Existence and number of solutions for a class of semilinear Schrödinger equations

Yanheng Ding
Institute of Mathematics, AMSS, Chinese Academy of Sciences
100080 Beijing, China

Andrzej Szulkin
Department of Mathematics, Stockholm University
106 91 Stockholm, Sweden

Dedicated to Djairo G. de Figueiredo on the occasion of his 70th birthday

Abstract

Using an argument of concentration-compactness type we study the problem $-\Delta u + \lambda V(x)u = |u|^{p-2}u$, $x \in \mathbb{R}^N$, where $2 < p < 2^*$ and the set $\{x \in \mathbb{R}^N : V(x) < b\}$ is nonempty and has finite measure for some $b > 0$. In particular, we show that if $V^{-1}(0)$ has nonempty interior, then the number of solutions increases with λ. We also study concentration of solutions on the set $V^{-1}(0)$ as $\lambda \to \infty$.

1 Introduction

The purpose of this paper is to present simple proofs of some results concerning the existence and the number of decaying solutions for the Schrödinger equation

$$-\Delta u + V(x)u = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

and for the related equations

$$-\Delta u + \lambda V(x)u = |u|^{p-2}u, \quad x \in \mathbb{R}^N \tag{1.2}$$

and

$$-\varepsilon^2 \Delta u + V(x)u = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

respectively as $\lambda \to \infty$ and $\varepsilon \to 0$. In a concluding section we shall also consider concentration of solutions as $\lambda \to \infty$ or $\varepsilon \to 0$. We shall assume throughout that V and p satisfy the following assumptions:

(V_1) $V \in C(\mathbb{R}^N)$ and V is bounded below.

(V_2) There exists $b > 0$ such that the set $\{x \in \mathbb{R}^N : V(x) < b\}$ is nonempty and has finite measure.

(P) $p \in (2, 2^*)$, where $2^* := 2N/(N-2)$ if $N \geq 3$ and $2^* := +\infty$ if $N = 1$ or 2.

Assumption (V_1) is only for simplicity. In Sections 2 and 3 it can be replaced by

(V_1') $V \in L^1_{loc}(\mathbb{R}^N)$ and $V^- := \max\{-V, 0\} \in L^q(\mathbb{R}^N)$, where $q = N/2$ if $N \geq 3$, $q > 1$ if $N = 2$ and $q = 1$ if $N = 1$

while in Section 4 we also need $V \in L^q_{loc}(\mathbb{R}^N)$. Such an extension requires nothing more than a simple modification of our arguments.

Note that if $\varepsilon^2 = \lambda^{-1}$, then u is a solution of (1.2) if and only if $v = \lambda^{-1/(p-2)}u$ is a solution of (1.3), hence as far as the existence and the number of solutions are concerned, these two problems are equivalent.

Problem (1.3) with $V \geq 0$ and a more general right-hand side has been studied extensively by several authors, see e.g. [5, 10, 11] and the references therein. For a problem similar to (1.2), again with $V \geq 0$ and a more general right-hand side, see [2]. In a recent work [6] it has been shown that for a certain class of functions V which may change sign, (1.1) has infinitely many solutions, see Remark 3.6 below. The results of the present paper extend and complement those mentioned above. In particular, our assumptions on V are rather weak, but perhaps more important, our proofs seem to be new and simpler. On the other hand, contrary to [5, 10, 11], we do not study single- or multispike solutions of (1.3) as $\varepsilon \to 0$. In a forthcoming paper we shall consider (1.2) for a much more general class of nonlinearities. However, this will be done at the expense of the simplicity of arguments.

Below $\|u\|_p$ will denote the usual $L^p(\mathbb{R}^N)$-norm and $V^\pm(x) := \max\{\pm V(x), 0\}$. B_ρ and S_ρ will respectively denote the open ball and the sphere of radius ρ and center at the origin.

It is well known that the functional

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\,dx$$

is of class C^1 in the Sobolev space

$$E = \{u \in H^1(\mathbb{R}^N) : \|u\|^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + V^+(x)u^2)\,dx < \infty\} \tag{1.4}$$

and critical points of Φ correspond to solutions u of (1.1). Moreover, $u(x) \to 0$ as $|x| \to \infty$. It is easy to see that if

$$M := \inf_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx}{\|u\|_p^2} \tag{1.5}$$

