Strategic complementarities, network games and endogenous network formation *

Andreas Lagerås[†]and David Seim[‡]

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Abstract

This paper investigates the role of strategic complementarities in the context of network games and network formation models. In the general model of static games on networks, we characterize conditions on the utility function that ensure the existence and uniqueness of a pure-strategy Nash equilibrium, regardless of the network structure. By applying the game to empirically-relevant networks that feature nestedness – Nested Split Graphs – we show that equilibrium strategies are non-decreasing in the degree. We extend the framework into a dynamic setting, comprising a game stage and a formation stage, and provide general conditions for the network process to converge to a Nested Split Graph with probability one, and for this class of networks to be an absorbing state. The general framework presented in the paper can be applied to models of games on networks, models of network formation, and combinations of the two.

1 Introduction

Social interactions are a key feature of everyday life and individuals' networks are widely recognized as imperative to performance. For instance, [8] finds that peer pressure, proxied by network centrality, boosts student performance and [16] argues that job-finding ability depends on weak ties. The empirical literature has focused on either the behavioral (peer) effects of networks, or on documenting the empirical regularities of social networks.[1] This

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[†]Department of Mathematical Statistics, Stockholm University

[‡]Department of Economics, University of Toronto, and Research Institute of Industrial Economics.

[1]For instance, [30] show that social networks are characterized by high clustering and short average path lengths and [3] argue that the degree distribution follows a power-law. Examples of networks that are nested, i.e. in which the neighborhood of nodes are contained within the neighborhoods of nodes with higher degrees, are identified by [23]. See also, e.g., [11, 9], for network effects.

has inspired two theoretical strands that seek to understand (i) how networks influence behavior and (ii) how networks are formed. The first strand considers games on networks and addresses how the nature of social interactions shape outcomes and how perturbations of the network structure affect agents' behavior. The second strand proposes frameworks for link formation and studies the equilibrium properties of the resulting networks.[2]

A key challenge within the first strand is that even simple games played on networks are often plagued by multiple equilibria. Since these equilibria typically possess very different properties, qualitative and generic statements about the game are difficult to make. Recent contributions have thus sought to identify general conditions that either ensure the existence of a unique equilibrium, or that all equilibria feature the same characteristics. Such conditions may then assist researchers interested in specifying theoretical models of network effects.

In [13] a general game-theoretic framework is constructed, where agents have private and incomplete information about the network structure and it is shown that (symmetric) equilibrium strategies are monotone in the number of connections, i.e. degree. In [5] a game with complete information and linear best-response functions is considered and it is shown that the existence and uniqueness of a Nash equilibrium depends on the lowest eigenvalue of the network. When the magnitude of the lowest eigenvalue is small, interaction effects are limited and one obtains uniqueness.

This paper contributes to this literature by providing a general framework for analyzing simultaneous-action games on networks that feature the canonical form of social interactions known as *strategic complementarities*. When an individual exerts more effort due to her friend increasing her effort, we say that actions are strategic complementarities.[3][4] We identify properties of the agent's utility function that are sufficient conditions for the existence and uniqueness of a Nash equilibrium, regardless of the network structure. The utility function is general, but the Hessian matrix, incorporating all network effects, must be strictly positive definite, i.e. its lowest eigenvalue must be bounded from below by a positive number.[5] The model thus ensures that own concavity effects always dominate interaction effects. Importantly, the framework is related to [5], but while they consider linear best-response functions and relate chosen parameter values to the smallest eigenvalue of the network, we do not impose assumptions about functional form beyond the aforementioned restrictions on the Hessian. In fact, the proposed framework encompasses the explicit model used to

[2]In [18], an endogenous network formation model that generates networks consistent with the empirically observed properties of many networks, except nestedness, is proposed. The model laid out by [23] is able to generate also nestedness.

[3]Under mild technical conditions the concept is equivalent to supermodularity, see for instance [28].

