, cf. [3, Theorem 6]. Assume without loss of generality that $0 \in \Omega$ and let $Q = Q_n$ be such that $Q_n > 0$ on the ball $B_{1/n}(0)$ and $Q_n < 0$ on $\Omega \setminus B_{2/n}(0)$. For each n there exists a solution $u_n \neq 0$, and in view of the discussion above it is natural to ask what happens with u_n as $n \to \infty$. It is the purpose of this paper to show that the functions u_n concentrate at $x = 0$. This concentration phenomenon does not seem to be earlier known.

There is also another aspect of equation (1.1), related to physics, or more specifically, to the propagation of electromagnetic waves which in our case is monochromatic light travelling through an optical cable (waveguide). The transport of light in dielectric media is controlled by Maxwell's equations (ME) and an important role is played by the dielectric response ε which may vary with location and light intensity, see e.g. [18]. In the following denote by $\omega > 0$ the frequency of light and by c the speed of light in a vacuum, and put $\widetilde{\varepsilon} = \frac{\omega^2}{c^2}\varepsilon$ for convenience.

Our equation (1.1) is inspired by two models of optical waveguides [6, pp. 67–68]. The first model concerns a stratified medium in $\mathbb{R}^3$ consisting of slabs of dielectric materials that are perpendicular to the x_1-axis. Here we assume that the light beam is a wave travelling in the direction of x_3, having polarization in the direction of x_2, and $\widetilde{\varepsilon}$ is a function of x_1 and $|u|^2$. With the ansatz $E(x,t) = u(x_1)\cos(kx_3 - \omega t)e_2$ for the electric field, where e_2 is the unit vector in the direction of x_2 and $k > 0$ is the wave number, one obtains a guided solution of ME in the form of a plane travelling wave if and only if $u \in H^1(\mathbb{R})$ is a solution of the equation

$$-u'' + \big(k^2 - \widetilde{\varepsilon}(x_1, |u|^2)\big)u = 0 \qquad \text{in } \mathbb{R}, \tag{1.2}$$

see [19, 20] and the references there. The total energy per unit length in x_3 of the wave is finite on each plane $\{x_2 \equiv \text{const.}\}$. Note how the x_1-dependence of ε exhibits the geometry of the waveguide. We remark that here and in what follows there is no term $i\partial u/\partial x_3$ which appears in [6]. The reason is that unlike in [6] we always assume that u is independent of x_3.

In the second model we assume $\widetilde{\varepsilon} = \widetilde{\varepsilon}(x_1, x_2, |u|^2)$ and make the ansatz $E(x,t) = u(x_1, x_2)\cos(kx_3 - \omega t)e_2$, the so-called *scalar approximation* for a linearly polarized wave propagating in the x_3-direction. Here one requires $u \in H^1(\mathbb{R}^2)$ to be a solution of

$$-\Delta u + \big(k^2 - \widetilde{\varepsilon}(x_1, x_2, |u|^2)\big)u = 0 \qquad \text{in } \mathbb{R}^2. \tag{1.3}$$

This ansatz does not yield solutions to ME, but it is nevertheless studied extensively in the relevant literature, cf. [6, p. 87], [16, 18] and the references given there. In this case the total energy per unit length in x_3 of the wave is finite on $\mathbb{R}^2$. One may also assume cylindrical symmetry, i.e., one puts $\widetilde{\varepsilon} = \widetilde{\varepsilon}(r, |u|^2)$ and looks for solutions of the form $u = u(r)$, where $r^2 = x_1^2 + x_2^2$.

In a nonlinear medium $\widetilde{\varepsilon}$ has a nontrivial dependence on $|u|^2$. The approximation

$$\widetilde{\varepsilon}(x, |u|^2) = A(x) + Q(x)|u|^{p-2}$$

is commonly used as long as $|u|$ is not too large, so our equation (1.1) is the direct analogue of (1.2) or (1.3) in arbitrary dimension, with $V := k^2 - A$. This approximation is called the Kerr nonlinearity if $p = 4$ and plays an important role in the physics literature [19]. However, also $p \neq 4$ is of interest (non-Kerr-like materials), as are dielectric response functions corresponding to saturation (which occurs when $|u|$ becomes large), see [18, note added in proof], [18] and the references there. In this latter case the response is of the form $A(x) + Q(x)g(|u|)$, with $g(0) = 0$, g increasing and $\lim_{|u|\to\infty} g(|u|)$ finite. This leads to the right-hand side $Q(x)g(|u|)u$ in (1.1). The part of the medium where $Q > 0$ is called self-focusing (the dielectric response increases with $|u|$) and the part where $Q < 0$ is called defocusing. So if $Q > 0$ on a set of small size, the medium has a self-focusing core and is defocusing outside of this core.

It is common to consider materials separately with Q positive or negative, see e.g. [6, Eq. (48)], which corresponds to investigating the existence of bright ($Q > 0$) or dark ($Q < 0$) solitons. However, also materials with sign-changing Q are considered. In this vein, see [11], or [8, Eq. (3)] for an example where a sharp localization of the self-focusing region is considered. There is also recent evidence that materials with a large range of prescribed optical properties can be created [13–15, 21]. Therefore it is reasonable to prescribe the nonlinear dielectric response almost at will for each material.