is attained at some $\bar{u}$ and M is positive, then $u = M^{1/(p-2)}\bar{u}/\|\bar{u}\|_p$ is a solution of (1.1) and $u(x) \to 0$ as $|x| \to \infty$. Such u is called a ground state. We note for further reference that (V_1), (V_2) and the Poincaré inequality imply E is continuously embedded in $H^1(\mathbb{R}^N)$. For basic critical point theory in a setting suitable for our purposes the reader is referred e.g. to [7, 14]. That $u(x) \to 0$ as $|x| \to \infty$ can be seen as follows. If $N = 1$ and $u \in H^1(\mathbb{R})$, then $u(x) \to 0$ as $|x| \to \infty$. Suppose $N \geq 2$, let u be a solution of (1.1) and set $W(x) = V(x) - |u(x)|^{p-2}$. Since V is continuous, bounded below and $|u|^{p-2} \in L^r(\mathbb{R}^N)$ for some $r > N/2$, it is easy to verify that $W^+ \in K_N^{loc}$ and $W^- \in K_N$, where K_N and K_N^{loc} are the Kato classes as defined in Section A2 of [13]. Since $-\Delta u + W(x)u = 0$, $u(x) \to 0$ according to Theorem C.3.1 in [13]. An alternative proof, for a much more general class of Schrödinger equations including those with V satisfying (V_1') instead of (V_1), may be found in [8].

2 Compactness

In this section we study the compactness of minimizing sequences and of Palais-Smale sequences. We adapt well known arguments (see e.g. [7, 14]) to our situation.

Let

$$V_b(x) := \max\{V(x), b\},$$

and

$$M_b := \inf_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + V_b(x)u^2)\, dx}{\|u\|_p^2}. \tag{2.1}$$

Denote the spectrum of $-\Delta + V$ in $L^2(\mathbb{R}^N)$ by $\sigma(-\Delta + V)$ and recall the definition (1.5) of M.

Theorem 2.1 *Suppose (V_1), (V_2), (P) are satisfied and $\sigma(-\Delta + V) \subset (0, \infty)$. If $M < M_b$, then each minimizing sequence for M has a convergent subsequence. So in particular, M is attained at some $u \in E \setminus \{0\}$.*

Proof Let (u_m) be a minimizing sequence. We may assume $\|u_m\|_p = 1$. Since $V < 0$ on a set of finite measure, (u_m) is bounded in the norm of E given by (1.4). Passing to a subsequence we may assume $u_m \rightharpoonup u$ in E and by the continuity of the embedding $E \hookrightarrow H^1(\mathbb{R}^N)$, $u_m \to u$ in $L^2_{loc}(\mathbb{R}^N)$, $L^p_{loc}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Let $u_m = v_m + u$. Then

$$\int_{\mathbb{R}^N} (|\nabla u_m|^2 + V(x)u_m^2)\, dx = \int_{\mathbb{R}^N} (|\nabla v_m|^2 + V(x)v_m^2)\, dx + \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx + o(1) \tag{2.2}$$

and by the Brézis-Lieb lemma [4], [14, Lemma 1.32],

$$\int_{\mathbb{R}^N} |u_m|^p\, dx = \int_{\mathbb{R}^N} |v_m|^p\, dx + \int_{\mathbb{R}^N} |u|^p\, dx + o(1). \tag{2.3}$$

Moreover, by (V_2) and since $v_m \rightharpoonup 0$,

$$\int_{\mathbb{R}^N} (V(x) - V_b(x))v_m^2\, dx \to 0. \tag{2.4}$$

Using (2.2)-(2.4) and the definitions of M, M_b we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx + \int_{\mathbb{R}^N} (|\nabla v_m|^2 + V(x)v_m^2)\, dx + o(1) = M \\
&= M\|u_m\|_p^2 = M(\|u\|_p^p + \|v_m\|_p^p)^{2/p} + o(1) \le M(\|u\|_p^2 + \|v_m\|_p^2) + o(1) \\
&\le \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx + MM_b^{-1} \int_{\mathbb{R}^N} (|\nabla v_m|^2 + V_b(x)v_m^2)\, dx + o(1) \\
&\le \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx + MM_b^{-1} \int_{\mathbb{R}^N} (|\nabla v_m|^2 + V(x)v_m^2)\, dx + o(1).
\end{aligned}$$

Since $MM_b^{-1} < 1$ and $\int_{\mathbb{R}^N} V^-(x)v_m^2\, dx \to 0$, it follows that $v_m \to 0$ and therefore $u_m \to u$. It is clear that $u \neq 0$. $\square$

Remark 2.2 If $M = M_b$, then all inequalities in the last formula above become equalities after passing to the limit. Therefore either $u = 0$ or $u_m \to u$ in $L^p(\mathbb{R}^N)$. In the latter case M is attained.