[4]There is ample empirical evidence of complementarities in networks. [6] finds that the probability of a youth being involved in criminal activities increases when the family moves to a neighborhood with more crime. Strategic complementarities in actions are also present in labor markets, in R & D activity and within families. See [9] for an extensive survey of neighborhood effects in economics.

[5]Our methodology applies the work by [21, 22] and [29] to a network setting, where payoffs depend only directly on own and neighbors' actions.

describe criminal behavior in [2].

While the existence and uniqueness result obtains even in a general setting, we proceed by applying the framework to a class of networks that are nested in order to make qualitative statements about equilibrium strategies. In such *Nested Split Graphs* (NSG) [6] the neighborhood of an agent is contained in the neighborhood of each node with higher degree. These graphs are sometimes referred to as *Hierarchy Graphs*, as the individuals' degrees are reflected in their ranking, in terms of centrality measures. Such hierarchical networks are common in the real world, yet have not been extensively studied from a theoretical point of view. The unambigous hierarchy of nodes according to degree that these networks exhibit harmonizes well with the sociology literature. [24, 25, 26] proposes a theory of social capital based on a given hierarchy of individuals in terms of valued attributes such as wealth, status or power. Connections with people in top positions are desirable because these people are able to influence decisions that may benefit the individual. Moreover, their popularity makes more people want to connect with them which makes them hubs of informational flows, which further amplifies their social capital Their social status thus becomes synonymous with their degree. [23] provide examples of other real-world networks that exhibit nestedness and negative assortativity, and build a network-formation model consistent with nestedness as well as other empirical regularities of social networks. Consistent with these works, when applying our general framework to NSGs, we find that, in the unique pure-strategy equilibrium, strategies are non-decreasing in degree.

In some applications, it is crucial to study games on networks and endogenous network formation simultaneously. Many complex real-world networks are the outcome of the interplay between strategic interactions on given networks and the endogenous creation and destruction of links. The final part of the paper extends the static model into a dynamic framework, where each period consists of a game stage, in which the static game is myopically played by all agents, and a formation stage, in which one arbitrarily chosen agent has the opportunity to add or delete a link. For games that feature strategic complementarities and increasing utility in neighbors' actions, our framework delivers the following results: the network structure converges to the class of NSGs with probability one and this class of networks is an absorbing state. This result contributes to an emerging literature that combines games on networks with a dynamic network formation process. In particular, our framework is a generalization of the model in [23]. This part of the paper is also related to the literature that combines strategic interactions with network formation, as in [4, 7, 19, 12, 15].[7] Relative to this literature, the framework that we provide is general and dynamic, i.e. allows for a wide range of applications and features a network structure that changes over time.

[6]Nested Split Graphs have been studied in the fields of physics and mathematics. They also go under the names *Threshold Graphs*, for a review see [27].

[7]This part is also related to dynamic models of network formation, such as [20, 1, 31, 10], where the shape of the network is dynamically changing. Although the resulting networks are the outcome of endogenous link choices, these models do not feature the combined element of strategic interactions and network formation present in this paper.

While maintaining generality, our framework is able to retain sharp predictions of (monotone) strategies in the unique equilibrium and characterize the properties of the network process.

The outline of the paper is as follows. Section 2 presents the static game. Section 3 introduces Nested Split Graphs and analyzes games played on such networks. Section 4 extends the framework into a dynamic model. Section 5 concludes.

2 The static model

We represent a network consisting of a finite set $N = \{1, 2, 3, \ldots, n\}$ of agents or nodes by the adjacency matrix G (of dimension $n \times n$). The agents may be individuals, business partners, students, firms etc. Entry $g_{ij} \in \{0, 1\}$ denotes whether a link between i and j is present or not and, conventionally, $g_{ii} = 0$. The network is undirected, implied by $g_{ij} = g_{ji}$. Further, let N_i be the neighborhood of i, i.e. $N_i = \{j \in N : g_{ij} = 1\}$, and denote the degree of i, i.e. the number of i's neighbors, by $d_i = |N_i|$.

