Generalized linking theorem with an application to semilinear Schrödinger equation

Wojciech Kryszewski*
Department of Mathematics, Nicholas Copernicus University
Chopina 12/18, 87-100 Toruń, Poland

Andrzej Szulkin**
Department of Mathematics, Stockholm University
106 91 Stockholm, Sweden

Abstract

Consider the semilinear Schrödinger equation (*) $-\Delta u + V(x)u = f(x,u)$, $u \in H^1(\mathbf{R}^N)$. It is shown that if f, V are periodic in the x-variables, f is superlinear at $u = 0$ and $\pm\infty$ and 0 lies in a spectral gap of $-\Delta + V$, then (*) has at least 1 nontrivial solution. If in addition f is odd in u, then (*) has infinitely many (geometrically distinct) solutions. The proofs rely on a degree theory and a linking-type argument developed in this paper.

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Introduction

In this paper we shall be concerned with the semilinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbf{R}^N). \end{cases} \tag{0.1}$$

We assume that f and V are continuous functions, periodic with respect to the x-variables, f is superlinear at $u = 0$, superlinear (but subcritical) at $|u| = \infty$ and

$$F(x,u) := \int_0^u f(x,\xi)\,\mathrm{d}\xi$$

is positive for $u \neq 0$. Under these hypotheses the functional

$$\Phi(u) := \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x - \int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x \tag{0.2}$$

is of class C^1 on the (real) Sobolev space $H^1(\mathbf{R}^N)$ and critical points of Φ correspond to weak solutions of (0.1). Let $L : H^1(\mathbf{R}^N) \to H^1(\mathbf{R}^N)$ be the self-adjoint operator given by

$$(Lu, v) := \int_{\mathbf{R}^N} (\nabla u \cdot \nabla v + V(x)uv)\,\mathrm{d}x, \tag{0.3}$$

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where here and below $(\cdot,\cdot)$ denotes the standard inner product in $H^1(\mathbf{R}^N)$. Assume that L has bounded inverse. The operator $-\Delta + V$ (on $L^2(\mathbf{R}^N)$) has purely continuous spectrum which is bounded below and consists of closed disjoint intervals [15, Theorem XIII.100]. Hence either $\sigma(L) \subset (0,\infty)$ and the quadratic form $u \mapsto (Lu,u)$ is positive definite, or 0 lies in a gap of the spectrum of L and $H^1(\mathbf{R}^N) = Y \oplus Z$, where Y, Z are infinite-dimensional L-invariant subspaces such that the above form is negative definite on Y and positive definite on Z. If $\sigma(L) \subset (0,\infty)$, then it is known by a result of Coti Zelati and Rabinowitz [8] that (0.1) has infinitely many solutions. If 0 is in a spectral gap of L, then the functional Φ has the so-called linking geometry. More precisely, if ρ is sufficiently small and R sufficiently large, then $\Phi \ge b > 0$ on the sphere $N := \{u \in Z : \|u\| = \rho\}$ and $\Phi \le 0$ on the boundary of the set $M := \{u = y + \lambda z_0 : \|u\| < R,\ \lambda > 0\}$, where $y \in Y$ and z_0 is a fixed element of $Z \setminus \{0\}$. One can therefore expect that (0.1) has a solution $\overline{u}$ such that $\Phi(\overline{u}) \ge b$. However, since the spaces Y and Z are infinite-dimensional and the gradient of the functional $u \mapsto \int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x$ is not compact, one cannot employ the usual argument based on the Brouwer or the Leray-Schauder degree (cf. [13, Section 5]) in order to show that ∂M and N actually link (in the sense that if $\eta : \overline{M} \times [0,T] \to E$ is in a suitably restricted class of deformations of $\overline{M}$ and $\eta(\partial M, t) \cap N = \emptyset$ for $t \in [0,T]$, then $\eta(M,t) \cap N) \ne \emptyset$ for such t). If the function F is strictly convex in u, then it is sometimes possible to circumvent this difficulty. One can either use the Legendre transform and look for critical points of a dual functional which has the simpler mountain pass geometry [1, 2, 10]. Or one can take advantage of the fact that Φ is concave on $z + Y$ (for each fixed $z \in Z$) and reduce the problem to that of finding critical points of a functional which is defined on Z and also has the mountain pass geometry [6]. In a recent paper [16] Troestler and Willem have proved that (0.1) possesses a solution $\overline{u} \ne 0$ under some additional hypotheses on f which imply that the functional Φ is of class C^2. However, they made no convexity assumption on F. In order to show that the sets ∂M and N link they used an extension of Smale's degree for proper Fredholm mappings. The fact that $\Phi \in C^2$ played an important role.

In the present paper we shall show that (0.1) (with 0 in a spectral gap of L) has a solution $\overline{u} \ne 0$ under weaker hypotheses which only imply that $\Phi \in C^1$. For this purpose we shall introduce a new degree of Leray-Schauder type, and the degree construction is in fact one of the main goals of our paper. Let U be a bounded subset of a Hilbert space E_0. The admissible mappings will be of the form $I - h$, where I denotes the identity and for each $u \in \overline{U}$ there exists a weak neighbourhood W such that $h(\overline{U} \cap W)$ is contained in a finite-dimensional space. The fact that $I - h$ is proper with respect to the weak topology of $\overline{U}$ will then lead to a correct definition of degree. To show that ∂M and N link we construct deformations by using the flow of a certain pseudogradient vector field V. It turns out that in order to have some control on the level sets of Φ and at the same time obtain the above finite-dimensional property one needs to construct V in such a way that it is continuous with respect to the weak topology of Y and the strong topology of Z. We also formulate our linking result in an abstract form that extends the linking theorem of Benci and Rabinowitz [5, 13].

In the second part of the paper we shall show that (0.1) has infinitely many solutions under the additional assumption that the function f is odd in u. The proof will use a variant of Benci's pseudoindex [4] and the above degree (which will be needed in order to find sets of arbitrarily large pseudoindex). Since oddness is not necessary if $\sigma(L) \subset (0,\infty)$ (see [8]), it would be interesting to know if our result on the existence of infinitely many solutions remains valid also for non-odd f.

Observe that the assumption that 0 lies in a gap of the spectrum of L excludes the possibility of having constant V (because then $\sigma(-\Delta+V)=[V,\infty)$ in $L^2(\mathbf{R}^N)$ and there are no gaps).

The paper is organized as follows: In Section 1 we study the properties of the functional Φ. In Section 2 we introduce the weak-strong topology mentioned above and construct the degree. In Section 3 we show that (0.1) has at least one solution $\overline{u}\neq 0$ and state an abstract version of our linking theorem. In Section 4 we prove the existence of infinitely many solutions for odd f, and in Section 5 we briefly consider homoclinic solutions for a second order system of ordinary differential equations.

Notation. $B(a,\rho)$ and $S(a,\rho)$ denote respectively the open ball and the sphere centered at a and having radius ρ. Furthermore, $B_\rho := B(0,\rho)$ and $S_\rho := S(0,\rho)$. The closure of a set A is denoted by $\overline{A}$. For $p\geq 1$, $|\cdot|_p$ is the usual norm in $L^p(\mathbf{R}^N)$. By $\to$ we denote the strong and by $\rightharpoonup$ the weak convergence. $\|u-A\|$ is the distance from the point u to the set A (in the topology induced by the norm $\|\cdot\|$). By K we denote the set of critical points of Φ, i.e. $K:=\{u\in H^1(\mathbf{R}^N):\Phi'(u)=0\}$, and $\Phi^\beta:=\{u\in H^1(\mathbf{R}^N):\Phi(u)\leq\beta\}$, $\Phi_\alpha:=\{u\in H^1(\mathbf{R}^N):\Phi(u)\geq\alpha\}$ are the sub- and superlevel sets of Φ; moreover, $\Phi_\alpha^\beta:=\Phi_\alpha\cap\Phi^\beta$.

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1 Properties of the functional

Let $E:=H^1(\mathbf{R}^N)$ and assume that the following conditions are satisfied:

(A1) The function $f:\mathbf{R}^N\times\mathbf{R}\to\mathbf{R}$ is continuous and 1-periodic with respect to each variable x_j, $j=1,\dots,N$.

(A2) There is a constant $c>0$ such that

$$|f(x,u)|\leq c(1+|u|^{p-1})$$

for all $x\in\mathbf{R}^N$ and $u\in\mathbf{R}$, where $p>2$ if $N=1,2$ and $2<p<2^*:=\frac{2N}{N-2}$ if $N\geq 3$.

(A3) $f(x,u)=o(|u|)$ uniformly with respect to x as $|u|\to 0$.

(A4) There is $\gamma>2$ such that for all $x\in\mathbf{R}^N$ and $u\in\mathbf{R}\setminus\{0\}$,

$$0<\gamma F(x,u)\leq uf(x,u).$$

(A5) The function $V:\mathbf{R}^N\to\mathbf{R}$ is continuous and 1-periodic with respect to each variable x_j, $j=1,\dots,N$.

(A6) 0 lies in a gap of the spectrum of L (where L is given by (0.3)).

Let us now make some remarks.

Remark 1.1

(i) It is clear that instead of 1-periodicity of f and V in x_j, $j = 1, \dots, N$, we may assume that these functions are T_j-periodic, where $T_j > 0$ for $j = 1, \dots, N$.

(ii) Assumptions (A1), (A2) and (A3) imply that $\Phi \in C^1(E, \mathbf{R})$. Indeed, for any $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that

$$|f(x,u)| \le \varepsilon |u| + c_\varepsilon |u|^{p-1} \tag{1.1}$$

and the conclusion follows from [8, Proposition 2.1] or [17, Lemma 3.10].

Assumptions (A1), (A5) imply that Φ is invariant with respect to the $\mathbf{Z}^N$-action on E given by the formula

$$(g * u)(x) := u(g + x), \tag{1.2}$$

where $g \in \mathbf{Z}^N$, $u \in E$ and $x \in \mathbf{R}^N$. Moreover (see (0.2), (0.3)),

$$\Phi(u) = \frac{1}{2}(Lu, u) - \int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x, \quad u \in E, \tag{1.3}$$

$$\Phi'(u)v = (Lu, v) - \int_{\mathbf{R}^N} f(x,u)v\,\mathrm{d}x, \quad u, v \in E, \tag{1.4}$$

and clearly, Φ is bounded on bounded sets.

(iii) According to (A6), 0 lies in a gap of the spectrum of L. Spectral theory asserts that the space E decomposes as a direct sum of two infinite-dimensional L-invariant orthogonal subspaces Y and Z on which L is respectively negative and positive definite. Let $P : E \to Y$ and $Q : E \to Z$ be the orthogonal projections. We may introduce a new inner product in E by the formula

$$\langle u, v \rangle := (L(Q - P)u, v), \quad u, v \in E$$

and the corresponding norm

$$\|u\| := \sqrt{\langle u, u \rangle}, \quad u \in E.$$

Clearly, the inner products $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ are equivalent and the spaces Y, Z are orthogonal with respect to $\langle \cdot, \cdot \rangle$. Moreover, by (1.3) and (1.4) we easily see that

$$\Phi(u) = \frac{1}{2}(\|Qu\|^2 - \|Pu\|^2) - \int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x, \tag{1.5}$$

$$\langle \nabla\Phi(u), v \rangle = \langle Qu, v \rangle - \langle Pu, v \rangle - \int_{\mathbf{R}^N} f(x,u)v\,\mathrm{d}x, \tag{1.6}$$

where as usual the gradient $\nabla\Phi(u)$ is given by the formula $\langle \nabla\Phi(u), v \rangle = \Phi'(u)v$ for all $v \in E$. In view of assumption (A2), $\nabla\Phi : E \to E$ is weakly sequentially continuous (i.e. if $u_n \rightharpoonup u$, then $\nabla\Phi(u_n) \rightharpoonup \nabla\Phi(u)$). Indeed, let $u_n \rightharpoonup u$. Then $u_n \to u$ in $L^p_{loc}(\mathbf{R}^N)$; therefore $f(x, u_n) \to f(x,u)$ in $L^{p/(p-1)}_{loc}(\mathbf{R}^N)$ and $\langle \nabla\Phi(u_n), v \rangle \to \langle \nabla\Phi(u), v \rangle$ for each $v \in E$.

(iv) Since the spaces Y and Z are L-invariant, they are $\mathbf{Z}^N$-invariant. Indeed, spectral theory asserts that the projectors P, Q commute with any operator which commutes with L; in particular, they commute with the $\mathbf{Z}^N$-action described in (1.2).