From the above theorem it follows that if $\sigma(-\Delta+V)\subset(0,\infty)$ and $M<M_b$, then there exists a ground state solution of (1.1).

We shall also need to work with the functional Φ. Recall that (u_m) is called a Palais-Smale sequence at the level c (a $(PS)_c$-sequence) if $\Phi'(u_m)\to 0$ and $\Phi(u_m)\to c$. If each $(PS)_c$-sequence has a convergent subsequence, then Φ is said to satisfy the $(PS)_c$-condition.

Theorem 2.3 *If (V_1), (V_2) and (P) hold, then Φ satisfies $(PS)_c$ for all*

$$c<\left(\frac{1}{2}-\frac{1}{p}\right)M_b^{p/(p-2)}.$$

Proof Let (u_m) be a $(PS)_c$-sequence with c satisfying the inequality above. First we show that (u_m) is bounded. We have

$$d_1+d_2\|u_m\|\geq\Phi(u_m)-\frac{1}{2}\langle\Phi'(u_m),u_m\rangle=\left(\frac{1}{2}-\frac{1}{p}\right)\|u_m\|_p^p \tag{2.5}$$

and

$$d_1+d_2\|u_m\|\geq\Phi(u_m)-\frac{1}{p}\langle\Phi'(u_m),u_m\rangle=\left(\frac{1}{2}-\frac{1}{p}\right)\|u_m\|^2-\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}V^-(x)u_m^2\,dx \tag{2.6}$$

for some constants $d_1,d_2>0$. Suppose $\|u_m\|\to\infty$ and let $w_m:=u_m/\|u_m\|$. Dividing (2.5) by $\|u_m\|^p$ we see that $w_m\to 0$ in $L^p(\mathbb{R}^N)$ and therefore $w_m\rightharpoonup 0$ in E after passing to a subsequence. Hence $\int_{\mathbb{R}^N}V^-(x)w_m^2\,dx\to 0$ (recall V^- is bounded; in fact it suffices that $V^-\in L^q(\mathbb{R}^N)$, where q is as in (V_1')). So dividing (2.6) by $\|u_m\|^2$, it follows that $w_m\to 0$ in E, a contradiction. Thus (u_m) is bounded.

As in the preceding proof, we may assume $u_m\rightharpoonup u$ in E and $u_m\to u$ in $L^2_{loc}(\mathbb{R}^N)$, $L^p_{loc}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Set $u_m=v_m+u$. Since $\Phi'(u)=0$ and $\Phi(u)=\Phi(u)-\frac{1}{2}\langle\Phi'(u),u\rangle=(\frac{1}{2}-\frac{1}{p})\|u\|_p^p\geq 0$, it follows from (2.2), (2.3) that

$$0=\langle\Phi'(u_m),u_m\rangle+o(1)=\langle\Phi'(v_m),v_m\rangle+\langle\Phi'(u),u\rangle+o(1)=\langle\Phi'(v_m),v_m\rangle+o(1) \tag{2.7}$$

and

$$c=\Phi(u_m)+o(1)=\Phi(v_m)+\Phi(u)+o(1)\geq\Phi(v_m)+o(1). \tag{2.8}$$

By (2.7),

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}(|\nabla v_m|^2+V(x)v_m^2)\,dx=\lim_{m\to\infty}\int_{\mathbb{R}^N}|v_m|^p\,dx=:\gamma, \tag{2.9}$$

possibly after passing to a subsequence, and therefore it follows from (2.8) that

$$c\geq\left(\frac{1}{2}-\frac{1}{p}\right)\gamma. \tag{2.10}$$

By (2.4),

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}(|\nabla v_m|^2+V_b(x)v_m^2)\,dx=\lim_{m\to\infty}\int_{\mathbb{R}^N}(|\nabla v_m|^2+V(x)v_m^2)\,dx=\gamma.$$

On the other hand,

$$\|v_m\|_p^2 \le M_b^{-1} \int_{\mathbb{R}^N} (|\nabla v_m|^2 + V_b(x)v_m^2)\,dx$$

by the definition (2.1) of M_b; therefore $\gamma^{2/p} \le M_b^{-1}\gamma$. Combining this with (2.10) we see that either $\gamma = 0$ or

$$c \ge \left(\frac{1}{2} - \frac{1}{p}\right) M_b^{p/(p-2)},$$

hence γ must be 0 by the assumption on c. So according to (2.9),

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} (|\nabla v_m|^2 + V^+(x)v_m^2)\,dx = \lim_{m\to\infty} \int_{\mathbb{R}^N} (|\nabla v_m|^2 + V(x)v_m^2)\,dx = 0.$$