Each agent i simultaneously undertakes action $x_i \in \mathbb{R}_+$ in order to maximize a utility or payoff function[8] $u_i(x_1, \ldots, x_n; G)$, i.e. the utility of node i on the graph G when node j exerts effort $x_j \in \mathbb{R}_+$ for $j = 1, \ldots, n$. All nodes have homogenous utility in the following sense. There exists a function $u(x_0, \ldots, x_n)$ from $\mathbb{R}_+^{n+1}$ to $\mathbb{R}$ symmetric in its last n arguments, i.e. $u(x_0, x_1, \ldots, x_n) = u(x_0, x_{\pi(1)}, \ldots, x_{\pi(n)})$ for all permutations π of $\{1, \ldots, n\}$, such that

$$u_i(x_1, \ldots, x_n; G) = u(x_i, g_{i1}x_1, \ldots, g_{in}x_n).$$

This means that the absence of a link between i and j has the same effect for i's utility as if there were a link between them and j's effort were zero.[9] Superficially x_i enters twice on the right hand side, but since $g_{ii} = 0$ and thus $g_{ii}x_i = 0$, only the first argument matters.

Before turning to the class of utility functions considered, we define the concept of *strongly positive definiteness.* Let $A(x)$ be a matrix-valued function defined on some set $D \subseteq \mathbb{R}^n$.

Definition 1. *$A(x)$ is* strongly positive definite *if there exists an $\epsilon > 0$ such that $y'A(x)y \geq \epsilon\|y\|^2$ for all $y \in \mathbb{R}^n \setminus \{0\}$ and all $x \in D$.*

We now proceed by postulating general conditions on the utility function which configure the nature of this non-cooperative game. Embedded in the utility function is the dependence on neighbors' actions. We start out by making the following assumptions regarding the complete graph.

Definition 2. *The utility of node i, $u_i(x_1, \ldots, x_n; K)$, where K is the adjacency matrix of the complete graph, $u_i : \mathbb{R}_+^n \to \mathbb{R}$ is three times continuously differentiable. To ease notation,*

[8] We will refer to this mapping as a utility function for simplicity, but as the framework is general and applicable to different economic and social settings, it may well represent other types of payoffs as well.

[9] This representation implies both that the utility of own effort is constant across players and that the utility from neighbors' actions is the same.

we define marginal utility of own effort as $h_i(x_1,\dots,x_n;K) = \frac{\partial}{\partial x_i} u_i(x_1,\dots,x_n;K)$*. It satisfies:*

C1. $h_i\left(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n;K\right) > 0$*.*

C2. $\frac{\partial h_i(x_1,\dots,x_n;K)}{\partial x_j} \geq 0$*,* $i \neq j$*, with equality only if* $x_j = 0$*.*

C3. $\left(-\frac{\partial h_i(x_1,\dots,x_n;K)}{\partial x_j}\right)_{i,j=1,\dots,n}$ *is strongly positive definite.*

C4. $\frac{\partial^2 h_i(x_1,\dots,x_n;K)}{\partial x_k \partial x_j} \geq 0$ $i \neq k$ *and for all* j*.*

These conditions may be described as follows. Condition 1 (C1) is simply a regularity condition, stating that as marginal utility of own effort is positive at zero, it is always worthwile to exert effort. The complementarity of neighbors' efforts is defined by C2, which states that marginal utility of own effort is non-decreasing in neighbor's actions, i.e. supermodularity.[10] In order to avoid positive feedback loops between neighbors pushing efforts toward infinity, C3 bounds potential complementarity effects. In particular, we assume that marginal utility in own effort is decreasing, $\frac{\partial}{\partial x_i} h_i(x_1,\dots,x_n;K) < 0$. At the same time, this condition ensures that own concavity always exceeds potential complementarity effects. C4 means that complementarity effects from any particular neighbor are non-decreasing in all neighbors' efforts ($i \neq j$), and that the concavity in own effort is non-increasing in neighbor's effort ($i = j$).

By imposing conditions on the utility functions on the complete graph, we ensure that the game is feasible in the case where positive or negative feedback loops are the greatest. The following lemma generalizes the definition to arbitrary adjacency matrices.