(v) Assumption (A4) implies that given $\delta > 0$, there exists $c_1 = c_1(\delta) > 0$ such that

$$F(x,u) \geq c_1|u|^\gamma - \delta|u|^2 \tag{1.7}$$

for any $u \in \mathbf{R}$ and $x \in \mathbf{R}^N$.

Indeed, by (A4), if $0 < c_1 \leq \min_{\mathbf{R}^N} F(x, \pm 1)$, then for any $\delta > 0$ and $|u| \geq 1$,

$$F(x,u) \geq c_1|u|^\gamma \geq c_1|u|^\gamma - \delta|u|^2.$$

We may assume that $c_1 \leq \delta$. Since $F \geq 0$, it follows that for $|u| < 1$,

$$F(x,u) \geq c_1|u|^\gamma - \delta|u|^2.$$

Lemma 1.2 $\int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x = o(\|u\|^2)$ *as* $\|u\| \to 0$.

Proof By (1.1), for any $\varepsilon > 0$ there is $\widetilde{c}_\varepsilon > 0$ such that

$$F(x,u) \leq \frac{\varepsilon}{2}|u|^2 + \widetilde{c}_\varepsilon|u|^p.$$

Hence

$$\int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x \leq \frac{\varepsilon}{2}|u|_2^2 + \widetilde{c}_\varepsilon|u|_p^p$$

and by the Sobolev embedding theorem there is a constant $C > 0$ such that

$$\int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x \leq C(\varepsilon\|u\|^2 + \|u\|^p).$$

Since ε was chosen arbitrarily, the conclusion follows. □

Lemma 1.3 [16, 17] *There is* $\rho > 0$ *such that*

$$b := \inf_{S_\rho \cap Z} \Phi > 0. \tag{1.8}$$

Proof By (1.5), for any $u \in Z$,

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x.$$

In view of Lemma 1.2, there is $\rho > 0$ such that $\int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x \leq \frac{1}{4}\|u\|^2$ for $\|u\| \leq \rho$; hence the assertion. □

Lemma 1.4 [16, 17] *Let* $z_0 \in Z$, $\|z_0\| = 1$. *There exists* $R > \rho$ *such that*

(i) $\max_{\partial M} \Phi = 0$*;*

(ii) $S := \sup_{\overline{M}} \Phi < \infty$*,*

where

$$M := \{u = y + \lambda z_0 : y \in Y,\ \|u\| < R,\ \lambda > 0\},$$

$$\partial M := \overline{M} \setminus M = \{u = y + \lambda z_0 : (\|u\| = R \text{ and } \lambda \geq 0) \text{ or } (\|u\| \leq R \text{ and } \lambda = 0)\}.$$

Proof By (1.7) and the Sobolev embedding theorem, for any $\delta > 0$,

$$\begin{aligned}\Phi(y+\lambda z_0) &\le -\frac{1}{2}\|y\|^2 + \frac{1}{2}\lambda^2 + \delta|y+\lambda z_0|_2^2 - c_1|y+\lambda z_0|_\gamma^\gamma \\ &\le \left(-\frac{1}{2}+c_2\delta\right)\|y\|^2 + \left(\frac{1}{2}+c_2\delta\right)\lambda^2 - c_1|y+\lambda z_0|_\gamma^\gamma,\end{aligned}$$

where c_2 is independent of δ. We may assume $c_2\delta = \frac{1}{4}$, so

$$\Phi(y+\lambda z_0) \le -\frac{1}{4}\|y\|^2 + \frac{3}{4}\lambda^2 - c_1|y+\lambda z_0|_\gamma^\gamma.$$

There exists a continuous projection from the closure of $Y \oplus \mathbf{R}z_0$ in L^γ to $\mathbf{R}z_0$; thus $|\lambda z_0|_\gamma \le c_3|y+\lambda z_0|_\gamma$ for some $c_3 > 0$. Hence

$$\Phi(y+\lambda z_0) \le -\frac{1}{4}\|y\|^2 + \frac{3}{4}\lambda^2 - c_4\lambda^\gamma,$$

where $c_4 > 0$. It follows that $\Phi(y+\lambda z_0) \to -\infty$ as $\|y+\lambda z_0\| \to \infty$. Since $\Phi \le 0$ on Y and $\Phi(0) = 0$, (i) is satisfied for each R large enough. The boundedness of $\overline{M}$ implies that $\sup_{\overline{M}} \Phi < \infty$. $\square$

Recall that by a $(PS)_\beta$*-sequence* – a *Palais-Smale sequence at the level* β – we mean a sequence $(u_m)_{m=1}^\infty \subset E$ such that $\Phi(u_m) \to \beta$ and $\nabla\Phi(u_m) \to 0$ as $m \to \infty$.

Lemma 1.5 *Let $(u_m) \subset E$ be a $(PS)_\beta$-sequence. Then (u_m) is bounded and $\beta \ge 0$.*

Proof For sufficiently large $m \in \mathbf{N}$ we have $\|\nabla\Phi(u_m)\| \le 1$ and $\Phi(u_m) \le \beta + 1$. Hence by (A4), for such m,

$$\beta + 1 + \|u_m\| \ge \Phi(u_m) - \frac{1}{2}\langle\nabla\Phi(u_m), u_m\rangle \ge \left(\frac{1}{2} - \frac{1}{\gamma}\right)\int_{\mathbf{R}^N} f(x,u_m)u_m\,\mathrm{d}x \ge 0. \tag{1.9}$$

Let $q := \frac{p}{p-1}$ be the conjugate exponent to p. Assumptions (A2), (A3) and (A4) imply that there is a constant $\widetilde{c} > 0$ such that

$$|f(x,u)|^2 \le \widetilde{c}|u||f(x,u)| = \widetilde{c}uf(x,u) \tag{1.10}$$

for $|u| \le 1$ and

$$|f(x,u)|^q \le \widetilde{c}|u|^{(p-1)(q-1)}|f(x,u)| = \widetilde{c}uf(x,u) \tag{1.11}$$

for $|u| \ge 1$. Fix m and let $\Gamma := \{x \in \mathbf{R}^N : |u_m(x)| \le 1\}$. By (1.9), (1.10) and (1.11), for some constant d we have

$$\beta + 1 + \|u_m\| \ge d\left(\int_\Gamma |f(x,u_m)|^2\,\mathrm{d}x + \int_{R^N\setminus\Gamma} |f(x,u_m)|^q\,\mathrm{d}x\right).$$

Therefore

$$a_1 := \left(\int_\Gamma |f(x,u_m)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \le \left[\frac{\beta+1+\|u_m\|}{d}\right]^{\frac{1}{2}} \tag{1.12}$$

and

$$a_2 := \left(\int_{R^N\setminus\Gamma} |f(x,u_m)|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \le \left[\frac{\beta+1+\|u_m\|}{d}\right]^{\frac{1}{q}}. \tag{1.13}$$

Let $y_m := Pu_m$, $z_m := Qu_m$. By the Hölder inequality,

$$\|y_m\|^2 = -\langle \nabla\Phi(u_m), y_m\rangle - \int_{\mathbf{R}^N} f(x,u_m)y_m \,\mathrm{d}x \le \|y_m\| + a_1|y_m|_2 + a_2|y_m|_p.$$

Hence, in view of (1.12), (1.13) and by the Sobolev embedding theorem,

$$\|y_m\| \le 1 + D[(\beta+1+\|u_m\|)^{\frac{1}{2}} + (\beta+1+\|u_m\|)^{\frac{1}{q}}] \tag{1.14}$$

for some $D > 0$. Similarly we obtain

$$\|z_m\| \le 1 + D[(\beta+1+\|u_m\|)^{\frac{1}{2}} + (\beta+1+\|u_m\|)^{\frac{1}{q}}]. \tag{1.15}$$

Inequalities (1.14) and (1.15) imply that $\|u_m\|^2 = \|y_m\|^2 + \|z_m\|^2$ is bounded.

By (1.9), we have

$$0 \le \left(\frac{1}{2} - \frac{1}{\gamma}\right)\int_{\mathbf{R}^N} f(x,u_m)u_m \,\mathrm{d}x \le \Phi(u_m) - \frac{1}{2}\langle\nabla\Phi(u_m), u_m\rangle \to \beta,$$

so $\beta \ge 0$. □

We shall need the following result due to P.L. Lions [8, Lemma 2.18], [11, Lemma I.1], [17, Lemma 1.21]:

Lemma 1.6 *Let $(u_m) \subset E$ be a bounded sequence. If there is $r > 0$ such that*

$$\lim_{m\to\infty} \sup_{a\in R^N} \int_{B(a,r)} |u_m|^2 \,\mathrm{d}x = 0,$$

then $u_m \to 0$ in $L^s(\mathbf{R}^N)$ for all $2 < s < 2^*$. □

As a corollary we obtain

Lemma 1.7 (comp. [8, Lemma 2.25]) *Let $(u_m) \subset E$ be a $(PS)_\beta$-sequence for Φ. Then either*

(i) $\liminf_{m\to\infty} \|u_m\| = 0$, or

(ii) there is a sequence $(a_m) \subset \mathbf{R}^N$ and $r, \eta > 0$ such that

$$\liminf_{m\to\infty} \int_{B(a_m,r)} |u_m|^2 \,\mathrm{d}x \ge \eta.$$

Proof In view of Lemma 1.5, the sequence (u_m) is bounded and $\beta \ge 0$. Suppose that condition (ii) is not satisfied. By Lemma 1.6, $u_m \to 0$ in $L^p(\mathbf{R}^N)$ after passing to a subsequence. Let $y_m = Pu_m$, $z_m = Qu_m$ and take $\varepsilon > 0$. By (1.1) there exists $c_\varepsilon > 0$ such that

$$|f(x,u)| \le \varepsilon|u| + c_\varepsilon|u|^{p-1}$$

for all $x \in \mathbf{R}^N$, $u \in \mathbf{R}$. Hence by the Hölder inequality,

$$\int_{\mathbf{R}^N} |f(x,u_m)z_m| \,\mathrm{d}x \le \varepsilon|u_m|_2|z_m|_2 + c_\varepsilon|u_m|_p^{p-1}|z_m|_p.$$

Since ε was chosen arbitrarily, it follows readily from the Sobolev embedding theorem that

$$\int_{\mathbf{R}^N} f(x, u_m) z_m \, \mathrm{d}x \to 0,$$

and by a similar argument,

$$\int_{\mathbf{R}^N} f(x, u_m) y_m \, \mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbf{R}^N} F(x, u_m) \, \mathrm{d}x \to 0.$$

Therefore

$$\Phi(u_m) - \frac{1}{2}\langle \nabla\Phi(u_m), u_m \rangle = \int_{\mathbf{R}^N} (\frac{1}{2} f(x, u_m) u_m - F(x, u_m)) \, \mathrm{d}x \to 0.$$

Since the left-hand side above tends to β, $\beta = 0$. Furthermore,

$$\|z_m\|^2 = \langle \nabla\Phi(u_m), z_m \rangle + \int_{\mathbf{R}^N} f(x, u_m) z_m \, \mathrm{d}x \to 0.$$

Hence $z_m \to 0$ and similarly, $y_m \to 0$. □

In Section 4 we shall study further properties of (PS)-sequences.

2 τ - topology and degree theory

In order to prove the main results of the next sections we shall introduce a new topology τ in the space E. In this topology it will be possible to construct a degree theory for a class of maps which are not necessarily of Leray-Schauder type. The present section consists of three short parts. In the first of them we define this new topology on E, in the second part we introduce the notion of admissible map, and in the third one a version of topological degree theory for such maps is constructed.

I. Let us consider a function

$$|\!|\!| \cdot |\!|\!| : E \to [0, \infty)$$

given by the formula

$$|\!|\!| u |\!|\!| := \max\{\|Qu\|, \sum_{j=1}^{\infty} \frac{1}{2^j} |\langle e_j, Pu \rangle|\},$$

where $\{e_j\}_{j=1}^{\infty}$ is a complete orthonormal system in Y. It is easy to see that $|\!|\!| \cdot |\!|\!|$ is a norm in E. The topology on E generated by $|\!|\!| \cdot |\!|\!|$ will be denoted by τ and all topological notions related to it will include this symbol.