The conditions we impose on the functions Q_n allow to model a composite of two materials where the size of the self-focusing core decreases as $n \to \infty$. In particular, we show for the plane travelling waves introduced above by way of (1.2) and for the Kerr nonlinearity that the field E concentrates on the x_1-axis in the sense of the H^1- and L^q-norms for all $q > 1$ as $n \to \infty$, see Theorem 3.1 and Remark 3.2. Concerning the scalar approximation (1.3) we obtain concentration at $(x_1, x_2) = (0, 0)$ in H^1 and L^q for every $q > 2$ as $n \to \infty$ but not in the physically relevant case $q = 2$. We do not know whether concentration in L^2 occurs here.

There are numerous rigorous mathematical results on the effect of a sign changing Q on the existence and properties of solutions of (1.1). E.g. in [4,10] Q takes the form $a_+ - \mu a_-$ with $a_\pm \geq 0$ continuous functions and $\mu \to \infty$. A similar analysis for $Q = \delta a_+ - a_-$ and $\delta \to 0$ is contained in [12]. Similarly as in our results the relative contribution of the negative and the positive part of Q varies with a changing parameter. Observe though that the change there occurs in the values of Q while the regions where $Q > 0$ and $Q < 0$ are fixed. The only result we are aware of that deals with changing the set $\{x : Q(x) > 0\}$ through a parameter is [1]. In that paper a small region of diameter $\delta > 0$ with $Q \equiv 0$ is enclosed in a region where $Q > 0$, and the behaviour as $\delta \to 0$ is considered. Nevertheless, this is different from our case, where a region with $Q < 0$ encloses a core with $Q > 0$.

Now we formulate our assumptions in a precise manner. Let Ω be a domain in $\mathbb{R}^N$ and assume without loss of generality that $0 \in \Omega$. Ω may be unbounded and we do not exclude the case $\Omega = \mathbb{R}^N$. We will be concerned with the problem

$$(\mathrm{P}_n) \qquad \begin{cases} -\Delta u + V_n(x)u = Q_n(x)|u|^{p-2}u, \quad x \in \Omega \\ u(x) = 0 \text{ as } x \in \partial\Omega, \quad u(x) \to 0 \text{ as } |x| \to \infty, \end{cases}$$

where $p \in (2, 2^*)$. Of course, the first condition in the second line of (P_n) is void if $\Omega = \mathbb{R}^N$ and the second condition is void if Ω is bounded. We make the following assumptions concerning V_n and Q_n:

(A1) $V \in L^\infty(\Omega)$, $V \geq 0$ and $\sigma(-\Delta + V) \subset (0, \infty)$, where the spectrum σ is realized in $H_0^1(\Omega)$. $V_n = V + K_n$, where $K_n \in L^\infty(\Omega)$, and there exists a constant B such that $\|K_n\|_\infty \leq B$ for all n. Moreover, for each $\varepsilon > 0$ there is N_ε such that $\operatorname{supp} K_n \subset B_\varepsilon(0)$ whenever $n \geq N_\varepsilon$.

(A2) $Q_n \in L^\infty(\Omega)$, $Q_n > 0$ on a set of positive measure and there exists a constant C such that $\|Q_n\|_\infty \leq C$ for all n. Moreover, for each $\varepsilon > 0$ there exist constants $\delta_\varepsilon > 0$ and N_ε such that $Q_n \leq -\delta_\varepsilon$ whenever $x \notin B_\varepsilon(0)$ and $n \geq N_\varepsilon$.

The following are two typical examples of Q_n which we have in mind.

Example 1.1. (a) We let $\varepsilon_n \to 0$ and

$$Q_n(x) := \begin{cases} 1, & |x| < \varepsilon_n \\ -1, & |x| > \varepsilon_n. \end{cases}$$

(b) Let Q be a bounded continuous function such that $Q(x) < Q(0)$ for all $x \neq 0$ and the diameter of the set $\{x : \lambda \leq Q(x) \leq Q(0)\}$ tends to 0 as $\lambda \nearrow Q(0)$. Put $Q_n(x) := Q(x) - \lambda_n$, where $\lambda_n \nearrow Q(0)$ as $n \to \infty$.

Remark 1.2. As we shall see, the property (A2) is the one which causes concentration. Concerning (A1), we do not exclude the case of $K_n = 0$, i.e., $V_n = V$ for all n.

Let $E := H_0^1(\Omega)$. According to (A1),

$$\|u\| := \left(\int_\Omega (|\nabla u|^2 + V u^2)\, dx \right)^{1/2}$$

is an equivalent norm in E. The notation $\|\cdot\|$ will always refer to this norm. We also set

$$\|u\|_n := \left(\int_\Omega (|\nabla u|^2 + V_n u^2)\, dx \right)^{1/2},$$

$$|u|_{q,A} := \left(\int_A |u|^q\, dx \right)^{1/q},$$

$|u|_{\infty,A} := \operatorname{ess\,sup}_A |u|$, and we abbreviate $|u|_{q,\Omega}$ to $|u|_q$. For $r > 0$ and $a \in \mathbb{R}^N$, we put

$$B_r(a) := \{x \in \mathbb{R}^N : |x - a| < r\}.$$

Weak convergence will be denoted by "$\rightharpoonup$".

In Section 2 we show that (P_n) has a ground state solution and that any sequence of solutions (u_n) to (P_n) concentrates at the origin in the H^1- and the L^p-norm. In Section 3 concentration in the L^q-norms for different q is considered and in Section 4 it is shown that if Q_n is positive in a neighbourhood of a finite number of points, then ground states concentrate at one of these points.