Lemma 1. *If the conditions in Definition 2 hold, then they also hold with* K *replaced by an arbitrary adjacency matrix.*

Proof. Assume the conditions in Definition 2 hold. The absence of a possible neighbor is equivalent to the neighbor exerting no effort. Since the conditions hold for all $x \in \mathbb{R}_+^n$, they in particular hold when some of the elements of x are zero. Hence C1, C2 and C4 hold for graphs in general. It remains to be shown that also C3 holds in general. We will need the following lemma, which lets us change the set $\mathbb{R}^n \setminus \{0\}$ in the definition of strongly positive definiteness to $\mathbb{R}_+^n \setminus \{0\}$ when we place some restrictions on $A(x)$.

Lemma 2. *Let* $A(x)$ *have positive elements on its diagonal and non-positive elements off its diagonal.* $A(x)$ *is strongly positive definite if there exists an* $\epsilon > 0$ *such that* $z'A(x)z \geq \epsilon \|z\|^2$ *for all* $z \in \mathbb{R}_+^n \setminus \{0\}$ *and all* $x \in D$*.*

Proof. Let $y = (y_1,\dots,y_n)'$ be an arbitrary element in $\mathbb{R}^n \setminus \{0\}$ and set $s = (s_1,\dots,s_n)'$ with $s_i = \text{sign}(y_i)$ and $z = (z_1,\dots,z_n)'$ with $z_i = |y_i|$. Let x be an arbitrary element in D

[10]We focus on differentiable utility functions allowing us to state supermodularity in partial derivatives.

and, for notational convenience, define the elements of $A(x)$: $A(x) = (a_{ij})$. We have $a_{ii} > 0$ and $a_{ij} \leq 0$ for $i \neq j$.

$$\begin{aligned}
y'A(x)y - \epsilon\|y\|^2 &= \sum_i a_{ii}y_i^2 + \sum_{i\neq j} a_{ij}y_iy_j - \epsilon\|y\|^2 \\
&= \sum_i a_{ii}z_i^2 + \sum_{i\neq j} a_{ij}s_is_jz_iz_j - \epsilon\|z\|^2 \\
&= \sum_i a_{ii}z_i^2 + \sum_{i\neq j, s_i=s_j} a_{ij}z_iz_j - \sum_{i\neq j, s_i=-s_j} a_{ij}z_iz_j - \epsilon\|z\|^2 \\
&\geq \sum_i a_{ii}z_i^2 + \sum_{i\neq j, s_i=s_j} a_{ij}z_iz_j + \sum_{i\neq j, s_i=-s_j} a_{ij}z_iz_j - \epsilon\|z\|^2 \\
&= z'A(x)z - \epsilon\|z\|^2.
\end{aligned}$$

Thus $w'A(x)w \geq \epsilon\|w\|^2$ for all $w \in \mathbb{R}^n_+ \setminus \{0\}$ implies that $w'A(x)w \geq \epsilon\|w\|^2$ for all $w \in \mathbb{R}^n \setminus \{0\}$, and hence that A is strongly positive definite.

□

We continue the proof of C3 holding for general graphs. Fix x and, to ease the notation, let $a_{ij} = -\frac{\partial}{\partial x_j}h_i(x;K)$ and $b_{ij} = -\frac{\partial}{\partial x_j}h_i(x;G)$. Note that (a_{ij}) fulfills the criteria of Lemma 2. We want to show that (b_{ij}) also does that. By C4 of Definition 2, $b_{ij} \geq a_{ij}$, once again due to the fact that absence of a possible neighbor is equivalent to that neighbor exerting no effort. Going from K to G thus implies less complementarity and more concavity. It follows that $\sum_{i,j} b_{ij}z_iz_j > \sum_{i,j} a_{ij}z_iz_j$ for all $z \in \mathbb{R}^n_+$, and by Lemma 2, we conclude that C3 holds for all graphs.

□

With this result in hand, we now turn to the main result of this section.