Therefore $v_m \to 0$ and $u_m \to u$ in E. □

3 Existence of solutions

Theorem 3.1 *Suppose* (V_1) *and* (P) *are satisfied,* $\sigma(-\Delta + V) \subset (0,\infty)$*,* $\sup_{x\in\mathbb{R}^N} V(x) = b > 0$ *and the measure of the set* $\{x \in \mathbb{R}^N : V(x) < b - \varepsilon\}$ *is finite for all* $\varepsilon > 0$*. Then the infimum in (1.5) is attained at some* $u \ge 0$*. If* $V \ge 0$*, then* $u > 0$ *in* $\mathbb{R}^N$*.*

Proof Since V^+ is bounded, $E = H^1(\mathbb{R}^N)$ here. Let u_b be the radially symmetric positive solution of the equation

$$-\Delta u + bu = |u|^{p-2}u, \quad x \in \mathbb{R}^N.$$

It is well known that such u_b exists, is unique and minimizes

$$N_b := \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + bu^2)\,dx}{\|u\|_p^2} \tag{3.1}$$

(see e.g. [7, Section 8.4] or [14, Section 1.7]). So if $V \equiv b$, we are done. Otherwise we may assume without loss of generality that $V(0) < b$. Then

$$\begin{aligned} M &= \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx}{\|u\|_p^2} \le \frac{\int_{\mathbb{R}^N} (|\nabla u_b|^2 + V(x)u_b^2)\,dx}{\|u_b\|_p^2} \\ &< \frac{\int_{\mathbb{R}^N} (|\nabla u_b|^2 + bu_b^2)\,dx}{\|u_b\|_p^2} = N_b = M_b, \end{aligned}$$

where the last equality follows from the fact that $V_b = b$. In order to apply Theorem 2.1 we need to show that $M < M_{b-\varepsilon}$ for some $\varepsilon > 0$ ($M < M_b$ does not suffice because the set $\{x \in \mathbb{R}^N : V(x) < b\}$ may have infinite measure). A simple computation shows that if $\lambda > 0$, then $N_{\lambda b}$ is attained at $u_{\lambda b}(x) = \lambda^{1/(p-2)}u_b(\sqrt{\lambda}x)$ and

$$N_{\lambda b} = \lambda^r N_b, \text{ where } r = 1 - \frac{N}{2} + \frac{N}{p} > 0. \tag{3.2}$$

Choosing $\lambda = (b-\varepsilon)/b$ we see that $N_{b-\varepsilon} < N_b$ and $N_{b-\varepsilon} \to N_b$ as $\varepsilon \to 0$. So for ε small enough we have

$$M < N_{b-\varepsilon} = \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + (b-\varepsilon)u^2)\,dx}{\|u\|_p^2} \le \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + V_{b-\varepsilon}(x)u^2)\,dx}{\|u\|_p^2} = M_{b-\varepsilon}.$$

Hence M is attained at some u. Since the expression on the right-hand side of (1.5) does not change if u is replaced by $|u|$, we may assume $u \geq 0$. By the maximum principle, if $V \geq 0$, then $u > 0$ in $\mathbb{R}^N$. $\square$

Theorem 3.2 *Suppose $V \geq 0$ and (V_1), (V_2), (P) are satisfied. Then there exists $\Lambda > 0$ such that for each $\lambda \geq \Lambda$ the infimum in (1.5) (with $V(x)$ replaced by $\lambda V(x)$) is attained at some $u_\lambda > 0$.*

Proof Here $V = V^+$. Let b be as in (V_2) and

$$M^\lambda := \inf_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda V(x) u^2)\, dx}{\|u\|_p^2}, \quad M_b^\lambda := \inf_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda V_b(x) u^2)\, dx}{\|u\|_p^2}. \tag{3.3}$$