Remark 2.1

(i) Observe that for each $u \in E$,

$$\|Qu\| \leq |\!|\!| u |\!|\!| \leq \|u\|.$$

Therefore the topology τ is weaker than the original one: any sequence $(u_m) \subset E$ such that $u_m \to u$ (in E) converges to u in the τ-topology ($u_m \xrightarrow{\tau} u$).

(ii) The space $(E, |\!|\!| \cdot |\!|\!|)$ is not complete. For instance, we have a Cauchy sequence (u_m), where $u_m = \sum_{j=1}^{m} je_j \in Y$, which does not τ-converge to any element of Y.

(iii) The topology τ is closely related to the topology on E which is weak on Y and strong on Z. More precisely, if a sequence (u_m) is bounded, then

$$u_m \xrightarrow{\tau} u \iff Pu_m \rightharpoonup Pu \text{ and } Qu_m \to Qu.$$

In particular, for any sequence $(u_m) \subset \Phi_r$, where $r \in \mathbf{R}$, the τ-convergence of (u_m) is equivalent to the weak convergence of (Pu_m) and the strong convergence of (Qu_m). Indeed, if $\Phi(u_m) \geq r$ and $u_m \xrightarrow{\tau} u$, then $(\|Qu_m\|)$ is bounded; since $\|Pu_m\|^2 \leq \|Qu_m\|^2 - 2r$, also $(\|u_m\|)$ is bounded. Conversely, if (Pu_m) converges weakly and (Qu_m) strongly, then again $(\|u_m\|)$ is bounded.

(iv) The functional Φ is τ-upper semicontinuous, i.e. for any $r \in \mathbf{R}$, Φ_r is τ-closed. Indeed, let $u_m \in \Phi_r$ and $u_m \xrightarrow{\tau} u$. Then $r \leq \Phi(u_m) = \frac{1}{2}\|z_m\|^2 - \frac{1}{2}\|y_m\|^2 - \int_{\mathbf{R}^N} F(x, u_m)\, \mathrm{d}x$. Since $u_m \rightharpoonup u$, $u_m \to u$ in L^2_{loc}; hence – passing to a subsequence if necessary – $u_m(x) \to u(x)$ almost everywhere on $\mathbf{R}^N$. By the Fatou lemma and the weak lower semicontinuity of $\|\cdot\|$, we obtain

$$r \leq \frac{1}{2}\|z\|^2 - \frac{1}{2}\|y\|^2 - \int_{\mathbf{R}^N} F(x, u)\, \mathrm{d}x = \Phi(u).$$

(v) The set of critical points K is τ-closed. For if $(u_m) \subset K$ and $u_m \xrightarrow{\tau} u$, then, by (1.9), $\Phi(u_m) \geq 0$ and by (iii), $u_m \rightharpoonup u$. So in view of the weak continuity of $\nabla\Phi$ we get that $\nabla\Phi(u) = 0$.

II. Let A be a closed subset of E. A map $h : A \to E$ will be called τ-*locally finite-dimensional* if each point $u \in A$ has a τ-neighborhood W_u such that $h(W_u \cap A)$ is contained in a finite-dimensional subspace of E. We say that a map $g : A \to E$ is *admissible* if it is τ-continuous (i.e. $g(u_m) \xrightarrow{\tau} g(u)$ provided $u_m, u \in A$ and $u_m \xrightarrow{\tau} u$) and the map $h = I - g$, where I stands for the identity map, is τ-locally finite-dimensional.

We say that a map $G : A \times [0, 1]$ is an *admissible homotopy* if it is τ-continuous (i.e. $G(u_m, t_m) \xrightarrow{\tau} G(u, t)$ provided $u_m \xrightarrow{\tau} u$ in A and $t_m \to t$ in $[0, 1]$) and for each $(u, t) \in A \times [0, 1]$ there is a neighborhood $W_{(u,t)}$ (in the product topology of (E, τ) and $[0, 1]$) such that the set $\{v - G(v, s) : (v, s) \in W_{(u,t)} \cap (A \times [0, 1])\}$ is contained in a finite-dimensional subspace of E.

Observe that an admissible map is continuous. Indeed, if $u_m \to u$, $u_m, u \in A$, then $u_m \xrightarrow{\tau} u$ and $h(u_m) \xrightarrow{\tau} h(u)$. This implies that $h(u_m) \to h(u)$ because, for all large m, $h(u_m), h(u)$ are contained in a finite-dimensional subspace of E on which both topologies – the original one and τ – agree.

Below we are going to give a useful example of admissible homotopy. Suppose that we are given a vector field $V : N \to E$, where N is τ-open, such that

• V is τ-locally τ-Lipschitzian (i.e. any $u \in N$ has a τ-neighborhood U such that $|\!|\!|V(u') - V(u'')|\!|\!| \leq L_u |\!|\!|u' - u''|\!|\!|$ for all $u', u'' \in U$ and some $L_u \geq 0$) and locally Lipschitzian;

• each point $u \in N$ has a τ-neighborhood W_u which is mapped by V into a finite-dimensional subspace of E.

Let $A \subset N$ be closed and consider the Cauchy problem

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = V(\eta), \quad \eta(u, 0) = u \in A.$$

For any $u \in A$, this problem admits a continuous solution $\eta(u, \cdot)$. Suppose that this solution exists on $[0,1]$. Since the space (E, τ) is not complete, it is not immediately clear that η is τ-continuous.

Proposition 2.2 *The map $\eta : A \times [0,1] \to E$ is an admissible homotopy.*

Proof Take any $u_0 \in A$, $t_0 \in [0,1]$. The set $\Gamma := \eta(\{u_0\} \times [0,1])$ is compact, hence τ-compact as well. Since V is τ-locally τ-Lipschtzian and τ-locally finite-dimensional, there are numbers $r, L > 0$ such that $\mathcal{U} := \{u \in E : |||u - \Gamma||| < r\} \subset N$ and if $u, v \in \mathcal{U}$, then $|||V(u)||| \le L$, $|||V(u) - V(v)||| \le L|||u - v|||$. Moreover, $V(\mathcal{U})$ is contained in a finite-dimensional subspace E_1 of E.

We shall show that η is τ-continuous at (u_0, t_0). Given $\delta > 0$, let $t \in [0,1]$ and $u \in A$, $|||u - u_0||| < \delta$. Suppose that $\eta(u, s) \in \mathcal{U}$ for $0 \le s \le t$. Then

$$\begin{aligned} |||\eta(u,t) - \eta(u_0,t)||| &\le |||u - u_0||| + \int_0^t |||V(\eta(u,s)) - V(\eta(u_0,s))|||\,\mathrm{d}s \\ &\le |||u - u_0||| + L\int_0^t |||\eta(u,s) - \eta(u_0,s)|||\,\mathrm{d}s. \end{aligned}$$

Hence by the Gronwall inequality,

$$|||\eta(u,t) - \eta(u_0,t)||| \le |||u - u_0|||e^{Lt} \le |||u - u_0|||e^{L}.$$

If $\delta < re^{-L}$, we obtain

$$|||\eta(u,t) - \eta(u_0,t)||| < r. \tag{2.1}$$

So $\eta(u,t) \in \mathcal{U}$ for each $t \in [0,1]$. Hence, if $|t - t_0| < \delta$, then

$$|||\eta(u,t) - \eta(u_0,t_0)||| \le |||\eta(u,t) - \eta(u_0,t)||| + |||\int_{t_0}^t V(\eta(u_0,s))\,\mathrm{d}s||| < (e^L + L)\delta.$$

Since δ may be chosen arbitrarily small, η is τ-continuous.

Clearly, for any $t \in [0,1]$ and $|||u - u_0||| < \delta$, $u - \eta(u,t) = -\int_0^t V(\eta(u,s))\,\mathrm{d}s \in E_1$. □

III. Let now Z_0 be a finite-dimensional subspace of Z and U an open subset of the space $E_0 := Y \oplus Z_0$. Suppose

(a) $g : \overline{U} \to E_0$ is an admissible map;

(b) $g^{-1}(0) \cap \partial U = \emptyset$ ($\overline{U}$ and ∂U denote the closure and the boundary of U in the original topology of E_0);

(c) $g^{-1}(0)$ is τ-compact.

Remark 2.3 If (a) holds, then assumption (c) is verified whenever $g^{-1}(0)$ is bounded and there is a τ-continuous extension $g^* : B \to E_0$ of g to a τ-closed set B such that $g^*(u) \neq 0$ for $u \in B \setminus U$. For in this case any sequence $(u_m) \subset g^{-1}(0)$ has a subsequence (denoted by the same symbol) such that $Pu_m \rightharpoonup y \in Y$ and $Qu_m \to z \in Z_0$. Thus $u_m \xrightarrow{\tau} u = y + z \in B$ and $g^*(u) = 0$. Hence $u \in U$ and $0 = g^*(u) = g(u)$, so $u \in g^{-1}(0)$. In particular, $g^{-1}(0)$ is τ-compact if $\overline{U}$ is τ-closed and bounded. This holds for instance whenever U is bounded and convex.

Clearly, $g^{-1}(0) \subset \bigcup_{u \in g^{-1}(0)} W_u$, where W_u is a τ-neighborhood of $u \in g^{-1}(0)$ which is mapped by $h = I - g$ into a finite-dimensional subspace of E_0 . Thus there are points $u_1, \ldots, u_m \in g^{-1}(0)$ such that $g^{-1}(0) \subset W := \bigcup_{i=1}^m W_{u_i} \cap U$. The set W is open and there is a finite-dimensional subspace $L \subset E_0$ such that $h(W) \subset L$. Let $W_L := W \cap L$.

Let us consider the map $g_L := g|W_L : W_L \to L$. It is clear that $g_L^{-1}(0) = g^{-1}(0)$; hence $g_L^{-1}(0)$ is compact (in L). Therefore we are in a position to define

$$\deg(g, U, 0) := \deg_B(g_L, W_L, 0), \tag{2.2}$$

where $\deg_B$ stands for the ordinary Brouwer degree (see e.g. [12]).

We shall show that definition (2.2) is correct, i.e. it does not depend on the choice of W and L.

If $\widetilde{L}$ is another finite-dimensional subspace of E_0 such that $h(W) \subset \widetilde{L}$, then we may assume that $L \subset \widetilde{L}$, and the equality

$$\deg_B(g_L, W_L, 0) = \deg_B(g_{\widetilde{L}}, W_{\widetilde{L}}, 0)$$

follows from the contraction property of the Brouwer degree [12, Lemma 4.2.3] since obviously, $W_{\widetilde{L}} \cap L = W_L$ and $h(W_{\widetilde{L}}) \subset h(W) \subset L$.

On the other hand, if $\widetilde{W}$ is a neighborhood of $g^{-1}(0)$ such that $h(\widetilde{W}) \subset L$, then assuming without loss of generality that $W \subset \widetilde{W}$, we see that

$$\deg_B(g_L, \widetilde{W} \cap L, 0) = \deg_B(g_L, W_L, 0)$$

in view of the excision property of the Brouwer degree.

Let us enumerate some useful properties of our degree. A set $U \subset E$ will be called symmetric (with respect to the origin) if $U = -U$.

Theorem 2.4

(i) If $\deg(g, U, 0) \neq 0$, *then* $g^{-1}(0) \neq \emptyset$.

(ii) If $g(u) = u - u_0$, *where* $u_0 \in U$, *then* $\deg(g, U, 0) = 1$.

(iii) Suppose that $G : \overline{U} \times [0,1] \to E_0$ *is an admissible homotopy such that* $G^{-1}(0) \cap (\partial U \times [0,1]) = \emptyset$ *and* $G^{-1}(0)$ *is* τ*-compact (in the product topology). Then the degree* $\deg(G(\cdot, t), U, 0)$ *is independent of* $t \in [0,1]$.

(iv) Suppose that U *is a symmetric neighborhood of the origin and let* $g : \overline{U} \to E_0$ *be an admissible odd map such that* $g^{-1}(0)$ *is* τ*-compact. If for each* $u \in \overline{U}$, $g(u) \in E_1$, *where* $E_1 = Y \oplus Z_1$ *and* Z_1 *is a proper subspace of* Z_0, *then* $g^{-1}(0) \cap \partial U \neq \emptyset$.

Proof (i) follows from the existence property of the Brouwer degree.

(ii) This is trivial in view of definition (2.2).

(iii) Clearly, for any $t \in [0,1]$, the map $G(\cdot, t)$ satisfies assumptions (a), (b) and (c) above which shows that $\deg(G(\cdot, t), U, 0)$ is well-defined.