Theorem 3. *There exists a unique internal pure strategy Nash equilibrium for the non-cooperative game on any network G featuring the utility functions defined in Definition 2.*

Proof. Condition 3 of Definition 2 implies, by Theorem 3.1(iii) in [21], that the function

$$h = -\left(h_1\left(x_1, \ldots, x_n\right), \ldots, h_n\left(x_1, \ldots, x_n\right)\right),$$

is strongly monotone (i.e. there exists a constant $\epsilon > 0$ such that $(x - y) \cdot (h(x) - h(y)) \geq \epsilon\|x - y\|^2$ for all x and y). The existence of a unique pure Nash equilibrium follows directly from Theorem 5.1 in [22]. The equilibrium is internal since, by Condition 1 of Definition 2, it is never optimal to exert no effort.

□

3 Nested split graphs

We now turn to a class of networks called *Nested Split Graphs* (NSG). These graphs are sometimes referred to as threshold graphs or hierarchy graphs, because of an apparent hierarchy of nodes according to degree being their key property. Before laying out the formal definition, we provide an illustrative example of these networks. Take a given network of 7 players and arrange the adjacency matrix according to degree in descending order, such that the nodes with the highest degree are placed in the first rows and so on. On a NSG, we obtain a characteristic step-wise pattern in the matrix (ignoring the diagonal which consists of zeros by convention). The top player(s) is connected to all other agents that have at least one link, the second most connected player(s) is connected to the top one(s) and everybody else except those with the second lowest degree of those that possess at least one link and so on. This feature is illustrated in the following example matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

To put this in a formal context, we first present notation for degree partitions. For any graph, we can define a *degree partition* $\{Z_1, \dots, Z_D\}$ of the nodes N, where all nodes in block Z_i have the same degree d_i and $d_1 > \cdots > d_D$. Distinguishing between networks with and without isolated nodes, we define the *positive degree partition* by $\{Z_1, \dots, Z_{D^+}\}$ with $D^+ = D$ if $d_D \geq 1$ and $D^+ = D - 1$ if $d_D = 0$. Hence, we always have $d_1 > \cdots > d_{D^+} \geq 1$, and the positive degree partition thus lacks the final block of the ordinary degree partition if and only if its members have zero degrees. Note that the positive degree sequence is empty if the graph itself is empty and the degree sequence only consists of a single block Z_1 with $d_1 = 0$.

Definition 3. *Both the empty and the complete graphs are* Nested Split Graphs. *A non-empty and non-complete graph is called a* Nested Split Graph *if and only if all nodes in block Z_i are connected to all other nodes in blocks $Z_1, \dots, Z_{D^+ + 1 - i}$ for $i = 1, \dots, D^+$.*

A direct consequence of the definition is that, on a NSG, nodes i and j are either friends, friends of friends (i.e. separated by one node) or there exists no path between them. Moreover, a NSG consists of at most one component and the degrees of linked nodes are always non-positively correlated. These graphs have been studied extensively in the graph theory literature. For further properties, see [27].

Considering the static game described above played on a NSG, we establish that equilibrium actions are non-decreasing in degree. Formally:

Theorem 4. *On a nested split graph, the ranking of efforts in the (symmetric) Nash equilibrium equals the ranking of degrees.*

Proof. Let d_i be the degree of node i. If $d_i = d_j$, both nodes i and j have the same neighborhood and thus the same strategy. Assume that $d_i > d_j$. We want to show that $x_i > x_j$ in equilibrium. This will be done by showing that for any common effort level $x_i = x_j = x$, i's marginal utility of exerting effort is higher than j's. There are two cases: either $g_{ij} = 0$ or $g_{ij} = 1$. In the first case, the neighborhood of j is a proper subset of i's neighborhood. Due to the complementarity, i's marginal utility is higher than j's and hence $x_i > x_j$ in equilibrium. In the second case, $N_j \setminus \{i\} \subsetneq N_i \setminus \{j\}$. Let $N_j^- = N_j \setminus \{i\}$, and $N_i^+ = N_i \setminus (N_j \cup \{j\})$ so that j's neighborhood is partitioned into $N_j = N_j^- \cup \{i\}$ and i's neighborhood is partitioned into $N_i = N_j^- \cup N_i^+ \cup \{j\}$. If both i and j exert effort x, then due to the complementarity, i's marginal effort is larger than j's, and in equilibrium we must have $x_i > x_j$. □