2 Concentration in the H^1- and L^p-norms

Proposition 2.1. *For all n large enough, $\|\cdot\|_n$ is a uniformly equivalent norm in E, i.e., there exist constants $c_1, c_2 > 0$ and $N_0 \ge 1$ such that*

$$c_1\|u\| \le \|u\|_n \le c_2\|u\| \quad \textit{for all } u \in E \textit{ and } n \ge N_0.$$

In what follows we always assume n is so large that the conclusion of this proposition holds.

Proof. Let $\mathcal{K}_n : E \to E$ be the linear operator given by

$$\langle \mathcal{K}_n u, v\rangle := \int_\Omega K_n uv\, dx.$$

Using the Hölder and Sobolev inequalities we see that for each $\varepsilon > 0$ there is N_ε such that

$$\begin{aligned} |\langle \mathcal{K}_n u, v\rangle| \le C_1 \int_{B_\varepsilon(0)} |uv|\, dx &\le C_1 |B_\varepsilon(0)|^{(q-2)/q} |u|_q |v|_q \\ &\le C_2 |B_\varepsilon(0)|^{(q-2)/q} \|u\|\,\|v\| \quad \text{for all } n \ge N_\varepsilon, \end{aligned}$$

where $q = 2^*$ if $N \ge 3$, $q > 2$ if $N = 1$ or 2, $|B_\varepsilon(0)|$ denotes the measure of $B_\varepsilon(0)$ and C_1, C_2 are constants independent of ε and n. Now the conclusion easily follows by taking ε small enough. □

Next we prove our main existence result for (P_n).

Theorem 2.2. *Suppose that V_n and Q_n satisfy* (A1), (A2) *above and $p \in (2, 2^*)$. Then for all sufficiently large n problem* (P_n) *has a positive ground state solution $u_n \in E$. Moreover, there exists a constant $\alpha > 0$, independent of n, such that $\|u_n\| \ge \alpha$.*

Proof. Let $J_n(v) := \int_\Omega Q_n|v|^p\, dx$ and

$$s_n := \inf_{J_n(v)>0} \frac{\|v\|_n^2}{J_n(v)^{2/p}} \equiv \inf_{J_n(v)>0} \frac{\int_\Omega(|\nabla v|^2 + V_n v^2)\, dx}{\left(\int_\Omega Q_n |v|^p\, dx\right)^{2/p}}.$$

If the infimum is attained at v_n, then it follows via the Lagrange multiplier rule that $u_n = c_n v_n$ is a solution of (P_n) for an appropriate $c_n > 0$. Moreover, since v_n may be replaced by $|v_n|$, we may assume $v_n \ge 0$ (and hence $u_n \ge 0$). To show that $u_n > 0$, we note that u_n satisfies

$$-\Delta v + (V(x) + Q_n^-(x)u_n(x)^{p-2})v = Q_n^+(x)u_n(x)^{p-1} \ge 0,$$

where $Q_n^\pm(x) := \max\{\pm Q_n(x), 0\}$. Since $V(x) + Q_n^-(x)u_n(x)^{p-2} \ge 0$, it follows from the strong maximum principle (see e.g. [9, Theorem 8.19]) that $v_n > 0$ (in fact it can be shown that all ground states have constant sign).

If $u_n \neq 0$ is a solution to (P_n), then, multiplying the equation by u_n, integrating by parts and using the Sobolev inequality, we obtain

$$\|u_n\|_n^2 = \int_\Omega Q_n |u_n|^p \, dx \le C_1 |u_n|_p^p \le C_2 \|u_n\|_n^p, \tag{2.1}$$

hence according to Proposition 2.1, $\|u_n\| \ge \alpha$ for some $\alpha > 0$ and all large n.

It remains to show that the infimum is attained. Let (v_k) be a minimizing sequence for s_n, normalized by $J_n(v_k) = 1$. Then $\|v_k\|_n$ is bounded, so we may assume passing to a subsequence that $v_k \rightharpoonup v$ in E and $v_k(x) \to v(x)$ a.e. in Ω. Since the norm is lower semicontinuous and $Q_n < 0$ on $|x| > 1$ for n large, it follows from the Rellich-Kondrachov theorem and Fatou's lemma (applied on the set $|x| > 1$) that

$$\begin{aligned} s_n = \lim_{k\to\infty} \|v_k\|_n^2 = \lim_{k\to\infty} & \frac{\|v_k\|_n^2}{\left(\int_{|x|<1} Q_n |v_k|^p \, dx + \int_{|x|>1} Q_n |v_k|^p \, dx\right)^{2/p}} \\ & \ge \frac{\|v\|_n^2}{J_n(v)^{2/p}} \ge s_n. \end{aligned}$$

Thus v is a minimizer. $\square$

Note that the only properties of V_n and Q_n which are essential in this proof are that $\|\cdot\|_n$ is a norm, $Q_n \in L^\infty(\Omega)$, $Q_n > 0$ on a set of positive measure and $Q_n(x) \le 0$ for all $|x|$ large enough.

Remark 2.3. (a) We see from (2.1) that $\|u\| \ge \alpha$ for any nontrivial solution u of (P_n) provided n is large enough.

(b) Since the Krasnoselskii genus of the manifold $J_n(v) = 1$ is infinite and the functional $v \mapsto \int_\Omega Q_n^+ |v|^p \, dx$ is weakly continuous, it is not difficult to see using standard minimax methods that (P_n) has infinitely many solutions. Since we shall not use this result, we leave out the details.

(c) The observation that $Q < 0$ outside a large ball implies compactness (and thus existence of solutions) seems to go back to [7].