Let $H(u,t) := u - G(u,t)$. Since $G^{-1}(0)$ is τ-compact, the set $K_0 \times [0,1]$, where K_0 is the projection of $G^{-1}(0)$ onto $\overline{U}$, is also τ-compact. Hence there exists an open set $\widetilde{W}$, $K_0 \times [0,1] \subset \widetilde{W} \subset U$, such that $H(\widetilde{W})$ is contained in a finite-dimensional subspace L of E_0. Since $K_0 \subset L$, K_0

is compact. One can therefore choose an open set $W \subset U$ such that $K_0 \subset W$ and $W \times [0,1] \subset \widetilde{W}$. By the definition, $\deg(G(\cdot,t),U,0) = \deg_B(G(\cdot,t)|W \cap L, W \cap L, 0)$ and the assertion follows from the homotopy invariance of the Brouwer degree.

(iv) Suppose $g^{-1}(0) \cap \partial U = \emptyset$. Then we may assume that W is a symmetric neighborhood of 0, so by the classical Borsuk theorem, $\deg(g,U,0) \neq 0$ (more precisely it is an odd integer). Take an arbitrary point $z \in Z_0 \setminus Z_1$ and consider a map $G : \overline{U} \times [0,1] \to E_0$ given by $G(u,t) := g(u) - tz$. Observe that G satisfies the assumptions of (iii); thus $\deg(G(\cdot,1),U,0) \neq 0$, so $z = g(u_0)$ for some $u_0 \in U$, a contradiction. □

3 Existence of nontrivial solution and abstract linking theorem

We are now going to prove that equation (0.1) has a solution $\overline{u} \neq 0$ (in addition to the trivial one $u \equiv 0$).

For a number $\varepsilon > 0$, let

$$T_\varepsilon := \{u \in E \,:\, \|\nabla\Phi(u)\| \leq \varepsilon\}.$$

Then either

(A) there is $\varepsilon > 0$ such that $T_\varepsilon \cap \Phi^{-1}[b-\varepsilon, S+\varepsilon] = \emptyset$,

or

(B) there is a $(PS)_c$-sequence (u_n) with $c \in [b,S]$

(the numbers b, S were defined in Lemmas 1.3 and 1.4 respectively).

In a moment we shall show that condition (A) leads to a contradiction – see Proposition 3.2. Hence (B) is satisfied. By Lemma 1.5, the $(PS)_c$-sequence (u_n) is bounded and clearly no subsequence of (u_n) converges to 0. Thus, by Lemma 1.7, there is a sequence $(a_n) \subset \mathbf{R}^N$ and numbers $r, \eta > 0$ such that

$$\liminf_{n\to\infty} \int_{B(a_n,r)} |u_n|^2 \,\mathrm{d}x \geq \eta.$$

Taking a subsequence if necessary we may suppose that

$$\|u_n\|_{L^2(B(a_n,r))} \geq \frac{\eta}{2} \tag{3.1}$$

for all $n \in \mathbf{N}$. Choose $g_n \in \mathbf{Z}^N$ such that $|g_n - a_n| = \min\{|g - a_n| \,:\, g \in \mathbf{Z}^N\}$. Thus $|g_n - a_n| \leq \frac{1}{2}\sqrt{N}$. Let

$$v_n := g_n * u_n \equiv u_n(\cdot + g_n)$$

(cf. (1.2)). In view of (3.1),

$$\|v_n\|_{L^2(B(0,r+\frac{1}{2}\sqrt{N}))} \geq \frac{\eta}{2}. \tag{3.2}$$

Observe that $\Phi(v_n) = \Phi(u_n)$ and $\|\nabla\Phi(v_n)\| = \|\nabla\Phi(u_n)\|$. Hence (v_n) is a $(PS)_c$-sequence, and by Lemma 1.5, (v_n) is bounded. Therefore a subsequence of (v_n) (again denoted by the same symbol) converges to some $v \in E$ weakly in E and strongly in $L^2_{loc}(\mathbf{R}^N)$. In view of (3.2), $\|v\|_{L^2(B(0,r+\frac{1}{2}\sqrt{N}))} \geq \frac{\eta}{2}$, so $v \neq 0$.

In view of the weak continuity of $\nabla\Phi$ we get that $\nabla\Phi(v) = 0$.

Assuming (B), we have proved the following result:

Theorem 3.1 *If assumptions (A1) – (A6) are satisfied, then (0.1) has a nontrivial solution.* □

Now it remains to show

Proposition 3.2 *Condition (A) does not hold.*

Proof Assume to the contrary that (A) holds.

I. First we shall construct a certain vector field $V : N \to E$, where N is a τ-open neighborhood of the set Φ^S. Denote $\alpha := b - \varepsilon$.

Let $u \in \Phi^S_\alpha$ and put

$$w(u) := \frac{2\nabla\Phi(u)}{\|\nabla\Phi(u)\|^2}.$$

Since $\nabla\Phi$ is weakly sequentially continuous, the function

$$\Phi^S_\alpha \ni v \mapsto \langle \nabla\Phi(v), w(u)\rangle \in \mathbf{R}$$

is τ-continuous, i.e. if $v_n \xrightarrow{\tau} v$ (in Φ^S_α), then $\langle \nabla\Phi(v_n), w(u)\rangle \to \langle \nabla\Phi(v), w(u)\rangle$ (cf. Remark 2.1 (iii)). Therefore u has a τ-open neighborhood $U_u \subset E$ such that

$$\langle \nabla\Phi(v), w(u)\rangle > 1 \tag{3.3}$$

for all $v \in U_u \cap \Phi^S_\alpha$. Additionally we let $U_0 := \Phi^{-1}(-\infty, \alpha)$. The set U_0 is τ-open in view of the τ-upper semicontinuity of Φ.

The family $\{U_u\}_{u\in\Phi^S_\alpha} \cup \{U_0\}$ is a τ-open covering of the (metric) space (Φ^S, τ). Therefore it has a τ-locally finite τ-open refinement $\{N_j\}_{j\in J}$. Clearly, $\Phi^S \subset N =: \bigcup_{j\in J} N_j$ and N is τ-open. Next let $\{\lambda_j\}_{j\in J}$ be a τ-Lipschitzian partition of unity subordinated to the cover $\{N_j\}_{j\in J}$. For each $j \in J$ there are two possibilities: either N_j is contained in some U_{u_j}, where $u_j \in \Phi^S_\alpha$, and in this case we put $w_j := w(u_j)$; or $N_j \subset U_0$ and then we put $w_j := 0$. Define

$$V(u) := \sum_{j\in J} \lambda_j(u) w_j \tag{3.4}$$

for any $u \in N$.

Let us collect some properties of V.

1. Since $\|\nabla\Phi(u_j)\| \geq \varepsilon$ in view of (A) and hence $\|w_j\| \leq \frac{2}{\varepsilon}$ for all $j \in J$, it follows that $|\!|\!| V(u) |\!|\!| \leq \|V(u)\| \leq \frac{2}{\varepsilon}$ for all $u \in N$.

2. The field V is τ-locally τ-Lipschitzian and locally Lipschitzian. Moreover, each point $u \in N$ has a τ-neighborhood which is mapped by V into a finite-dimensional subspace.

Indeed, by the construction we see that any point $u \in N$ has a τ-open neighborhood $W_u \subset N$ such that the set $J_u := \{j \in J : N_j \cap W_u \neq \emptyset\}$ is finite. Hence $V(W_u)$ is contained in a finite-dimensional subspace. Since for each j there is a constant L_j such that $|\lambda_j(u') - \lambda_j(u'')| \leq L_j |\!|\!| u' - u'' |\!|\!| \leq L_j \|u' - u''\|$, it is easy to see that V is τ-locally τ-Lipschitzian and locally Lipschitzian.

3. By (3.3), $\langle \nabla\Phi(u), V(u)\rangle \geq 0$ for all $u \in N$ and

$$\langle \nabla\Phi(u), V(u)\rangle > 1$$

for all $u \in \Phi^S_\alpha$.

II. Consider the Cauchy problem

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -V(\eta), \quad \eta(u,0) = u \in \Phi^S. \tag{3.5}$$

The classical theory of ordinary differential equations asserts that (3.5) has a unique solution $\eta(u,\cdot)$ which exists for all $t \geq 0$ because, by Property 1 above, V is bounded.

Let $T := S - b + 2\varepsilon$ and consider $\eta : \Phi^S \times [0,T] \to \Phi^S$. In view of Proposition 2.2, η is an admissible homotopy. Observe that $\overline{M} \subset \Phi^S$.

Lemma 3.3 $\sup_{u\in\overline{M}} \Phi(\eta(u,T)) < b$.

Proof Suppose $u \in \overline{M}$ and $\Phi(\eta(u,t)) \geq b - \varepsilon$. We shall show that $t < T$. By Property 3 of V,

$$\Phi(\eta(u,t)) - \Phi(u) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\Phi(\eta(u,s))\,\mathrm{d}s = -\int_0^t \langle \nabla\Phi(\eta(u,s)), V(\eta(u,s))\rangle\,\mathrm{d}s \leq -t.$$

Hence

$$S \geq \Phi(u) \geq t + \Phi(\eta(u,t)) \geq t + b - \varepsilon$$

and $t \leq S - b + \varepsilon < T$. □

III. Now we shall prove that

$$\sup_{u\in\overline{M}} \Phi(\eta(u,T)) \geq b. \tag{3.6}$$

The achieved contradiction to Lemma 3.3 will complete the proof of Proposition 3.2.

Consider a map $G : \overline{M} \times [0,T] \to Y \oplus \mathbf{R}z_0$ given by

$$G(u,t) := P\eta(u,t) + (\|Q\eta(u,t)\| - \rho)z_0.$$

Clearly, G is an admissible homotopy since η is. Next observe that $G(u,t) = 0$ if and only if $\eta(u,t) \in S_\rho \cap Z$. Then, by Lemma 1.3,

$$\Phi(u) \geq \Phi(\eta(u,t)) \geq b. \tag{3.7}$$

This implies that $u \notin \partial M$, i.e. $G^{-1}(0) \cap (\partial M \times [0,T]) = \emptyset$. Since M is convex and bounded, we infer that $G^{-1}(0)$ is τ-compact (see Remark 2.3). Hence we may employ here the degree theory developed in Section 2. Since $G(u,0) = u - \rho z_0$ and $\rho z_0 \in M$, we see that

$$\deg(G(\cdot,T), M, 0) = \deg(G(\cdot,0), M, 0) = 1$$

in view of Theorem 2.4 (ii), (iii). Therefore, by Theorem 2.4 (i), there exists $\overline{u} \in M$ such that $G(\overline{u}) = 0$ and, by (3.7), $\Phi(\eta(\overline{u},T)) \geq b$.

So we have (3.6) and this completes the proof of Proposition 3.2. □

Below we state two generalized linking theorems which extend a result by Benci and Rabinowitz [5, Theorem 0.1], [13, Section 5]. The proofs follow by inspection of the argument employed in Proposition 3.2.

Theorem 3.4 *Let E be a real Hilbert space and suppose that $\Phi \in C^1(E, \mathbf{R})$ satisfies the following hypotheses:*

(i) $\Phi(u) = \frac{1}{2}\langle Lu, u\rangle - \psi(u)$, where L is a bounded selfadjoint linear operator, ψ is bounded below, weakly sequentially lower semicontinuous and $\nabla\psi$ is weakly sequentially continuous;

(ii) there exists a closed separable L-invariant subspace Y such that the quadratic form $u \mapsto \langle Lu, u\rangle$ is negative definite on Y and positive semidefinite on $Y^\perp$;

(iii) there are constants $b, \rho > 0$ such that $\Phi|S_\rho \cap Y^\perp \geq b$;

(iv) there is $z_0 \in S_1 \cap Y^\perp$ and $R > \rho$ such that $\Phi|\partial M \leq 0$, where $M := \{u = y + \lambda z_0 : y \in Y,\ \|u\| < R,\ \lambda > 0\}$.

Then there exists a sequence (u_n) such that $\nabla\Phi(u_n) \to 0$ and $\Phi(u_n) \to c$ for some $c \in [b, \sup_{\overline{M}} \Phi]$.

Note that hypothesis (ii) can be slightly weakened: we may assume $Y = Y_0 \oplus Y_1$, where Y_0 is a finite-dimensional subspace of the nullspace of L and $u \mapsto \langle Lu, u\rangle$ is negative definite on Y_1.