4 Network formation games

We now extend the model above into a dynamic setting and study the properties of the network formation process. Time is discrete and the formation process continues for an infinite number of periods. The set of players, N, is constant over time.

Each period consists of three stages. In the first stage, all agents myopically play the game outlined above. In the second stage, an exogenous and potentially stochastic process chooses one player i from the network and decides whether this player gets the opportunity to add a link or sever one of the existing ones.[11] The chosen player then chooses which link to create or destroy, although link formation requires mutual consent.[12] After the link has been formed, i.e. after the second stage, the agent that initiated a change in the network may reoptimize her action given the new structure. The payoffs of each period are then realized.

The utility function obeys Definition 2 and, in addition, we impose the following supplemental condition.

Definition 4. *The utility function exhibits* strict positive externalities *if, for all i and j and all $x_j > x'_j$, $u_i(x_1, \ldots, x_j, \ldots, x_n; G) > u_i(x_1, \ldots, x'_j, \ldots, x_n; G)$ whenever i and j are friends.*

Analyzing the dynamic features, we find the following.

[11] In case the process prescribes adding a link to an agent who is connected to everyone, or removing a link from an agent who has no connections, payoffs are realized and the period ends.

[12] If remaining idle when getting the opportunity to form a link were part of the agents' choice sets, our results would not be affected, but endowing players with the possibility of not removing a link when removal is imposed, would. The theorems would still hold under these assumptions, but the resulting network formation process would be uninteresting since no agent would ever want to remove a link if given the opportunity to remain idle. Due to strict positive externalities, the network would converge to the complete network if this choice were endogenous, as a player's utility is increasing in degree.

Theorem 5. *In the dynamic game described above, let $g^0, g^1, \ldots, g^t$ be a sequence of graphs such that g^{t+1} is obtained from g^t by letting a node either add or delete an edge, and let g^0 be a nested split graph. All $g^0, g^1, \ldots, g^{t+1}$ are nested split graphs if the utility functions obeys Definition 2 and exhibits strict positive externalities.*

Proof. Define M_i as the set of nodes with the highest degree that are not neighbors of i, i.e. $M_i = \arg\max\{d_k : k \notin N_i\}$, and, similarily, let m_i be the neighbors of i of lowest degree, i.e. $m_i = \arg\min\{d_k : k \in N_i\}$. The network remains in the class of NSGs if i targets $j \in M_i$ when i gets to add a link and $l \in m_i$ when i must sever a link. Recall the degree partitioning that preceded Definition 3, in which D^+ denotes the index of the lowest ranked nodes with positive degree. Combining our knowledge of the degree partitioning and the sets M_i and m_i we have that on a nested split graph, if $j \in Z_k$ then $M_j = Z_{D^+ + 2-k}$ and $m_j = Z_{D^+ + 1-k}$. By symmetry we have

$$j \in M_i \iff i \in M_j,$$
$$j \in m_i \iff i \in m_j.$$

Lemma 6. *Let $g^0, g^1, \ldots$ be a sequence of graphs such that g^{t+1} is obtained from g^t by adding or deleting an edge, and g^0 is a nested split graph. All $g^0, g^1, \ldots$ are nested split graphs if and only if*

(i) An edge ij is only added whenever $j \in M_i$ (and thus $i \in M_j$).

(ii) An edge ij is only removed whenever $j \in m_i$ (and thus $i \in m_j$).