It suffices to show that $M^\lambda < M_b^\lambda$ for all λ large enough. We may assume $V(0) < b$ and choose $\varepsilon, \delta > 0$ so that $V(x) < b - \varepsilon$ whenever $|x| < 2\delta$. Let $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be a function such that $\varphi(x) = 1$ for $|x| \leq \delta$ and $\varphi(x) = 0$ for $|x| \geq 2\delta$. Set $w_{\lambda b}(x) := \varphi(x) u_{\lambda b}(x) \equiv \lambda^{1/(p-2)} u_b(\sqrt{\lambda} x) \varphi(x)$, where u_b is as in the proof of Theorem 3.1. Then for all sufficiently large λ and some $c_0 > 0$,

$$\begin{aligned} M^\lambda &\leq \frac{\int_{\mathbb{R}^N} (|\nabla w_{\lambda b}|^2 + \lambda V(x) w_{\lambda b}^2)\, dx}{\|w_{\lambda b}\|_p^2} \leq \frac{\int_{\mathbb{R}^N} (|\nabla w_{\lambda b}|^2 + \lambda (b-\varepsilon) w_{\lambda b}^2)\, dx}{\|w_{\lambda b}\|_p^2} \\ &= \lambda^r \left(\frac{\int_{\mathbb{R}^N} (|\nabla u_b|^2 + b u_b^2)\, dx - \varepsilon \int_{\mathbb{R}^N} u_b^2\, dx}{\|u_b\|_p^2} + o(1) \right) \leq \lambda^r (N_b - c_0 \varepsilon) \end{aligned}$$

(N_b is defined in (3.1) and r in (3.2)). Using (3.2) and (3.3) we also see that

$$M_b^\lambda \geq \inf_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda b u^2)\, dx}{\|u\|_p^2} = N_{\lambda b} = \lambda^r N_b, \tag{3.4}$$

hence $M^\lambda < M_b^\lambda$ (the infimum above is equal to $N_{\lambda b}$ also when E is a proper subspace of $H^1(\mathbb{R}^N)$ because $C_0^\infty(\mathbb{R}^N)$, and hence also E, is dense in $H^1(\mathbb{R}^N)$). By the argument at the end of the proof of Theorem 3.1, the infimum is attained at some $u_\lambda > 0$. $\square$

Remark 3.3 If (V_1) is replaced by (V_1'), then we need to assume that the set $\{x \in \mathbb{R}^N : V(x) < b - \varepsilon\}$ appearing in Theorem 3.1 has nonempty interior for each $\varepsilon > 0$. Likewise, in Theorem 3.2 the set $\{x \in \mathbb{R}^N : V(x) < b\}$ should have nonempty interior.

Next we shall consider the existence of multiple solutions under the hypothesis that $V^{-1}(0)$ has nonempty interior.

Theorem 3.4 *Suppose $V \geq 0$, $V^{-1}(0)$ has nonempty interior and (V_1), (V_2), (P) are satisfied. For each $k \geq 1$ there exists $\Lambda_k > 0$ such that if $\lambda \geq \Lambda_k$, then (1.2) has at least k pairs of nontrivial solutions in E.*

Proof For a fixed k we can find $\varphi_1, \ldots, \varphi_k \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp} \varphi_j$, $1 \leq j \leq k$, is contained in the interior of $V^{-1}(0)$ and $\operatorname{supp} \varphi_i \cap \operatorname{supp} \varphi_j = \emptyset$ whenever $i \neq j$. Let

$$F_k := \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}.$$

Since $V \ge 0$, $\Phi(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\|u\|_p^p$ and therefore there exist $\alpha, \rho > 0$ such that $\Phi|_{S_\rho} \ge \alpha$. Denote the set of all symmetric (in the sense that $-A = A$) and closed subsets of E by Σ, for each $A \in \Sigma$ let $\gamma(A)$ be the Krasnoselski genus and

$$i(A) := \min_{h\in\Gamma} \gamma(h(A) \cap S_\rho),$$

where Γ is the set of all odd homeomorphisms $h \in C(E, E)$. Then i is a version of Benci's pseudoindex [1, 3]. Let

$$\Phi_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda V(x)u^2)\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\,dx, \quad \lambda \ge 1$$

and

$$c_j := \inf_{i(A)\ge j}\ \sup_{u\in A} \Phi_\lambda(u), \quad 1 \le j \le k.$$

Since $\Phi_\lambda(u) \ge \Phi(u) \ge \alpha$ for all $u \in S_\rho$ and since $i(F_k) = \dim F_k = k$ (see [1, 3]),

$$\alpha \le c_1 \le \ldots \le c_k \le \sup_{u\in F_k} \Phi_\lambda(u) =: C.$$

It is clear that C depends on k but not on λ. As in (3.4), we have

$$M_b^\lambda \ge N_{\lambda b} = \lambda^r N_b,$$

where $r > 0$, and therefore $M_b^\lambda \to \infty$. Hence $C < (\frac{1}{2} - \frac{1}{p})(M_b^\lambda)^{p/(p-2)}$ whenever λ is large enough and it follows from Theorem 2.3 that for such λ the Palais-Smale condition is satisfied at all levels $c \le C$. By the usual critical point theory, all c_j are critical levels and Φ_λ has at least k pairs of nontrivial critical points. □