In the sequel suppose for each n that u_n is a nontrivial solution of (P_n) and set $w_n := u_n / \|u_n\|_n$.

Lemma 2.4. $\|u_n\| \to \infty$ *as* $n \to \infty$.

Proof. Assuming the contrary, $u_n \rightharpoonup u$ in E and $u_n \to u$ in $L^p_{loc}(\Omega)$ after passing to a subsequence. Multiplying (P_n) (with $u = u_n$) by u_n, integrating and using the fact that

$Q_n < 0$ for each $\varepsilon > 0$ and $n \geq N_\varepsilon$, we obtain

$$\limsup_{n\to\infty} \|u_n\|_n^2 = \limsup_{n\to\infty} \int_\Omega Q_n |u_n|^p \, dx$$
$$\leq \limsup_{n\to\infty} \int_{|x|<\varepsilon} Q_n |u_n|^p \, dx \leq C \int_{|x|<\varepsilon} |u|^p \, dx.$$

Letting $\varepsilon \to 0$ and using Proposition 2.1, we see that $u_n \to 0$ in E, a contradiction because $\|u_n\| \geq \alpha > 0$. □

Lemma 2.5. *$w_n \rightharpoonup 0$ in E as $n \to \infty$.*

Proof. Passing to a subsequence we may assume that $w_n \rightharpoonup w$ in E. Multiplying (P_n) (with $u = u_n$) by $u_n/\|u_n\|_n^2$, we obtain

$$1 = \|w_n\|_n^2 = \|u_n\|_n^{p-2} \int_\Omega Q_n |w_n|^p \, dx. \tag{2.2}$$

By Lemma 2.4, $\int_\Omega Q_n |w_n|^p \, dx \to 0$. Let $0 < \varepsilon < \varepsilon_1$. Then

$$0 = \lim_{n\to\infty} \int_\Omega Q_n |w_n|^p \, dx = \lim_{n\to\infty} \left(\int_{|x|<\varepsilon} Q_n |w_n|^p \, dx + \int_{|x|>\varepsilon} Q_n |w_n|^p \, dx \right)$$
$$\leq \lim_{n\to\infty} \left(\int_{|x|<\varepsilon} Q_n |w_n|^p \, dx + \int_{|x|>\varepsilon_1} Q_n |w_n|^p \, dx \right)$$
$$\leq C \int_{|x|<\varepsilon} |w|^p \, dx - \delta_{\varepsilon_1} \int_{|x|>\varepsilon_1} |w|^p \, dx.$$

If $w \neq 0$, we may chose ε_1 so small that the second integral on the right-hand side above is positive. Letting $\varepsilon \to 0$, we obtain a contradiction. □

Now we can study concentration of (u_n) as $n \to \infty$. Let $\varepsilon > 0$ be given and let $\chi \in C^\infty(\Omega, [0,1])$ be such that $\chi(x) = 0$ for $x \in B_{\varepsilon/2}(0)$ and $\chi(x) = 1$ for $x \notin B_\varepsilon(0)$. Multiplying (P_n) (with $u = u_n$) by χu_n we obtain

$$\int_\Omega (\nabla u_n \cdot \nabla(\chi u_n) + \chi V_n u_n^2) \, dx = \int_\Omega \chi Q_n |u_n|^p \, dx,$$

or equivalently,

$$\int_\Omega \chi(|\nabla u_n|^2 + V_n u_n^2) \, dx - \int_\Omega \chi Q_n |u_n|^p \, dx = -\int_\Omega u_n \nabla\chi \cdot \nabla u_n \, dx.$$

Given $\varepsilon > 0$, we have $Q_n \le -\delta_\varepsilon$ and $V_n = V \ge 0$ on $\operatorname{supp}\chi$ provided n is large enough. Hence for all such n,

$$
\begin{aligned}
(2.3)\quad \int_{\Omega\setminus B_\varepsilon(0)} (|\nabla u_n|^2 + V_n u_n^2)\, dx + \delta_\varepsilon \int_{\Omega\setminus B_\varepsilon(0)} |u_n|^p\, dx \\
&\le \int_\Omega \chi(|\nabla u_n|^2 + V_n u_n^2)\, dx - \int_\Omega \chi Q_n |u_n|^p\, dx \\
&\le d_\varepsilon \int_{B_\varepsilon(0)\setminus B_{\varepsilon/2}(0)} |u_n|\, |\nabla u_n|\, dx,
\end{aligned}
$$

where d_ε is a constant independent of n. Since $w_n = u_n/\|u_n\|_n \to 0$ in $L^2_{loc}(\Omega)$ according to Lemma 2.5, it follows from Hölder's inequality that

$$\int_{B_\varepsilon(0)\setminus B_{\varepsilon/2}(0)} |w_n|\, |\nabla w_n|\, dx \to 0.$$

So (2.3) implies

$$(2.4)\qquad \lim_{n\to\infty} \left(\int_{\Omega\setminus B_\varepsilon(0)} (|\nabla w_n|^2 + V_n w_n^2)\, dx + \delta_\varepsilon \|u_n\|_n^{p-2} \int_{\Omega\setminus B_\varepsilon(0)} |w_n|^p\, dx \right) = 0.$$