Proof. By induction, it suffices to verify that g^1 is a NSG if and only if (i) and (ii) hold. Let $Z_1^k, Z_2^k, \ldots$ be the degree partition of g^k and D^{k+} the number of distinct positive degrees in g^k for $k = 0, 1$. Consider two nodes: i in some block Z_k^0 and j in some block Z_l^0 of the degree partition of g^0.

Addition of a link ij increases both i's and j's degrees by one. If i's degree in g^1 equals the degree of some nodes in g^0, then i is "promoted" to their block in the degree partition: $i \in Z_{k-1}^1$, otherwise i will constitute a block of its own in the degree partition of g^1: $\{i\} = Z_k^1$. The same argument applies to node j, either $j \in Z_{l-1}^1$ or $\{j\} = Z_l^1$. The rest of the blocks in the degree partition are unchanged in the transition from g^0 and g^1. $D^{1+} = D^{0+} + 0, 1$ or 2 corresponding to whether both, one or none of i and j are in blocks of their own in the new block partition.

If (i) holds, then $l = D^{0+} + 2 - k$ and it is straightforward to check that g^1 is a nested split graph by Definition 3 in the three cases $D^{1+} = 0, 1$ or 2. If, on the other hand, $l \neq D^{0+} + 2 - k$, i.e. (i) does not hold, the blocks in g^1 are no longer connected in accordance with the definition. For example, if $i \in Z_{k-1}^1$, the nodes in that block are no longer connected to the same nodes: j is connected to i but not to the other nodes of Z_{k-1}^1.

The argument for the necessity and sufficiency of (ii) in the case of link removal is similar to the proof for link addition. □

Now, in order to show that the network structure of the game outlined above remains in the class of NSGs if played on a NSG, we need to show that i optimally targets $j \in M_i$ when given the opportunity to add a link (and that i's target is $j \in m_i$ if removing a link). If nodes connected to all other nodes get to add a link or nodes not connected to any nodes get to remove a link, nothing happens and the network continues to be a NSG. Due to strict positive externalities, a new link always generates higher utility than abstaining from forming a link. Node i will thus choose to send out a link and any receiver will accept. By Theorem 4, the nodes in M_i are those outside i's neighborhood who exert the most effort. Again by strict positive externalities, choosing one of them as a neighbor increases utility the most both before and after i changes (increases) effort. Correspondingly, the nodes in m_i are those agents who exert the least effort, and losing them as neighbors decreases utility the least due to strict positive externalities. □

The degree structure of NSGs connotes a well defined hierarchy, which means that conditions on the formation process under which the degree structure - and hence the graph structure - is kept intact, are easy to formulate. In principle, a link emanating from node i may only be added as long as it is formed with a node j with the highest degree that i is not yet linked to (denote this set M_i). Moreover, by the symmetry of NSGs, if j belongs to M_i, i is an element of the set of nodes with highest degree that j does not have an edge with, i.e. M_j. From Theorem 4, we know that the myopic game played on a NSG implies that the set of nodes with the highest degree that i is not linked to is equivalent to the set of nodes with highest effort that i does not possess an edge with. As utilities exhibit strict positive externalities, and because utilities are homogenous in all players' efforts (a permutation of the arguments in the utility function does not affect utility), it is always optimal to target the nodes with highest effort that are not a part of one's own neighborhood.

Analogous arguments hold for link destruction. However, unlike link removal, the addition of links requires mutual consent. An interesting feature of this network formation process is that a link offer from player i to player $j \in M_i$ is not only acceptable for agent j, but also optimal in the sense that j would have targeted i, or any other node of the same degree as agent i, if j had given the opportunity to add a link.

Due to strict positive externalities agent i targets $j \in M_i$ when endowed with the possibility of adding a link. Now, suppose i has the opportunity of reoptimizing her effort after creating the link. Strategic complementarities implies that a link with $j \in M_i$ increases marginal utility of own effort the most, and in combination with strict positive externalities also i's utility the most, both before and after i changes her effort.

Note that a simple hierarchy model in which agents strive to be connected to nodes with the highest degree would fit in the framework above. Such a model would approximate the preferential-attachment model that emerges in different forms in the social networks literature.[13] The main difference is that links cannot be generated randomly in our model.