Next we extend the above result to the case of $V^- \not\equiv 0$. As in [9], we consider the eigenvalue problem

$$-\Delta u + \lambda V^+(x)u = \mu\lambda V^-(x)u, \quad u \in E \tag{3.5}$$

(here $\lambda \ge 1$ is fixed). An equivalent norm $\|u\|_\lambda$ in E is given by the inner product

$$\langle u, v\rangle_\lambda := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + \lambda V^+(x)uv)\,dx.$$

Since $V^- > 0$ on a set of finite measure, the linear operator $u \mapsto \int_{\mathbb{R}^N} \lambda V^-(x)u \cdot dx$ is compact. It follows that there are finitely many eigenvalues $\mu \le 1$ and the quadratic form

$$u \mapsto \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda V(x)u^2)\,dx$$

is negative semidefinite on the space E^- spanned by the corresponding eigenfunctions. It is easy to see that $\dim E^- \to \infty$ as $\lambda \to \infty$.

Theorem 3.5 *Suppose $V^- \not\equiv 0$, $V^{-1}(0)$ has nonempty interior and (V_1), (V_2), (P) are satisfied. For each $k \ge 1$ there exists $\Lambda_k > 0$ such that if $\lambda \ge \Lambda_k$, then (1.2) has at least k pairs of nontrivial solutions in E.*

Proof We need to modify the argument of Theorem 3.4. Let φ_j and F_k be as before. If e is an eigenfunction of (3.5) and μ a corresponding eigenvalue, then

$$\langle e, \varphi_j \rangle_\lambda = \mu\lambda \int_{\mathbb{R}^N} V^-(x) e \varphi_j \, dx = 0 \tag{3.6}$$

because $\operatorname{supp} \varphi_j \subset V^{-1}(0)$. Hence $E_k := E^- + F_k = E^- \oplus F_k$. Let $l = \dim E^-$ and

$$c_j := \inf_{i(A) \geq l+j} \sup_{u \in A} \Phi_\lambda(u), \quad 1 \leq j \leq k.$$

Write $u = e + f$, $e \in E^-$, $f \in F_k$. By (3.6) and since there exists a continuous projection $L^p(\mathbb{R}^N) \to F_k$,

$$\Phi_\lambda(u) \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla f|^2 \, dx - \frac{\tilde{C}}{p} \int_{\mathbb{R}^N} |f|^p \, dx$$

for some $\tilde{C} \leq 1$. Thus

$$c_k \leq \sup_{u \in E_k} \Phi_\lambda(u) = C,$$

where C is independent of λ. If $i(A) \geq l + 1$, then $\gamma(h(A) \cap S_\rho) \geq l + 1$ for each $h \in \Gamma$ and therefore $h(A) \cap S_\rho$ intersects any subspace of codimension $\leq l$. The space E has an orthogonal decomposition $E = E^+ \oplus E^- \oplus F$ (with respect to the inner product $\langle .\,,. \rangle_\lambda$), where E^+ corresponds to the eigenvalues $\mu > 1$ of (3.5) and F is the subspace of functions $u \in E$ whose support is contained in $V^{-1}([0, \infty))$. It is clear that the quadratic part of Φ_λ is positive definite on E^+, and it is also positive definite on F because $V^{-1}(0)$ has finite measure. Hence there exist $\alpha, \rho > 0$ (possibly depending on λ) such that $\Phi_\lambda|_{S_\rho \cap (E^+ \oplus F)} \geq \alpha$. Since $\operatorname{codim}(E^+ \oplus F) = l$, it follows that $h(A) \cap S_\rho \cap (E^+ \oplus F) \neq \emptyset$ and $c_1 \geq \alpha$. Now it remains to repeat the argument at the end of the preceding proof. $\square$

Remark 3.6 (i) If $V(x) \to \infty$ as $|x| \to \infty$, then it is well known that the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, $2 \leq q < 2^*$ is compact, see e.g. [2]. Therefore the Palais-Smale condition holds at all levels and (1.1) has infinitely many solutions.

(ii) It has been shown in [6] that if $V \in C^1(\mathbb{R}^N)$ and satisfies certain growth conditions at infinity (which are much weaker than the requirement that $V(x) \to \infty$ as $|x| \to \infty$), then (1.1) has infinitely many solutions.