In [5, 13] Y is not assumed to be separable, there are no conditions on the (semi)definiteness of the quadratic form and ψ is not necessarily bounded below. On the other hand, the assumptions made there imply that $\nabla\psi$ is a compact map and Φ satisfies the Palais–Smale condition.

Theorem 3.5 *Let E be a real Hilbert space and suppose that $\Phi \in C^1(E, \mathbf{R})$ satisfies the following hypotheses:*

(i) $\nabla\Phi$ is weakly sequentially continuous and there exists a closed separable subspace Y such that Φ is τ-upper semicontinuous (where τ is the topology on $E = Y \oplus Y^\perp$ introduced in the preceding section);

(ii) conditions (iii) and (iv) of Theorem 3.4 are satisfied.

Then the same conclusion remains valid.

Observe that it follows from the hypotheses of Theorems 3.4 and 3.5 that $\sup_{\overline{M}} \Phi < \infty$. Let us also remark that it is possible to replace conditions (iii), (iv) of Theorem 3.4 by a more general linking condition of a similar type as in [5, Theorem 1.4], [13, Theorem 5.29].

4 Infinite number of solutions

If u is a solution of (0.1), then so is $g * u$ for each $g \in \mathbf{Z}^N$. Let $\mathcal{O}(u) := \{g * u \,:\, g \in \mathbf{Z}^N\}$ denote the orbit of u with respect to the $\mathbf{Z}^N$-action $*$. Clearly, $\mathcal{O}(u)$ is an infinite set if $u \neq 0$. Two solutions u_1, u_2 of (0.1) will be called *geometrically distinct* if $\mathcal{O}(u_1) \neq \mathcal{O}(u_2)$.

In this section we are going to show that if the functional Φ given by (1.5) is even, then (0.1) has infinitely many geometrically distinct solutions. More precisely, let us suppose that the following two additional assumptions are satisfied:

(A7) For all $x \in \mathbf{R}^N$ and $u \in \mathbf{R}$, $f(x, -u) = -f(x, u)$.

(A8) There are $\overline{c}$ and $\varepsilon_0 > 0$ such that

$$|f(x, u + v) - f(x, u)| \leq \overline{c}|v|(1 + |u|^{p-1})$$

for all $x \in \mathbf{R}^N$ and $u, v \in \mathbf{R}$ such that $|v| \leq \varepsilon_0$.

(A8) implies that f is locally Lipschitzian with respect to u; consequently, $f(x,u) = f(x,0) + \int_0^u f'_u(x,\xi)\,\mathrm{d}\xi$ for each $x \in \mathbf{R}^N$. It is therefore easy to see that (A8) is equivalent to f being locally Lipschitzian in u and satisfying

$$|f'_u(x,u)| \le \overline{c}(1 + |u|^{p-1})$$

for some $\overline{c}$ and all $x \in \mathbf{R}^N$, $u \in \mathbf{R}$ for which the derivative $f'_u(x,u)$ exists. Note that the exponent above is $p-1$ and not $p-2$, so Φ may not be of class C^2 even if f is differentiable.

Theorem 4.1 *If assumptions (A1) – (A8) are satisfied, then (0.1) has infinitely many geometrically distinct nontrivial solutions.*

Proof The proof will contain several steps. We shall proceed by contradiction. Namely let us suppose (to the contrary) that $K/\mathbf{Z}^N$ is a finite set, i.e. K contains finitely many orbits.

Let $\mathcal{F}$ be a set consisting of arbitrarily chosen representatives of the (finitely many) orbits of K. In view of (A7) we may assume that $\mathcal{F} = -\mathcal{F}$.

Since

$$\Phi(v) = \Phi(v) - \frac{1}{2}\langle \nabla\Phi(v), v\rangle \ge (\frac{1}{2} - \frac{1}{\gamma})\int_{\mathbf{R}^N} f(x,v)v\,\mathrm{d}x > 0 \tag{4.1}$$

if $v \in \mathcal{F} \setminus \{0\}$ and $\mathcal{F}$ is a finite set, there are numbers $\alpha, \beta > 0$ such that

$$\alpha < \min_{\mathcal{F}\setminus\{0\}} \Phi = \min_{K\setminus\{0\}} \Phi \le \max_{K\setminus\{0\}} \Phi = \max_{\mathcal{F}\setminus\{0\}} \Phi < \beta. \tag{4.2}$$

Clearly, we may assume that

$$\alpha < b \le \beta \tag{4.3}$$

(b was defined in Lemma 1.3).

In order to continue the proof of Theorem 4.1, we shall need a number of prerequisites. First we study Palais-Smale sequences more carefully.

Denote the integer part of $r \in \mathbf{R}$ by $[r]$.

Proposition 4.2 *Let $(u_m)_{m=1}^\infty$ be a $(PS)_c$-sequence. Then either*

(i) $\liminf_{m\to\infty} \|u_m\| = 0$ (and then $c = 0$)

or

(ii) $c \ge \alpha$ and there exist a positive integer $l \le [\frac{c}{\alpha}]$, points $\overline{u}_1, \dots, \overline{u}_l \in \mathcal{F} \setminus \{0\}$ (not necessarily distinct), a subsequence of (u_m) (still denoted by the same symbol) and sequences $(g_m^i)_{m=1}^\infty \subset \mathbf{Z}^N$, $i = 1, \dots, l$, such that

$$\left\| u_m - \sum_{i=1}^{l} (g_m^i * \overline{u}_i) \right\| \to 0$$

and

$$\sum_{i=1}^{l} \Phi(\overline{u}_i) = c.$$

Proof Our argument is modelled on [8]. By Lemma 1.5, the sequence (u_m) is bounded and $c \geq 0$. Suppose that (i) is not satisfied. As before, by Lemma 1.7 (ii), there is a sequence $(a_m) \subset \mathbf{R}^N$ and constants $r, \eta > 0$ such that

$$\|u_m\|_{L^2(B(a_m,r))} \geq \frac{\eta}{2} \tag{4.4}$$

for almost all $m \in \mathbf{N}$. We may choose $g_m \in \mathbf{Z}^N$ such that setting $v_m := (g_m * u_m)$ and passing to a subsequence,

$$\|v_m\|_{L^2(B(0,r+\frac{1}{2}\sqrt{N}))} \geq \frac{\eta}{2} \tag{4.5}$$

for all m (cf. (3.1), (3.2)). Moreover, $\Phi(v_m) = \Phi(u_m)$, $\|\nabla\Phi(v_m)\| = \|\nabla\Phi(u_m)\|$ and $\|v_m\| = \|u_m\|$. Hence (v_m) is bounded, so a subsequence of (v_m) – still denoted by the same symbol – converges to some $v \in E$ both weakly in E and strongly in $L^s_{loc}(\mathbf{R}^N)$ for all $s \in [2, 2^*)$. Therefore

$$v \in K \setminus \{0\} \tag{4.6}$$

by the argument following (3.2). We shall show that

$$\Phi(v) \leq c. \tag{4.7}$$

Let $w_m := v_m - v$. We claim that

$$\Phi(w_m) \to c - \Phi(v) \tag{4.8}$$

and

$$\nabla\Phi(w_m) \to 0. \tag{4.9}$$

If (4.8), (4.9) hold, then also (4.7) does because (w_m) is a (PS)-sequence and, by Lemma 1.5, $c - \Phi(v) \geq 0$.

Observe that

$$\begin{aligned}\Phi(v_m) = \Phi(w_m + v) &= \Phi(w_m) + \Phi(v) + \langle Qw_m - Pw_m, v\rangle \\ &\quad - \int_{\mathbf{R}^N} [F(x, w_m + v) - F(x, w_m) - F(x, v)]\,\mathrm{d}x.\end{aligned} \tag{4.10}$$

As $\langle (Q-P)w_m, v\rangle \to 0$ (because $w_m \rightharpoonup 0$) and $\Phi(v_m) \to c$, we shall obtain (4.8) provided we prove that

$$\int_{\mathbf{R}^N} [F(x, w_m + v) - F(x, w_m) - F(x, v)]\,\mathrm{d}x \to 0. \tag{4.11}$$

Since v is a solution of (0.1), we have $-\Delta v + q(x)v = 0$, where $q(x) = V(x) - f(x,v)/v$. By (1.1), $q \in L^t_{loc}(\mathbf{R}^N)$ for some $t > \frac{N}{2}$. Hence $v(x) \to 0$ as $|x| \to \infty$ [9, 14]. Now take any $\varepsilon > 0$ and a bounded domain $\Omega \subset \mathbf{R}^N$ such that

$$\|v\|_{H^1(\mathbf{R}^N \setminus \Omega)} \leq \varepsilon, \qquad \int_{\mathbf{R}^N \setminus \Omega} F(x, v)\,\mathrm{d}x \leq \varepsilon, \tag{4.12}$$

and

$$\|v\|_{L^\infty(\mathbf{R}^N \setminus \Omega)} \leq \varepsilon. \tag{4.13}$$

Since $w_m \to 0$ in $L^p(\Omega)$, we get

$$\left| \int_\Omega [F(x, w_m + v) - F(x, w_m) - F(x, v)]\,\mathrm{d}x \right| < \varepsilon \tag{4.14}$$

for sufficiently large m. By the mean value theorem, (1.1), the Hölder and the Sobolev inequalities and (4.12),

$$\begin{aligned}\int_{\mathbf{R}^N\setminus\Omega}|F(x,w_m+v)-F(x,w_m)|\,\mathrm{d}x &\le \int_{\mathbf{R}^N\setminus\Omega}[(|w_m|+|v|)+c_1(|w_m|+|v|)^{p-1}]|v|\,\mathrm{d}x\\ &\le c_2(\|w_m\|+\|v\|)\|v\|_{H^1(\mathbf{R}^N\setminus\Omega)}+c_3(\|w_m\|^{p-1}+\|v\|^{p-1})\|v\|_{H^1(\mathbf{R}^N\setminus\Omega)}\le c_4\varepsilon.\end{aligned} \tag{4.15}$$

Since c_4 is independent of ε, (4.12), (4.14) and (4.15) show that (4.11), and hence also (4.8), is satisfied.

Now we shall verify (4.9). For any $\varphi\in E$,

$$\langle\nabla\Phi(w_m),\varphi\rangle=\langle\nabla\Phi(v_m),\varphi\rangle-\int_{\mathbf{R}^N}[f(x,w_m)-f(x,v_m)+f(x,v)]\varphi\,\mathrm{d}x.$$

Therefore, since $\nabla\Phi(v_m)\to 0$, it suffices to show that

$$\sup_{\|\varphi\|\le 1}\left|\int_{\mathbf{R}^N}[f(x,w_m)-f(x,v_m)+f(x,v)]\varphi\,\mathrm{d}x\right|\to 0. \tag{4.16}$$

Again, let $\varepsilon>0$ be given and let Ω be such that (4.12), (4.13) hold. Since $w_m\to 0$ and $v_m\to v$ in $L^p(\Omega)$, we have

$$\left|\int_\Omega[f(x,w_m)-f(x,v_m)+f(x,v)]\varphi\,\mathrm{d}x\right|\le\varepsilon \tag{4.17}$$

for large $m\in\mathbf{N}$. Next by (1.1) and (4.12), if $\|\varphi\|\le 1$, then

$$\left|\int_{\mathbf{R}^N\setminus\Omega}f(x,v)\varphi\,\mathrm{d}x\right|\le c_5(\|v\|_{H^1(\mathbf{R}^N\setminus\Omega)}+\|v\|^{p-1}_{H^1(\mathbf{R}^N\setminus\Omega)})\|\varphi\|\le c_6\varepsilon. \tag{4.18}$$

Hence it remains to show that

$$\sup_{\|\varphi\|\le 1}\left|\int_{\mathbf{R}^N\setminus\Omega}[f(x,w_m)-f(x,w_m+v)]\varphi\,\mathrm{d}x\right|\to 0. \tag{4.19}$$

By (4.12), (4.13), (A8) and the Hölder inequality, if $\|\varphi\|\le 1$, then

$$\begin{aligned}\int_{\mathbf{R}^N\setminus\Omega}|f(x,w_m)-f(x,w_m+v)||\varphi|\,\mathrm{d}x &\le\int_{\mathbf{R}^N\setminus\Omega}\overline{c}(1+|w_m|^{p-1})|v||\varphi|\,\mathrm{d}x\\ &\le c_7\|\varphi\|(\|v\|_{H^1(\mathbf{R}^N\setminus\Omega)}+\|w_m\|^{p-1}_{H^1(\mathbf{R}^N\setminus\Omega)}\|v\|_{L^\infty(\mathbf{R}^N\setminus\Omega)})\le c_8\varepsilon.\end{aligned} \tag{4.20}$$

In view of (4.20), (4.19) is satisfied. This, together with (4.18) and (4.17), shows that (4.16), and therefore also (4.9) and (4.7), are satisfied.