Theorem 2.6. *Suppose that V_n and Q_n satisfy* (A1), (A2) *and $p \in (2, 2^*)$. Let u_n be a nontrivial solution for* (P_n) *and let $w_n = u_n/\|u_n\|_n$. Then for every $\varepsilon > 0$ it holds that*

$$(2.5)\qquad \lim_{n\to\infty} \int_{\Omega\setminus B_\varepsilon(0)} (|\nabla w_n|^2 + V_n w_n^2)\, dx = 0$$

and

$$(2.6)\qquad \lim_{n\to\infty} \|u_n\|_n^{p-2} \int_{\Omega\setminus B_\varepsilon(0)} |w_n|^p\, dx = 0.$$

Moreover,

$$\lim_{n\to\infty} \frac{\int_{\Omega\setminus B_\varepsilon(0)} (|\nabla u_n|^2 + V_n u_n^2)\, dx}{\int_\Omega (|\nabla u_n|^2 + V_n u_n^2)\, dx} = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{\int_{\Omega\setminus B_\varepsilon(0)} |u_n|^p\, dx}{\int_\Omega |u_n|^p\, dx} = 0.$$

Proof. The first conclusion is an immediate consequence of (2.4). Since

$$\int_\Omega (|\nabla w_n|^2 + V_n w_n^2)\, dx \equiv \|w_n\|_n^2 = 1,$$

it follows from (2.5) that

$$\lim_{n\to\infty} \frac{\int_{\Omega\setminus B_\varepsilon(0)} (|\nabla u_n|^2 + V_n u_n^2)\, dx}{\int_\Omega (|\nabla u_n|^2 + V_n u_n^2)\, dx} = \lim_{n\to\infty} \frac{\int_{\Omega\setminus B_\varepsilon(0)} (|\nabla w_n|^2 + V_n w_n^2)\, dx}{\int_\Omega (|\nabla w_n|^2 + V_n w_n^2)\, dx} = 0.$$

By (2.2)

$$C\|u_n\|_n^{p-2}\int_\Omega |w_n|^p\,dx \ge \|u_n\|_n^{p-2}\int_\Omega Q_n|w_n|^p\,dx = \|w_n\|_n^2 = 1.$$

This and (2.6) imply

$$\lim_{n\to\infty}\frac{\int_{\Omega\setminus B_\varepsilon(0)}|u_n|^p\,dx}{\int_\Omega |u_n|^p\,dx} = \lim_{n\to\infty}\frac{\|u_n\|_n^{p-2}\int_{\Omega\setminus B_\varepsilon(0)}|w_n|^p\,dx}{\|u_n\|_n^{p-2}\int_\Omega |w_n|^p\,dx} = 0.$$

□

3 Concentration in the L^q-norm

Here we consider concentration in other norms. Note that

$$\frac{N(p-2)}{2} < p \qquad \text{and} \qquad \text{if } N\ge 3, \text{ then } \frac{2N-2}{N-2} < 2^*.$$

Theorem 3.1. *Suppose that* (A1) *and* (A2) *hold and there exist* $R,\lambda > 0$ *such that* $V \ge \lambda$ *whenever* $x \in \Omega\setminus B_R(0)$. *For every* $n\in\mathbb{N}$ *let* u_n *denote a nontrivial solution to* (P_n). *If* $\varepsilon > 0$ *is such that* $\overline{B}_\varepsilon(0)\subset\Omega$, *then the following hold:*

(a) *For every* $q\in[1,\infty]$ *the norm* $|u_n|_{q,\Omega\setminus B_\varepsilon(0)}$ *remains bounded, uniformly in* n.

(b) *If* $\delta = \delta_\varepsilon > 0$ *in* (A2) *can be chosen independently of* $\varepsilon > 0$, *if* $N\ge 3$ *and* $p \in \left[\frac{2N-2}{N-2}, 2^*\right)$, *then* $\lim_{n\to\infty}|u_n|_{q,\Omega\setminus B_\varepsilon(0)} = 0$, *for every* $q\in[1,\infty]$.

(c) *For every* $q\ge 1$, $q\in\left(\frac{N(p-2)}{2},\infty\right]$ *it holds that* $\lim_{n\to\infty}|u_n|_q = \infty$ *and*

$$\lim_{n\to\infty}\frac{|u_n|_{q,\Omega\setminus B_\varepsilon(0)}}{|u_n|_q} = 0. \tag{3.1}$$

(d) *If* $\frac{N(p-2)}{2}\ge 1$, *then for* $q = \frac{N(p-2)}{2}$ *it holds that*

$$\liminf_{n\to\infty}|u_n|_q > 0. \tag{3.2}$$

If the hypotheses in (b) *are satisfied, then* (3.1) *holds for this* q.

Note that $V\ge\lambda>0$ for $x\in\Omega\setminus B_R(0)$ is trivially satisfied if Ω is bounded and R large enough. Note also that it follows from the Poincaré inequality that the above condition and $V\ge 0$ for all x imply $\sigma(-\Delta+V)\subset(0,\infty)$.

Remark 3.2. It holds that

$$\frac{N(p-2)}{2} < 2 \qquad \text{iff} \qquad p < 2 + \frac{4}{N}. \tag{3.3}$$

In that case (u_n) concentrates with respect to the L^q-norm for every $q \in [2, \infty]$, covering the physically interesting L^2-concentration. In particular, for the travelling planar waves considered in the introduction we have $N = 1$ or 2. In a Kerr medium, where $p = 4$, (3.3) and Theorem 3.1 yield concentration near 0 with respect to the L^2-norm for $N = 1$ but not for $N = 2$.