4 Concentration of solutions

Theorem 4.1 *Suppose (V_1), (V_2), (P) are satisfied and $V^{-1}(0)$ has nonempty interior Ω. Let $u_m \in E$ be a solution of the equation*

$$-\Delta u + \lambda_m V(x) u = |u|^{p-2} u, \quad x \in \mathbb{R}^N. \tag{4.1}$$

If $\lambda_m \to \infty$ and $\|u_m\|_{\lambda_m} \leq C$ for some $C > 0$, then, up to a subsequence, $u_m \to \bar{u}$ in $L^p(\mathbb{R}^N)$, where $\bar{u}$ is a weak solution of the equation

$$-\Delta u = |u|^{p-2} u, \quad x \in \Omega, \tag{4.2}$$

and $\bar{u} = 0$ a.e. in $\mathbb{R}^N \setminus V^{-1}(0)$. If moreover $V \geq 0$, then $u_m \to \bar{u}$ in E.

We note that $\bar{u} \in H_0^1(\Omega)$ if $V^{-1}(0) = \overline{\Omega}$ and $\partial\Omega$ is locally Lipschitz continuous (cf. [2]). Before proving the above theorem we point out some of its consequences.

Corollary 4.2 *Suppose (V_1), (V_2), (P) are satisfied, $V^{-1}(0)$ has nonempty interior, $V \geq 0$, $u_m \in E$ is a solution of (4.1), $\lambda_m \to \infty$ and $\Phi_{\lambda_m}(u_m)$ is bounded and bounded away from 0. Then the conclusion of Theorem 4.1 is satisfied and $\bar{u} \neq 0$.*

Proof We have $\Phi_{\lambda_m}(u_m) = \frac{1}{2}\|u_m\|_{\lambda_m}^2 - \frac{1}{p}\|u_m\|_p^p$ and

$$\Phi_{\lambda_m}(u_m) = \Phi_{\lambda_m}(u_m) - \frac{1}{2}\langle \Phi'_{\lambda_m}(u_m), u_m\rangle = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_m\|_p^p.$$

Hence $\|u_m\|_p$, and therefore also $\|u_m\|_{\lambda_m}$ is bounded. So the conclusion of Theorem 4.1 holds. Moreover, as $\|u_m\|_p$ is bounded away from 0, $\bar{u} \neq 0$. $\square$

Note that as a consequence of this corollary, if k is fixed, then any sequence of solutions u_m of (1.2) with $\lambda = \lambda_m \to \infty$ obtained in Theorem 3.4 contains a subsequence concentrating at some $\bar{u} \neq 0$. Moreover, it is possible to obtain a positive solution for each λ, either via Theorem 3.1 or by the mountain pass theorem. It follows that each sequence (u_m) of such solutions with $\lambda_m \to \infty$ has a subsequence concentrating at some $\bar{u}$ which is positive in Ω. Corresponding to u_m are solutions $v_m = \varepsilon_m^{2/(p-2)} u_m$ of (1.3), where $\varepsilon_m^2 = \lambda_m^{-1}$. Then $v_m \to 0$ and $\varepsilon_m^{-2/(p-2)} v_m \to \bar{u}$. This should be compared with (iii) of Theorem 1 in [5] where it was shown that $\lim_{m\to\infty} \varepsilon_m^{-2/(p-2)}\|v_m\|_\infty > 0$.

It will become clear from the proof of Theorem 4.1 that if $V^{-1}(0)$ has empty interior, then $\bar{u} \equiv 0$ which is impossible under the assumptions of Corollary 4.2. Since $\sigma(-\Delta + \lambda V) \subset (a, \infty)$ for some $a > 0$ (independent of λ if λ is bounded away from 0), $u = 0$ is the only critical point of Φ_λ in B_r for some $r > 0$. Hence in this case $\Phi_{\lambda_m}(u_m) \to \infty$ and $\|u_m\| \to \infty$ if u_m is a nontrivial solution of (1.2) with $\lambda = \lambda_m \to \infty$.

If $V^- \neq 0$, we do not know whether $u_m \to \bar{u}$ in E or whether a result corresponding to Corollary 4.2 is true. However, if $V^{-1}(0)$ has empty interior, then it follows from Theorem 4.1 that either $u_m \to 0$ in $L^p(\mathbb{R}^N)$ or $\|u_m\|_{\lambda_m} \to \infty$.