By (4.2), (4.6) and (4.7), $\alpha<\Phi(v)\le c$. There are now two possibilities to consider:

– If $c=\Phi(v)$, then $w_m\to 0$. Indeed, if $w_m\not\to 0$, then arguing as in (4.4)-(4.7) but replacing (u_m) and c by (w_m) and $c'=c-\Phi(v)=0$, we obtain $\overline{v}\in K\setminus\{0\}$ such that $\Phi(\overline{v})\le 0$ – a contradiction to (4.1). Hence our proposition holds with $l=1$, $\overline{u}_1=(g*v)$ (where $g\in\mathbf{Z}^N$ is chosen to ensure that $\overline{u}_1\in\mathcal{F}$) and $g^1_m=(gg_m)^{-1}$.

– If $c>\Phi(v)$, then we argue as in (4.4)-(4.7) again, with (u_m) and c replaced by (w_m) and $c'=c-\Phi(v)$ respectively, and we obtain $v'\in K$ with $\alpha<\Phi(v')\le c-\alpha$. After at most $[\frac{c}{\alpha}]$ steps, we obtain the conclusion. □

Given $l \in \mathbf{N}$ and a finite set $\mathcal{A} \subset E$, let

$$[\mathcal{A}, l] := \left\{ \sum_{i=1}^{j} g_i * a_i \ : \ 1 \le j \le l, \ g_i \in \mathbf{Z}^N, \ a_i \in \mathcal{A} \right\}.$$

Proposition 4.3 [7, Proposition 1.55] *For any* $l \in \mathbf{N}$,

$$\inf\{\|a - a'\| \ : \ a, a' \in [\mathcal{A}, l], \ a \ne a'\} > 0. \qquad \square$$

In view of Proposition 4.2 we have:

Corollary 4.4 *If* (u_m) *is a* $(PS)_c$*-sequence,* $c \ge \alpha$, *then*

$$0 \le |||u_m - [\mathcal{F}, l]||| \le \|u_m - [\mathcal{F}, l]\| \to 0$$

provided that $l \ge [\frac{c}{\alpha}]$. $\square$

Note that $K \subset [\mathcal{F}, l]$ and both sets are symmetric with respect to the origin.

Let $\Sigma := \{A \subset E \ : \ A \text{ is closed and } A = -A\}$. For each $A \in \Sigma$ we define a class $\mathcal{H}(A)$ of all maps $g : A \to E$ such that:

(a) $g(A)$ is closed and g is a homeomorphism of A onto $g(A)$ (in the original topology of E);

(b) g is an odd admissible map;

(c) for any $u \in A$, $\Phi(g(u)) \le \Phi(u)$.

Remark 4.5

(i) Clearly, $\mathcal{H}(A)$ is nonempty: it contains the identity $I : A \to A \subset E$.

(ii) Observe that $\mathcal{H}(A)$ is closed under composition. More precisely, let $g_i \in \mathcal{H}(A_i)$, where $A_i \in \Sigma$, $i = 1, 2$, and suppose that $g_1(A_1) \subset A_2$. Then $g = g_2 \circ g_1 \in \mathcal{H}(A_1)$. In particular, if $A \subset B \in \Sigma$, then for any $g \in \mathcal{H}(B)$, $g|A \in \mathcal{H}(A)$.

For $B \in \Sigma$, denote by $\gamma(B)$ the Krasnoselskii genus of B [13, Section 7], i.e.

$$\gamma(B) := \min\{k \in \mathbf{N} \ : \ \exists \text{ odd continuous } \varphi : B \to \mathbf{R}^k \setminus \{0\}\}; \quad \gamma(\emptyset) := 0.$$

In our minimax argument we shall need the following deformation lemma:

Lemma 4.6 *Let* $\xi > \beta + 2$. *There exist* $\varepsilon > 0$, *a symmetric* τ*-open set* $\mathcal{N}$ *with* $\gamma(\overline{\mathcal{N}}) = 1$ *and a map* $g \in \mathcal{H}(\Phi^\xi)$ *such that:*

(i) for any $d \in [b, \xi - 1]$, $g(\Phi^{d+\varepsilon} \setminus \mathcal{N}) \subset \Phi^{d-\varepsilon}$;

(ii) if moreover $d \ge \beta + 1$, *then* $g(\Phi^{d+\varepsilon}) \subset \Phi^{d-\varepsilon}$.

Proof Let

$$l := \left[\frac{\xi + 1}{\alpha}\right]$$

and

$$0 < \mu < \inf\{\|z - z'\| \ : \ z, z' \in [Q\mathcal{F}, l], \ z \ne z'\}.$$

By Proposition 4.3 (which also holds in the space Z), such μ exists. For any $z \in [Q\mathcal{F}, l] \setminus \{0\}$, let $A_z := Y \oplus B_Z(z, \frac{\mu}{4})$ (where $B_Z(z, \frac{\mu}{4}) := B(z, \frac{\mu}{4}) \cap Z$). It is clear that $0 \notin A_z$. If $z \neq z'$, $z, z' \neq 0$, then $A_z \cap A_{z'} = \emptyset$; in particular, $A_z \cap A_{-z} = \emptyset$.

Let

$$\mathcal{N} := \bigcup_{z \in [Q\mathcal{F}, l] \setminus \{0\}} A_z \equiv Y \oplus \bigcup_{z \in [Q\mathcal{F}, l] \setminus \{0\}} B_Z\left(z, \frac{\mu}{4}\right). \tag{4.21}$$

Since $[Q\mathcal{F}, l] = Q[\mathcal{F}, l]$, $\mathcal{N}$ is a τ-open symmetric neighborhood of $[\mathcal{F}, l] \setminus \{0\}$. Since for $z \in [Q\mathcal{F}, l] \setminus \{0\}$, $\overline{A}_z = Y \oplus \overline{B}_Z(z, \frac{\mu}{4})$ is contractible and $\overline{\mathcal{N}} = \bigcup_{z \in [Q\mathcal{F}, l] \setminus \{0\}} \overline{A}_z$, $\gamma(\overline{\mathcal{N}}) = 1$.

Let

$$\mathcal{N}_0 := Y \oplus \bigcup_{z \in [Q\mathcal{F}, l]} B_Z\left(z, \frac{1}{8}\mu\right).$$

We easily see that $\mathcal{N}_0$ is a τ-neighborhood of $[\mathcal{F}, l]$, hence by Corollary 4.4, there is $\delta > 0$ such that if $u \in \Phi_\alpha^{\xi+1} \setminus \mathcal{N}_0$, then $\|\nabla\Phi(u)\| \geq \delta$.

Take $\varepsilon > 0$ such that

$$\varepsilon < \min\left\{\frac{1}{2},\, b - \alpha,\, \frac{1}{32}\delta\mu\right\}. \tag{4.22}$$

Now we shall define a vector field V by slightly modifying the construction in Section 3. For any $u \in \Phi_\alpha^{\xi+1} \setminus K$ (note that K is τ-closed), let

$$w(u) := \frac{2\nabla\Phi(u)}{\|\nabla\Phi(u)\|^2}.$$

In view of the τ-continuity of the function $\Phi_\alpha^{\xi+1} \ni v \mapsto \langle \nabla\Phi(v), w(u)\rangle \in \mathbf{R}$, there is a τ-open neighborhood U_u of u (in E) such that for any $v \in \Phi_\alpha^{\xi+1} \cap U_u$,

$$\langle \nabla\Phi(v), w(u)\rangle > 1 \tag{4.23}$$

and

$$|||v - u||| < \frac{1}{16}\mu. \tag{4.24}$$

Additionally let $U_0 := \Phi^{-1}(-\infty, \alpha)$. Take a τ-locally finite τ-open refinement $\{N_j\}_{j \in J}$ of the τ-open covering $\{U_u\}_{u \in \Phi_\alpha^{\xi+1}} \cup \{U_0\}$ of $\Phi^{\xi+1}$ and a τ-locally τ-Lipschitzian partition of unity $\{\lambda_j\}_{j \in J}$ subordinated to $\{N_j\}_{j \in J}$. Put $w_j := w(u_j)$ if $N_j \subset U_{u_j}$ for some $u_j \in \Phi_\alpha^{\xi+1}$ and $w_j := 0$ if $N_j \subset U_0$. Let N be a symmetric τ-open set such that $\Phi^{\xi+1} \subset N \subset \bigcup_{j \in J} N_j$ and set

$$\widetilde{V}(u) := \sum_{j \in J} \lambda_j(u) w_j, \tag{4.25}$$

$$V(u) := \frac{1}{2}[\widetilde{V}(u) - \widetilde{V}(-u)] \tag{4.26}$$

for $u \in N$. Below we collect some properties of V.

1. V is odd; it is τ-locally τ-Lipschitzian and thus locally Lipschitzian.

2. By (4.23) and since $\nabla\Phi$ is an odd map, for all $u \in N$, $\langle \nabla\Phi(u), V(u)\rangle \geq 0$ and $\langle \nabla\Phi(u), V(u)\rangle > 1$ when $u \in \Phi_\alpha^{\xi+1}$.

3. Each point $u \in \Phi^{\xi+1}$ has a τ-neighborhood W_u on which V is τ-Lipschitzian and such that $V(W_u)$ is contained in a finite-dimensional subspace of E.

Finally, let $\psi : E \to [0,1]$ be an even τ-locally τ-Lipschitzian function such that $\psi(u) = 0$ if $\|u - K\| \le \frac{1}{10}\mu$ and $\psi(u) = 1$ if $\|u - K\| \ge \frac{1}{8}\mu$. Note in particular that $\psi(u) = 1$ if $u \notin \mathcal{N}_0$. Diminishing μ if necessary we may assume that

$$\psi(u) = 1 \text{ if } u \in \Phi_\beta. \tag{4.27}$$

Consider the Cauchy problem

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\psi(\eta)V(\eta), \quad \eta(u,0) = u \in \Phi^{\xi+1}. \tag{4.28}$$

Since ψV is locally Lipschitzian, (4.28) has a unique continuous solution $\eta(u,\cdot)$ defined on a right maximal neighborhood $I_u := [0, \omega_+(u))$ of $t = 0$. Since the field ψV is odd, we have that $\omega_+(u) = \omega_+(-u)$ and $\eta(-u,t) = -\eta(u,t)$ for all $u \in \Phi^{\xi+1}$, $t \in I_u$.

Claim.
(a) For any $u \in \Phi^{\xi+1}$, $\omega_+(u) = \infty$;
(b) $\eta : \Phi^{\xi+1} \times [0,1] \to \Phi^{\xi+1}$ is an admissible homotopy and $g := \eta(\cdot,1)|\Phi^\xi$ is an admissible odd map;
(c) $g(\Phi^{d+\varepsilon} \setminus \mathcal{N}) \subset \Phi^{d-\varepsilon}$; moreover, $g(\Phi^{d+\varepsilon}) \subset \Phi^{d-\varepsilon}$ if $d \ge \beta + 1$.

Proof of Claim.
(a) Assume that $\omega_+(u) = \omega_+(-u) < \infty$ for some $u \in \Phi^{\xi+1}$. If there is a constant $C > 0$ such that $\|V(\eta(u,t_m))\| \le C$ as $t_m \nearrow \omega_+(u)$, then

$$\|\eta(u,t_n) - \eta(u,t_m)\| = \left\| \int_{t_m}^{t_n} \psi(\eta(u,t))V(\eta(u,t))\,\mathrm{d}t \right\| \le C|t_n - t_m|.$$

Thus $(\eta(u,t_m))_{m=1}^\infty$ is a Cauchy sequence and it is easy to see that $\eta(u,\cdot)$ may be extended beyond $\omega_+(u)$. Therefore there must exist a sequence $t_m \nearrow \omega_+(u)$ such that $\psi(\eta(u,t_m)) > 0$ and $\|V(\eta(u,t_m))\| \to \infty$. Set $v_m := \eta(u,t_m)$. Since $V(v_m) = \frac{1}{2}[\widetilde{V}(v_m) - \widetilde{V}(-v_m)]$ and $\widetilde{V}(v_m) = \sum_{j\in J} \lambda_j(v_m)w_j$, we get that for all m there is an element $j(m) \in J$ such that $\|w_{j(m)}\| = \frac{2}{\|\nabla\Phi(u_m)\|} \to \infty$ as $m \to \infty$, where $u_m := u_{j(m)}$, and $\lambda_{j(m)}(v_m) \ne 0$ or $\lambda_{j(m)}(-v_m) \ne 0$. So $\nabla\Phi(u_m) \to 0$ (and $u_m \in \Phi_\alpha^{\xi+1}$).