[13] An important contribution to the network-formation literature is [3], who provide a framework in which links are created randomly with the probability of creating a link proportional to the degree. Preferential

Such link creation would eventually destroy the particular structure of NSGs.

Let us now turn to a convergence result of the dynamic game. Allowing for any initial network structure, the graph converges to the class of NSGs with probability one. Moreover, as shown above, this is an absorbing state.

Theorem 7. *Consider the dynamic model above. Let $g^0, g^1, \ldots, g^t$ be a sequence of graphs such that g^{t+1} is obtained by letting a node either create or remove one link. The network structure converges to the absorbing class of NSGs with probability one if the utility function obeys Definition 2, strategic complementarities, strict positive externalities and there exists some $\epsilon > 0$, such that the probability of creating a link is $p > \epsilon$.*

Proof. Consider an arbitrary period in the process. If the current graph is a NSG, Theorem 5 implies that the process will continue to consist of NSGs and we are done. Assume otherwise, in particular that the current graph is non-empty and non-complete, i.e. the number of links in the graph is larger than zero and less than $\binom{n}{2}$. Let ϵ be the least probability of creating (deleting) a link as the case may be. The probability that all next $\binom{n}{2}$ link actions are additions (removals) is at least $\epsilon^{\binom{n}{2}}$, and since this number is positive, though typically very small, it will eventually happen that a stretch of $\binom{n}{2}$ link actions are all additions (removals), which would create a complete (empty) graph. From that moment on, all graphs will be NSGs.

□

5 Concluding Remarks

There is ample empirical evidence suggesting that social networks affect individual behavior with a number of studies finding support for complementarities in actions. By focusing on such interaction effects, we establish sufficient conditions regarding the utility function for existence and uniqueness of Nash equilibrium independent of the network structure. In essence, we let utility depend on neighbors' actions and put restrictions on the Hessian matrix, which ensure that own concavity dominates interaction effects. This approach not only keeps the formulation general, but at the same time resolves the inconvenience that many games on networks exhibit: multiple equilibria. We thus understand the mechanisms at play in static games on a deeper level and give a framework that can be applied to a number of specific research questions.

By applying the game to a particular class of networks — Nested Split Graphs — we are able to provide even more precise predictions regarding outcomes of the static game. Games of the types outlined above played on NSGs have clear predictions regarding monotonicity

attachment models have the feature that as nodes obtain more links, their probability of obtaining a new link increases, thereby generating a positive feedback loop. The virtue of these networks is that they exhibit scale-free degree distributions - a feature common in empirically observed networks. See [17] and [14] for a review of preferential-attachment models.

of equilibrium actions. We are also able to provide stability and convergence results of a dynamic network process that features strategic interactions. With a general utility function, we obtain absolute convergence to the absorbing state of NSGs. More importantly, these networks have features in common with empirically observed networks. The regularities that social networks display are, e.g. short average distance (where distance means the shortest path between two nodes in a network), large clustering coefficient, the power law[14] characterizing the degree distribution and nestedness.[15] Among these characteristics, by definition, NSGs feature short average distance and nestedness. Properties of degree distributions and clustering coefficients are context specific.

However, the tractability of Nested Split Graphs comes at a cost. The degree distributions can be extreme in the sense that there must be at least one vertex which is connected to every node with positive degree. The maximum distance on a NSG between any two nodes cannot exceed two. Despite these complications in describing the real world, these networks appear important from an economic perspective. Imposing general and empirically motivated conditions on the utility function, we provide a microfounded framework in which we obtain clear predictions regarding static games on NSGs as well as dynamic results. In order to fully understand social networks, it is important to combine network formation models with models of strategic interactions and this is an attempt in doing so while retaining mathematical tractability.

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[14] A continuous random variable X satisfies the power law if its density is of the form $f(x) = \kappa x^{-\gamma}$ for positive scalars κ and γ.

[15] Empirical properties of networks - the ones above included - are recapitulated in [18, 23].

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