Proof of Theorem 3.1. To prove (a), fix $\delta_{\varepsilon/2} > 0$ and $N_{\varepsilon/2}$ as in (A2). By [2, Sect. 1.6] there is a positive classical solution w of the equation

$$-\Delta u = -\delta_{\varepsilon/2}|u|^{p-2}u$$

on $\mathbb{R}^N \backslash \overline{B}_{\varepsilon/2}(0)$ that satisfies $\lim_{|x|\to\varepsilon/2} w(x) = \infty$ and $\lim_{|x|\to\infty} w(x) = 0$. Fixing $n \geq N_{\varepsilon/2}$, setting $z_n := w - u_n$ and

$$\varphi_n(x) := (p-1)\int_0^1 |sw(x) + (1-s)u_n(x)|^{p-2}\,\mathrm{d}s \geq 0$$

we obtain

$$\begin{aligned}
\varphi_n z_n &= (p-1)\int_0^1 |sw + (1-s)u_n|^{p-2}(w - u_n)\,ds \\
&= \int_0^1 \frac{d}{ds}\left(|sw + (1-s)u_n|^{p-2}(sw + (1-s)u_n)\right)ds = w^{p-1} - |u_n|^{p-2}u_n
\end{aligned}$$

and hence from (A2)

$$(-\Delta + V - Q_n\varphi_n)z_n = -\Delta w + Vw - Q_n w^{p-1} \geq -\Delta w + \delta_{\varepsilon/2}w^{p-1} = 0.$$

Note that $V - Q_n\varphi_n \geq 0$ in $\Omega\backslash\overline{B}_{\varepsilon/2}(0)$ since $n \geq N_{\varepsilon/2}$. By the continuity of u_n and since $w_n(x) \to \infty$ as $x \to \partial B_{\varepsilon/2}(0)$, there is $r \in (\varepsilon/2, \varepsilon)$ such that $z_n \geq 0$ on $\partial B_r(0)$. Moreover, $z_n \geq 0$ on $\partial\Omega$. If Ω is bounded then we may apply the maximum principle for weak supersolutions [9, Theorem 8.1] to z_n and obtain $z_n \geq 0$ in $\Omega\backslash B_r(0)$. If Ω is unbounded, we consider any $\gamma > 0$ and pick $\widetilde{R} > 0$ such that $z_n \geq -\gamma$ in $\Omega\backslash B_{\widetilde{R}}(0)$. This is possible since $w(x)$ tends to 0 as $|x| \to \infty$ by construction. Moreover, $u_n \in E$ and standard estimates from regularity theory imply that also $u_n(x) \to 0$ as $|x| \to \infty$. Now the same maximum principle, applied on $\Omega \cap (B_{\widetilde{R}}(0)\backslash\overline{B}_r(0))$, implies $z_n \geq -\gamma$ in all of $\Omega\backslash B_r(0)$. Letting $\gamma \to 0$ we obtain $z_n \geq 0$ also in this case. In an analogous way we obtain $u_n \geq -w$ (take $z_n := w + u_n$), and hence

$$|u_n| \leq w \qquad \text{in } \Omega\backslash B_\varepsilon(0), \text{ for all } n \geq N_{\varepsilon/2}.$$

Note that w is continuous in $\overline{\Omega}\backslash B_\varepsilon(0)$. Hence (a) follows if Ω is bounded. For unbounded Ω, according to Lemma 3.3 below, setting $M := \max_{|x|=R} w$ we obtain $|u_n| \le Me^{-\alpha|x-R|}$ whenever $x \in \Omega \setminus B_R(0)$. So the conclusion in (a) holds also in this case.

Next we consider (b). The hypotheses imply that there is $\delta > 0$ such that $Q_n \le -\delta$ on $\Omega\backslash B_{1/n}(0)$ for every n large enough. Denote by w_n a positive solution of

$$-\Delta u = -\delta|u|^{p-2}u \tag{3.4}$$

on $\mathbb{R}^N\backslash B_{1/n}(0)$ with boundary conditions $\lim_{|x|\to 1/n} w_n(x) = \infty$ and $\lim_{|x|\to\infty} w_n(x) = 0$, as before. Then the sequence w_n is monotone decreasing since $w_n \ge w_{n+1}$ on $B_{1/n}(0)$ for every $n \in \mathbb{N}$ by the maximum principle (using similar arguments as before). Therefore w_n converges locally uniformly to a nonnegative solution w of (3.4) on $\mathbb{R}^N\backslash\{0\}$. Our hypotheses on N and p, and [5, Theorem 2] imply that w extends to an entire solution of (3.4). By [2, Theorem 1.3] $w \equiv 0$. On the other hand, the function w_n dominates the solution u_n on $\overline{\Omega}\backslash B_r(0)$ for some $r \in (\varepsilon/2, \varepsilon)$ and large n, as seen in the proof of (a). Therefore also u_n converges to 0 locally uniformly in $\Omega\backslash B_r(0)$. Together with Lemma 3.3 (take $M := \max_{|x|=R} w_n$) we obtain $\lim_{n\to\infty}|u_n|_{q,\Omega\backslash\overline{B}_\varepsilon(0)} = 0$.