Taking a subsequence if necessary, we may assume that for all m, $\lambda_{j(m)}(v_m) \ne 0$. Hence $v_m \in N_{j(m)} \subset U_{u_m}$ and by (4.24), $\|v_m - u_m\| < \frac{\mu}{16}$. By Corollary 4.4, $\|u_m - [\mathcal{F}, l]\| \to 0$. Therefore either

(i) there is $z \in [Q\mathcal{F}, l]$ such that, for almost all m, $u_m \in Y \oplus B_Z(z, \frac{\mu}{16})$

or

(ii) the sequence (u_m) enters infinitely many such sets and therefore (v_m) enters infintely many sets of the form $Y \oplus B_Z(z, \frac{\mu}{8})$, where $z \in [Q\mathcal{F}, l]$.

If (i) holds, then $Qu_m \to z$. Since $\Phi(u_m) \ge \alpha$, the sequence (Pu_m) is bounded and therefore $Pu_m \rightharpoonup y \in Y$ (taking a subsequence if necessary). Consequently, $u_m \xrightarrow{\tau} y + z$. By the weak continuity of $\nabla\Phi$, we get $\nabla\Phi(y+z) = 0$, so $y + z \in K$. Hence $\|u_m - K\| \to 0$ and thus $\|v_m - K\| \le \frac{\mu}{10}$ for almost all m. Therefore $\psi(v_m) = 0$ for such m, a contradiction.

Suppose (ii) is satisfied. Outside $\mathcal{V} := Y \oplus \bigcup_{z\in[Q\mathcal{F},l]} B_Z(z, \frac{\mu}{16})$, if $u \in \Phi_\alpha^{\xi+1}$, then, again by Corollary 4.4, $\|\nabla\Phi(u)\| \ge \delta_0$ for some $\delta_0 > 0$.

Let $t_1 < t_2 < \omega_+(u)$ be such that $\eta(u,t) \notin \mathcal{V}$ for $t \in (t_1, t_2)$; moreover, $\eta(u,\cdot)$ leaves $Y \oplus \overline{B}_Z(z_1, \frac{\mu}{8})$ for $t = t_1$ and enters $Y \oplus \overline{B}_Z(z_2, \frac{\mu}{8})$ for $t = t_2$, where $z_1, z_2 \in [Q\mathcal{F}, l]$, $z_1 \neq z_2$. Then $\|\eta(u,t_1) - \eta(u,t_2)\| \geq \frac{3}{4}\mu$.

Now let $v = \eta(u,t)$ for some $t \in (t_1,t_2)$. For such v, $\widetilde{V}(v) = \sum_{j \in J_0} \lambda_j(v) w_j$, where J_0 is a finite set. By (4.24), for all $j \in J_0$, $\|u_j - v\| < \frac{\mu}{16}$. Hence $u_j \notin \mathcal{V}$ and

$$\|\widetilde{V}(v)\| \leq \sup_{j \in J_0} \|w_j\| = \sup_{j \in J_0} \frac{2}{\|\nabla\Phi(u_j)\|} \leq \frac{2}{\delta_0}.$$

So

$$\frac{3}{4}\mu \leq \|\eta(u,t_2) - \eta(u,t_1)\| \leq \int_{t_1}^{t_2} \|V(\eta(u,s)\| \,\mathrm{d}s \leq \frac{2}{\delta_0}(t_2 - t_1). \tag{4.29}$$

Since t_1, t_2 may be chosen arbitrarily close to $\omega_+(u)$, this is a contradiction. Hence $\omega_+(u) = \infty$ for all $u \in \Phi^{\xi+1}$.

(b) The admissibility of the homotopy $\eta : \Phi^{\xi+1} \times [0,1] \to \Phi^{\xi+1}$ follows from Proposition 2.2. Thus $g = \eta(\cdot,1)|\Phi^{\xi}$ is also admissible and it is clearly odd.

(c) Take any $d \leq \xi - 1$ and let $u \in \Phi^{d+\varepsilon} \setminus \mathcal{N}$. Suppose there is $t \geq 0$ such that $\eta(u,t) \in \overline{\mathcal{N}}_0$ and $\eta(u,s) \notin \mathcal{N}_0$ for $s \in [0,t)$ (otherwise the argument is simpler). Then $\psi(\eta(u,s)) = 1$ and

$$\|Q\eta(u,t) - z\| = \frac{1}{8}\mu$$

for some $z \in [Q\mathcal{F}, l]$. Using an argument similar to (4.29), we obtain

$$\frac{1}{8}\mu \leq \|Qu - Q\eta(u,t)\| \leq \int_0^t \|V(\eta(u,s)\| \,\mathrm{d}s \leq \frac{2}{\delta}t.$$

Since $\psi(\eta(u,s)) = 1$,

$$d + \varepsilon - \Phi(\eta(u,t)) \geq \Phi(u) - \Phi(\eta(u,t)) = \int_0^t \langle \nabla\Phi(\eta(u,s)), V(\eta(u,s))\rangle \,\mathrm{d}s \geq t, \tag{4.30}$$

and therefore $\Phi(\eta(u,t)) \leq d + \varepsilon - \frac{\delta\mu}{16} < d - \varepsilon$ (the last inequality follows from (4.22)). So if $t \leq 1$, then $\Phi(\eta(u,1)) \leq \Phi(\eta(u,t)) \leq c - \varepsilon$. If $t > 1$, then (4.30) implies $\Phi(\eta(u,1)) \leq d - \varepsilon$ (since $\varepsilon < \frac{1}{2}$).

If $d \geq \beta + 1$, then clearly, for all $u \in \Phi_{d-\varepsilon}^{d+\varepsilon}$ and $t \in [0,1]$, $\psi(\eta(u,t)) = 1$ in view of (4.27). Therefore it is easy to see using (4.30) again that $\Phi(\eta(u,1)) \leq d - \varepsilon$ and $\eta(\Phi^{d+\varepsilon}, 1) \subset \Phi^{d-\varepsilon}$.

So far we have proved that $g : \Phi^{\xi} \to \Phi^{\xi}$ is an odd admissible map satisfying (i) and (ii). Clearly, $\Phi(g(u)) \leq \Phi(u)$ and g is a homeomorphism of Φ^{ξ} into Φ^{ξ}. It remains to show that $g(\Phi^{\xi})$ is a closed set. To this end we introduce an even Lipschitzian function $\varphi : E \to [0,1]$ such that $\varphi|\Phi_{\xi+1} \equiv 0$ and $\varphi|\Phi^{\xi} \equiv 1$. Consider the Cauchy problem

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = -\varphi(\chi)\psi(\chi)V(\chi), \quad \chi(u,0) = u \in E.$$

Using a similar argument as above we show that $\chi(u,t)$ exists for all $t \in \mathbf{R}$. Now assume $v_m := g(u_m) = \eta(u_m,1) = \chi(u_m,1) \to v$, where $u_m \in \Phi^{\xi}$. Then $u_m = \chi(v_m,-1) \to \chi(v,-1) = u \in \Phi^{\xi}$. Hence $g(u) = v$ and $v \in g(\Phi^{\xi})$.

This completes the proof of Lemma 4.6. □

For each $A \in \Sigma$ we define a *pseudoindex* $\gamma^*(A)$ of A by setting

$$\gamma^*(A) := \min_{g \in \mathcal{H}(A)} \gamma(g(A) \cap S_\rho \cap Z)$$

(recall γ denotes the genus). Observe that since $g(A)$ is closed, $g(A) \cap S_\rho \cap Z$ is an element of Σ and the above definition is correct. Our pseudoindex is similar to that of Benci [4]; however, it does not have all properties required in [4].

Lemma 4.7 *Let $A, B \in \Sigma$.*
(i) If $\gamma^*(A) \neq 0$, then $A \neq \emptyset$.
(ii) If $A \subset B$, then $\gamma^*(A) \leq \gamma^*(B)$.
(iii) If $h \in \mathcal{H}(A)$, then $\gamma^*(h(A)) \geq \gamma^*(A)$.

Proof Property (i) is obvious.

(ii) If $A \subset B$, then $g(A) \subset g(B)$ for any $g \in \mathcal{H}(B)$. Hence, by the monotonicity of the genus, $\gamma(g(A) \cap S_\rho \cap Z) \leq \gamma(g(B) \cap S_\rho \cap Z)$. By Remark 4.5 (ii),

$$\gamma^*(A) = \min_{g \in \mathcal{H}(A)} \gamma(g(A) \cap S_\rho \cap Z) \leq \min_{g \in \mathcal{H}(B)} \gamma(g(A) \cap S_\rho \cap Z) \leq \min_{g \in \mathcal{H}(B)} \gamma(g(B) \cap S_\rho \cap Z) = \gamma^*(B).$$

(iii) For any $\overline{g} \in \mathcal{H}(h(A))$, $\overline{g} \circ h \in \mathcal{H}(A)$ according to Remark 4.5 (ii). Hence

$$\gamma^*(A) = \min_{g \in \mathcal{H}(A)} \gamma(g(A) \cap S_\rho \cap Z) \leq \min_{\overline{g} \in \mathcal{H}(h(A))} \gamma(\overline{g} \circ h(A) \cap S_\rho \cap Z) = \gamma^*(h(A)). \qquad \square$$

Now we shall show that in Σ there are sets of arbitrarily large pseudoindex. To this end suppose that Z_k is a k-dimensional subspace of Z and let $E_k := Y \oplus Z_k$. Arguing as in Lemma 1.4, one proves easily that

$$\Phi(u) \to -\infty \text{ as } u \in E_k \text{ and } \|u\| \to \infty.$$

Therefore there is a number $R_k > \rho$ (ρ was determined in Lemma 1.3) such that for $u \in E_k$, $\|u\| \geq R_k$,

$$\Phi(u) < \inf_{\|u\| \leq \rho} \Phi(u). \tag{4.31}$$

Let us put

$$A := \overline{B}(0, R_k) \cap E_k = \{u \in E_k \,:\, \|u\| \leq R_k\}.$$

Lemma 4.8 *$\gamma^*(A) \geq k$.*

Proof Suppose to the contrary that $\gamma^*(A) = l$, $0 \leq l < k$. Hence there is $g \in \mathcal{H}(A)$ such that $\gamma(g(A) \cap S_\rho \cap Z) = l$.

Let $U := g^{-1}(B_\rho) \cap E_k$ and $B := g^{-1}(\overline{B}_\rho) \cap E_k$. Since g is odd, the sets U, B are symmetric and $0 \in U$. Since g is τ-continuous and $\overline{B}_\rho$ is τ-closed, B is τ-closed. Moreover, U is open in E_k because g is continuous and for all $u \in U$, $\|u\| < R_k$ (for if $\|u\| = R_k$, then by (4.31), $\Phi(g(u)) \leq \Phi(u) < \inf_{\overline{B}_\rho} \Phi$, and $u \notin g^{-1}(B_\rho)$). Clearly, $\overline{U} \subset B \subset A$ and $g(B \setminus U) \subset S_\rho$.

Assume that $g(A) \cap S_\rho \cap Z \neq \emptyset$. There is a continuous odd map $\varphi : g(A) \cap S_\rho \cap Z \to \mathbf{R}^l \setminus \{0\} \subset \mathbf{R}^{k-1} \setminus \{0\}$. Since $\mathbf{R}^{k-1}$ is isomorphic to a $(k-1)$-dimensional subspace Z_{k-1} of Z_k, we may assume that $\varphi : g(A) \cap S_\rho \cap Z \to Z_{k-1} \setminus \{0\}$. Let $\varphi^* : \overline{B}_\rho \to Z_{k-1}$ be an odd extension of φ to $\overline{B}_\rho$ (take any continuous extension $\widetilde{\varphi}$ with values in Z_{k-1} and put $\varphi^*(x) = \frac{1}{2}[\widetilde{\varphi}(x) - \widetilde{\varphi}(-x)]$, $x \in \overline{B}_\rho$).