Proof of Theorem 4.1 We modify the argument in [2]. Since $\lambda_m \geq 1$, $\|u_m\| \leq \|u_m\|_{\lambda_m} \leq C$. Passing to a subsequence, $u_m \rightharpoonup \bar{u}$ in E and $u_m \to \bar{u}$ in $L^p_{loc}(\mathbb{R}^N)$. Since $\langle \Phi'_{\lambda_m}(u_m), \varphi\rangle = 0$, we see that $\int_{\mathbb{R}^N} V(x) u_m \varphi \, dx \to 0$ and $\int_{\mathbb{R}^N} V(x)\bar{u}\varphi \, dx = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Therefore $\bar{u} = 0$ a.e. in $\mathbb{R}^N \setminus V^{-1}(0)$.

We claim that $u_m \to \bar{u}$ in $L^p(\mathbb{R}^N)$. Assuming the contrary, it follows from P.L. Lions' vanishing lemma (see [12, Lemma I.1] or [14, Lemma 1.21]) that

$$\int_{B_\rho(x_m)} (u_m - \bar{u})^2 \, dx \geq \gamma$$

for some $(x_m) \subset \mathbb{R}^N$, $\rho, \gamma > 0$ and almost all m ($B_\rho(x)$ denotes the open ball of radius ρ and center x). Since $u_m \to \bar{u}$ in $L^2_{loc}(\mathbb{R}^N)$, $|x_m| \to \infty$. Therefore the measure of the set $B_\rho(x_m) \cap \{x \in \mathbb{R}^N : V(x) < b\}$ tends to 0 and

$$\|u_m\|_{\lambda_m}^2 \geq \lambda_m b \int_{B_\rho(x_m)\cap\{V\geq b\}} u_m^2 \, dx = \lambda_m b \left(\int_{B_\rho(x_m)} (u_m - \bar{u})^2 \, dx + o(1) \right) \to \infty,$$

a contradiction.

Let now $V \geq 0$. Since u_m satisfies (4.1), $\langle \Phi'_{\lambda_m}(u_m), \bar{u} \rangle = 0$ and $\bar{u}(x) = 0$ whenever $V(x) > 0$, it follows that

$$\|u_m\|^2 \leq \|u_m\|_{\lambda_m}^2 = \|u_m\|_p^p$$

and

$$\|\bar{u}\|^2 = \|\bar{u}\|_{\lambda_m}^2 = \|\bar{u}\|_p^p.$$

Hence $\limsup_{m\to\infty} \|u_m\|^2 \leq \|\bar{u}\|_p^p = \|\bar{u}\|^2$ and therefore $u_m \to \bar{u}$ in E. □

References

[1] P. Bartolo, V. Benci and D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity*, Nonlin. Anal. 7 (1983), 981–1012.

[2] T. Bartsch, A. Pankov and Z.Q. Wang, *Nonlinear Schrödinger equations with steep potential well*, Comm. Contemp. Math. 3 (2001), 549–569.

[3] V. Benci, *On critical point theory of indefinite functionals in the presence of symmetries*, Trans. Amer. Math. Soc. 274 (1982), 533–572.

[4] H. Brézis, E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. 88 (1983), 486–490.

[5] J. Byeon, Z.Q. Wang, *Standing waves with a critical frequency for nonlinear Schrödinger equations, II*, Calc. Var. PDE 18 (2003), 207–219.

[6] G. Cerami, G. Devillanova and S. Solimini, *Infinitely many bound states for some nonlinear scalar field equations*, Preprint.

[7] J. Chabrowski, *Variational Methods for Potential Operator Equations*, de Gruyter, Berlin 1997.

[8] J. Chabrowski and A. Szulkin, *On the Schrödinger equation involving a critical Sobolev exponent and magnetic field*, Top. Meth. Nonl. Anal., to appear.

[9] S.W. Chen and Y.Q. Li, *Nontrivial solution for a semilinear elliptic equation in unbounded domain with critical Sobolev exponent*, J. Math. Anal. Appl. 272 (2002), 393–406.

[10] M. del Pino and P. Felmer, *Semi-classical states of nonlinear Schrödinger equations: a variational reduction method*, Math. Ann. 324 (2002), 1–32.

[11] L. Jeanjean and K. Tanaka, *Singularly perturbed elliptic problems with superlinear or asymptotically linear nonlinearities*, Calc. Var. PDE 21 (2004), 287–318.

[12] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. Part I*, Ann. IHP, Analyse Non Linéaire 1 (1984), 109–145.

[13] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. (New Series) 7, (1982), 447–526.

[14] M. Willem, *Minimax Theorems*, Birkhäuser, Boston 1996.