In the proof of (c) first consider the case $q \ge 1$, $q \in (N(p-2)/2, p]$. Since u_n is a solution, by (A1), Hölder's inequality, the Sobolev embedding, and Proposition 2.1 we have

$$\|u_n\|_n^2 = \int_\Omega Q_n|u_n|^p \le C_1|u_n|_p^p \le C_1|u_n|_q^{p\theta}|u_n|_{2^*}^{p(1-\theta)} \le C_2|u_n|_q^{p\theta}\|u_n\|_n^{p(1-\theta)}. \tag{3.5}$$

Here C_1, C_2 are independent of n, and θ satisfies

$$\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{2^*}.$$

From Lemma 2.4 we see that it is sufficient to impose $p(1-\theta) < 2$ or, equivalently, $q > N(p-2)/2$. This and (a) prove the case $q \in (N(p-2)/2, p]$.

Since we already know from (3.5) that $|u_n|_p \to \infty$, (a) yields $|u_n|_{p,B_\varepsilon(0)} \to \infty$ and hence $|u_n|_{q,B_\varepsilon(0)} \to \infty$ as $n \to \infty$, for every $q \in [p, \infty]$. Now (3.1) follows from (a).

To prove (d) we note that (3.5) implies (3.2) for $q = \frac{N(p-2)}{2}$. The other claim is obvious. $\square$

Lemma 3.3. *Suppose Ω is unbounded and $V(x) \ge \lambda > 0$ for $x \in \Omega \setminus B_R(0)$. If u_n is a nontrivial solution to* (P_n) *and $|u_n| \le M$ on $\partial B_R(0)$, then $|u_n| \le Me^{-\alpha|x-R|}$ for $x \in \Omega \setminus B_R(0)$, where $\alpha := \sqrt{\lambda}$.*

Proof. We follow the argument in [17, Proposition 4.4]. Write $u = u_n$ and let

$$W(x) := Me^{-\alpha(|x|-R)},$$
$$\Omega_S := \{x \in \Omega : R < |x| < S \text{ and } u(x) > W(x)\}.$$

Condition (A2) implies that there is $\delta > 0$ such that for $x \in \Omega_S$ we have $u(x) > 0$ and

$$-\Delta u \le -V(x)u - \delta|u|^{p-2}u \le -\lambda u,$$

hence

$$\Delta(W - u) = \left(\alpha^2 - \frac{\alpha(N-1)}{|x|}\right) W - \Delta u \le \alpha^2(W - u) \le 0 \tag{3.6}$$

for such x. By the maximum principle,

$$W(x) - u(x) \ge \min_{x \in \partial\Omega_S}(W - u) \ge \min\left\{0, \min_{|x|=S}(W - u)\right\}.$$

Since $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} W(x) = 0$, letting $S \to \infty$ we obtain

$$u(x) \le W(x) = Me^{-\alpha(|x|-R)} \quad \text{for} \quad x \in \Omega \setminus B_R(0).$$

Replacing $u(x) > W(x)$ by $u(x) < -W(x)$ in the definition of Ω_S and $W - u$ by $W + u$ in (3.6), we see that $u \ge -W$ for $x \in \Omega \setminus B_R(0)$. □

Remark 3.4. In the proof of (b) it was essential that (3.4) has no nontrivial solution $w \ge 0$ in $\mathbb{R}^N \setminus \{0\}$. If $2 < p < (2N-2)/(N-2)$, this argument cannot be used because $w = c_p|x|^{-2/(p-2)}$ is a solution of (3.4) for a suitable constant $c_p > 0$ (note that if $p > (2N-2)/(N-2)$, then w solves equation (3.4) with δ replaced by $-\delta$).

4 Concentration at several points

In this section we assume that the functions Q_n are positive in a neighbourhood of two distinct points $x_1, x_2 \in \Omega$ and V_n may not be equal to V in this neighbourhood. More precisely, we assume

(A3) $V \in L^\infty(\Omega)$, $V \ge 0$ and $\sigma(-\Delta + V) \subset (0, \infty)$, where the spectrum σ is realized in $H_0^1(\Omega)$. $V_n = V + K_n$, where $K_n \in L^\infty(\Omega)$, and there exists a constant B such that $\|K_n\|_\infty \le B$ for all n. Moreover, for each $\varepsilon > 0$ there is N_ε such that $\operatorname{supp} K_n \subset B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$ whenever $n \ge N_\varepsilon$.

(A4) $Q_n \in L^\infty(\Omega)$, $Q_n > 0$ in a neighbourhood of $\{x_1\} \cup \{x_2\}$ and there exists a constant C such that $\|Q_n\|_\infty \le C$ for all n. Moreover, for each $\varepsilon > 0$ there exist constants $\delta_\varepsilon > 0$ and N_ε such that $Q_n \le -\delta_\varepsilon$ whenever $x \notin B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$ and $n \ge N_\varepsilon$.

We have taken two points x_1, x_2 for notational convenience only. The arguments below are valid for any finite number of points in Ω.

It is clear that the arguments of Section 2 go through with obvious changes if one replaces (A1)-(A2) by (A3)-(A4). Our purpose here is to show that if (A3)-(A4) hold, then each ground state u_n for n large concentrates exactly at one of the points x_1, x_2. In Section 2 u_n could be any nontrivial solution to (P_n). To the contrary, in Theorem 4.1 below it is important that u_n is a ground state.