If $g(A) \cap S_\rho \cap Z = \emptyset$, we put $\varphi^* \equiv 0$.

We now consider a map $\overline{g} : B \to E_k$ given by

$$\overline{g}(u) := Pg(u) + \varphi^*(Qg(u)).$$

It is clear that $\overline{g}$ is odd and admissible. If $\overline{g}(u) = 0$, then $g(u) \in Z$, so $Qg(u) = g(u)$ and $\varphi^*(g(u)) = 0$. Since for $u \in B \setminus U$, $\varphi^*(g(u)) = \varphi(g(u)) \neq 0$, we obtain $\overline{g}^{-1}(0) \cap (B \setminus U) = \emptyset$. Moreover, $\overline{g}^{-1}(0)$ is τ-compact (recall B is τ-closed). Hence, in view of Theorem 2.4 (iv), $\overline{g}^{-1}(0) \cap \partial U \neq \emptyset$, a contradiction because $\partial U \subset B \setminus U$.

This proves that $\gamma^*(A) \geq k$. □

Now we can return to the proof of Theorem 4.1. Let

$$c_k := \inf_{\gamma^*(A) \geq k} \sup_{u \in A} \Phi(u).$$

By Lemma 4.8, c_k is a well-defined real number for each $k \geq 1$.

Lemma 4.9 *For any integer $k \geq 1$, $b \leq c_k \leq c_{k+1}$.*

Proof The second inequality is obvious since $\{A \in \Sigma : \gamma^*(A) \geq k+1\} \subset \{A \in \Sigma : \gamma^*(A) \geq k\}$. If $\gamma^*(A) \geq k$, then $\gamma(g(A) \cap S_\rho \cap Z) \geq k$ and $g(A) \cap S_\rho \cap Z \neq \emptyset$ for any $g \in \mathcal{H}(A)$. Hence there is $u \in A$ such that $g(u) \in S_\rho \cap Z$; so $\Phi(u) \geq \Phi(g(u)) \geq b$. □

There are two possibilities:

(j) There is an integer $k \geq 1$ such that $c := c_k \geq \beta + 1$.

(jj) For all $k \geq 1$, $\alpha < b \leq c_k \leq \beta + 1$.

We are now going to show that both conditions (j) and (jj) lead to a contradiction. This will complete the proof of Theorem 4.1.

(A) Suppose that condition (j) is satisfied. Let $\xi > c + 1$. Then $\xi > \beta + 2$. For each $\varepsilon > 0$ there is $A \subset \Phi^{c+\varepsilon}$ such that $\gamma^*(A) \geq k$. By Lemma 4.7 (ii),

$$\gamma^*(\Phi^{c+\varepsilon}) \geq \gamma^*(A) \geq k.$$

Further, by Lemma 4.6 (ii) (with $d = c$), there exist $\varepsilon > 0$ and $g \in \mathcal{H}(\Phi^\xi)$ such that $g(\Phi^{c+\varepsilon}) \subset \Phi^{c-\varepsilon}$. So by Lemma 4.7 (iii),

$$k \leq \gamma^*(\Phi^{c+\varepsilon}) \leq \gamma^*(g(\Phi^{c+\varepsilon})) \leq \gamma^*(\Phi^{c-\varepsilon})$$

and thus $c \equiv c_k \leq c - \varepsilon$, a contradiction.

(B) Suppose that (jj) is satisfied. Now in order to proceed further we need to introduce another pseudoindex.

Let X be an arbitrary but fixed member of Σ and let

$$\Sigma_X := \{A \in \Sigma : A \subset X\}.$$

We define

$$\gamma_X^*(A) := \min_{g \in \mathcal{H}(X)} \gamma(g(A) \cap S_\rho \cap Z).$$

Lemma 4.10 *Let $A, B \in \Sigma_X$.*

(i) $\gamma_X^*(A) \geq \gamma^*(A)$.

(ii) If $A \subset B$, *then* $\gamma_X^*(A) \leq \gamma_X^*(B)$.

(iii) If $h \in \mathcal{H}(X)$ *and* $h(X) \subset X$, *then* $\gamma_X^*(h(A)) \geq \gamma_X^*(A)$.

(iv) $\gamma_X^*(A \cup B) \leq \gamma_X^*(A) + \gamma(B)$.

Proof (i) For any $g \in \mathcal{H}(X)$, $g|A \in \mathcal{H}(A)$. Hence $\gamma^*(A) \leq \gamma_X^*(A)$.

(ii) $\gamma(g(A) \cap S_\rho \cap Z) \leq \gamma(g(B) \cap S_\rho \cap Z)$ for each $g \in \mathcal{H}(X)$. Hence $\gamma_X^*(A) \leq \gamma_X^*(B)$.

(iii) For any $g \in \mathcal{H}(X)$, $g \circ h \in \mathcal{H}(X)$. Thus

$$\gamma_X^*(A) \leq \min_{g \in \mathcal{H}(X)} \gamma(g \circ h(A) \cap S_\rho \cap Z) = \gamma_X^*(h(A)).$$

(iv) Take any $g \in \mathcal{H}(X)$. Then

$$\gamma_X^*(A \cup B) \leq \gamma(g(A \cup B) \cap S_\rho \cap Z) \leq \gamma(g(A) \cap S_\rho \cap Z) + \gamma(g(B))$$

in view of the subadditivity and monotonicity of the genus. Since $g(B)$ is closed and homeomorphic to B, $\gamma(B) = \gamma(g(B))$. Therefore $\gamma_X^*(A \cup B) \leq \gamma(g(A) \cap S_\rho \cap Z) + \gamma(B)$. This implies that $\gamma_X^*(A \cup B) \leq \gamma_X^*(A) + \gamma(B)$. □

The sequence (c_k) being nondecreasing and bounded is convergent, say $b \leq c := \lim_{k\to\infty} c_k \leq \beta + 1$. It follows from the definition of c_k and Lemma 4.7 (ii) that $\gamma^*(\Phi^{c+\nu}) \geq k$ for all $\nu > 0$ and $k \geq 1$. Take any $\xi > \beta + 2$ and let

$$X := \Phi^{\beta+2} \subset \Phi^\xi.$$

Since

$$\gamma_X^*(\Phi^{c+\nu}) \geq \gamma^*(\Phi^{c+\nu}) = \infty \tag{4.32}$$

for each $0 < \nu < 1$, we may define a new sequence of real numbers $(d_k)_{k=1}^\infty$ by the formula

$$d_k := \inf_{\gamma_X^*(A) \geq k} \sup_{u \in A} \Phi(u).$$

Lemma 4.11 *For any* $k \geq 1$, $b \leq d_k \leq d_{k+1} \leq c$.

Proof The first two inequalities are established similarly as in Lemma 4.9. The last one follows from (4.32) and the definition of d_k. □

The sequence (d_k) is nondecreasing and bounded, so $b \leq d := \lim_{k\to\infty} d_k \leq c < \xi - 1$. For all $\varepsilon > 0$ sufficiently small, $X \equiv \Phi^{\beta+2} \supset \Phi^{d+\varepsilon} \supset \Phi^{d_k+\varepsilon}$ and thus $\gamma_X^*(\Phi^{d+\varepsilon}) = \infty$.

By Lemma 4.6, there are $\varepsilon > 0$ and $g \in \mathcal{H}(X)$ such that $g(\Phi^{d+\varepsilon} \setminus \mathcal{N}) \subset \Phi^{c-\varepsilon}$ and $g(X) \subset X$. Since $\Phi^{d+\varepsilon} = (\Phi^{d+\varepsilon} \setminus \mathcal{N}) \cup (\overline{\mathcal{N}} \cap \Phi^{d+\varepsilon})$, we get by Lemma 4.10 (iv), (iii) and (ii) that

$$\infty = \gamma_X^*(\Phi^{d+\varepsilon}) \leq \gamma_X^*(\Phi^{d+\varepsilon} \setminus \mathcal{N}) + \gamma(\overline{\mathcal{N}} \cap \Phi^{d+\varepsilon}) \leq \gamma_X^*(g(\Phi^{d+\varepsilon} \setminus \mathcal{N})) + 1 \leq \gamma_X^*(\Phi^{d-\varepsilon}) + 1,$$

so $\gamma_X^*(\Phi^{d-\varepsilon}) = \infty$. Therefore $d_k \leq d - \varepsilon$ for all k, contradicting the fact that $d_k \to d$.

The proof of Theorem 4.1 is complete. □

5 Existence of homoclinic solutions

In this final section we indicate how our results can be carried over to the problem of existence of homoclinic solutions for the second order system of differential equations

$$(5.1) \qquad -\ddot{q} + A(t)q = F_q(t,q), \quad t \in \mathbf{R},\ q \in \mathbf{R}^N.$$

Let $E := H^1(\mathbf{R}, \mathbf{R}^N)$ and suppose that A and F satisfy the following conditions:

(B1) The functions $F : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ and $F_q : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ are continuous and 1-periodic in t.

(B2) $\liminf_{q\to 0} q \cdot F_q(t,q)/|F_q(t,q)|^2 > 0$ and $\liminf_{|q|\to\infty} q \cdot F_q(t,q)/|F_q(t,q)| > 0$, uniformly with respect to t.

(B3) $F_q(t,q) = o(|q|)$ uniformly with respect to t as $|q| \to 0$.

(B4) There is $\gamma > 2$ such that for all $t \in \mathbf{R}$ and $q \in \mathbf{R}^N \setminus \{0\}$,

$$0 < \gamma F(t,q) \le q \cdot F_q(t,q).$$

(B5) $A : \mathbf{R} \to \mathbf{R}^{N^2}$ is a symmetric $N \times N$ matrix with continuous 1-periodic entries.

(B6) 0 lies in a gap of the spectrum of the operator $L : E \to E$ given by

$$(Lq, v) := \int_{\mathbf{R}} (\dot{q} \cdot \dot{v} + A(t)q \cdot v)\,\mathrm{d}t.$$

(B7) For all $t \in \mathbf{R}$ and $q \in \mathbf{R}^N$, $F(t,-q) = F(t,q)$.

(B8) F_q is locally Lipschitzian with respect to q.

Note that (B2) follows from (B3) and (B4) if $N = 1$ (a single equation).

Let

$$\Phi(q) := \frac{1}{2}\int_{\mathbf{R}} (|\dot{q}|^2 + A(t)q \cdot q)\,\mathrm{d}t - \int_{\mathbf{R}} F(t,q)\,\mathrm{d}t.$$

It is well-known [7] that $\Phi \in C^1(E, \mathbf{R})$ and nontrivial critical points of Φ correspond to homoclinic solutions of (5.1) whenever (B1), (B3), (B5) are satisfied.

Theorem 5.1 *If assumptions (B1)–(B6) are satisfied, then (5.1) has a homoclinic solution.*

Theorem 5.2 *If assumptions (B1)–(B8) are satisfied, then (5.1) has infinitely many homoclinic solutions.*

The proofs use the same arguments as for the Schrödinger equation. Note that since the space E is continuously embedded in $L^\infty(\mathbf{R}, \mathbf{R}^N)$, it is not necessary to have a growth restriction like (A2) here. By the same reason no growth condition in (B8) is needed (cf. [7, Proposition 1.24]). Hypothesis (B2) (which replaces (A2)) is used in order to show that $(PS)_\beta$-sequences are bounded. More precisely, in the proof of Lemma 1.5 we replace (1.9) by

$$\beta + 1 + \varepsilon_m\|q_m\| \ge \Phi(q_m) - \frac{1}{2}\langle \nabla\Phi(q_m), q_m\rangle \ge \left(\frac{1}{2} - \frac{1}{\gamma}\right)\int_{\mathbf{R}^N} q_m \cdot F_q(t,q_m)\,\mathrm{d}t \ge 0,$$

where $\varepsilon_m \to 0$ as $m \to \infty$. It follows from (B2), (B4) that there is a constant $\widetilde{c} > 0$ such that

$$|F_q(t,q)|^2 \le \widetilde{c}\, q \cdot F_q(t,q) \qquad \text{if } |q| \le 1$$

and

$$|F_q(t,q)| \le \widetilde{c}\, q \cdot F_q(t,q) \qquad \text{if } |q| \ge 1.$$

The remaining part of the proof is similar to that of Lemma 1.5 (with obvious changes).

A homoclinic solution for (5.1) has also been found in [3] by a different argument (constructing subharmonics and passing to the limit) and under somewhat more restrictive conditions than in Theorem 5.1.

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