As in Section 2, we put $J_n(u) = \int_\Omega Q_n|u|^p\,dx$ and

$$s_n := \inf_{J_n(u)>0} \frac{\|u\|_n^2}{J_n(u)^{2/p}} \equiv \inf_{J_n(u)>0} \frac{\int_\Omega(|\nabla u|^2 + V_n u^2)\,dx}{\left(\int_\Omega Q_n|u|^p\,dx\right)^{2/p}}.$$

Theorem 4.1. *Suppose that V_n and Q_n satisfy* (A3), (A4) *and $p \in (2, 2^*)$. Let u_n be a ground state solution for* (P_n). *Then, for n large, u_n concentrates at x_1 or x_2. More precisely, for each $\varepsilon > 0$ we have, passing to a subsequence,*

$$\lim_{n\to\infty} \frac{\int_{\Omega\setminus B_\varepsilon(x_j)}(|\nabla u_n|^2 + V_n u_n^2)\,dx}{\int_\Omega(|\nabla u_n|^2 + V_n u_n^2)\,dx} = 0 \text{ and } \lim_{n\to\infty} \frac{\int_{\Omega\setminus B_\varepsilon(x_j)} Q_n|u_n|^p\,dx}{\int_\Omega Q_n|u_n|^p\,dx} = 0 \tag{4.1}$$

for $j = 1$ or 2 (but not for $j = 1$ and 2).

Remark 4.2. Note that in view of the obvious modification of Theorem 2.6 the limits in (4.1) are 0 if $\Omega \setminus B_\varepsilon(x_j)$ is replaced by $\Omega \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2))$. So if $j = 1$ in (4.1), then concentration occurs at x_1 and if $j = 2$, it occurs at x_2.

Proof of Theorem 4.1. Renormalizing, we may assume that $J_n(u_n) = \int_\Omega Q_n|u_n|^p\,dx = 1$ (then u_n may not be a solution of (P_n) but we still have $s_n = \|u_n\|^2/J_n(u_n)^{2/p}$). Let $\xi_j \in C_0^\infty(\Omega, [0,1])$ be a function such that $\xi_j = 1$ on $B_{\varepsilon/2}(x_j)$ and $\xi_j = 0$ on $\Omega \setminus B_\varepsilon(x_j)$, $j = 1, 2$, where ε is so small that $\overline{B}_\varepsilon(x_j) \subset \Omega$ and $\overline{B}_\varepsilon(x_1) \cap \overline{B}_\varepsilon(x_2) = \emptyset$. Set $v_n := \xi_1 u_n$, $w_n := \xi_2 u_n$, $z_n := u_n - v_n - w_n$. Since $\operatorname{supp} z_n \subset \Omega \setminus (B_{\varepsilon/2}(x_1) \cup B_{\varepsilon/2}(x_2))$ and the conclusion of Theorem 2.6 remains valid after an obvious modification, we have

$$\begin{aligned}
\|u_n\|_n^2 &= \int_\Omega (|\nabla u_n|^2 + V_n u_n^2)\,dx \\
&= \left(\int_\Omega (|\nabla v_n|^2 + V_n v_n^2)\,dx + \int_\Omega (|\nabla w_n|^2 + V_n w_n^2)\,dx\right)(1 + o(1)) \\
&= (\|v_n\|_n^2 + \|w_n\|_n^2)(1 + o(1))
\end{aligned}$$

and

$$\begin{aligned}
1 = J_n(u_n) = \int_\Omega Q_n|u_n|^p\,dx &= \int_\Omega Q_n|v_n|^p\,dx + \int_\Omega Q_n|w_n|^p\,dx + o(1) \\
&= J_n(v_n) + J_n(w_n) + o(1).
\end{aligned}$$

Assume first that $\limsup_{n\to\infty} J_n(v_n) \geq 0$ and $\limsup_{n\to\infty} J_n(w_n) \geq 0$. Then, passing to a subsequence, $J_n(v_n) \to c_0 \in [0,1]$ and $J_n(w_n) \to 1 - c_0 \in [0,1]$. Suppose $c_0 \in (0,1)$. Since

$p > 2$, for n large enough we have

$$\begin{aligned} s_n &= \frac{\|u_n\|_n^2}{J_n(u_n)^{2/p}} = \frac{(\|v_n\|_n^2 + \|w_n\|_n^2)(1+o(1))}{(J_n(v_n) + J_n(w_n) + o(1))^{2/p}} \\ &> \frac{\|v_n\|_n^2 + \|w_n\|_n^2}{J_n(v_n)^{2/p} + J_n(w_n)^{2/p}} \geq \min\left\{ \frac{\|v_n\|_n^2}{J_n(v_n)^{2/p}}, \frac{\|w_n\|_n^2}{J_n(w_n)^{2/p}} \right\} \geq s_n, \end{aligned}$$

a contradiction. So $c_0 = 0$ or 1. If $c_0 = 1$ (say), then the second limit in (4.1) is 0 for $j = 1$ because $\operatorname{supp} v_n \subset B_\varepsilon(x_1)$. Also the first limit is 0 since otherwise $\|w_n\|_n^2/\|v_n\|_n^2$ is bounded away from 0 for large n, and we obtain

$$s_n = \frac{(\|v_n\|_n^2 + \|w_n\|_n^2)(1+o(1))}{(J_n(v_n) + J_n(w_n) + o(1))^{2/p}} > \frac{\|v_n\|_n^2}{J_n(v_n)^{2/p}} \geq s_n, \tag{4.2}$$

a contradiction again.

Finally, suppose $\limsup_{n\to\infty} J_n(w_n) < 0$ (the case $\limsup_{n\to\infty} J_n(v_n) < 0$ is of course analogous). Passing to a subsequence, $J_n(w_n) \leq -\eta$ for some $\eta > 0$ and all n large enough. Then (4.2) holds for such n because $J_n(v_n) > J_n(v_n) + J_n(w_n) + o(1)$. □

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