On Higher Inductive Types in Cubical Type Theory

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LICS, July 11, 2018

Univalent Type Theory (UTT)

Aims at providing a foundations for mathematics built on type theory

Founded by Vladimir Voevodsky around 2006–2009 and is being actively developed in various proof assistants (Agda, Coq, Lean, ...)

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Theorem (Cohen, Coquand, Huber, M. 2015)

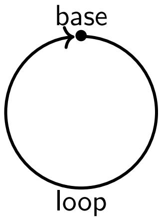
UTT has a (constructive) model in Kan cubical sets with diagonals and connections

Based on this we developed a **cubical type theory** in which we can prove and compute with the **univalence theorem**

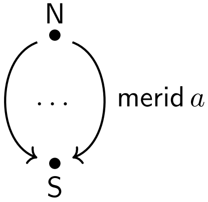
Homotopy Type Theory = UTT + Higher Inductive Types

Higher inductive types (~ 2011) allow us to directly represent topological spaces in type theory, using univalence we can also reason about these internally

$\mathbb{S}^1$:



$\mathsf{Susp}(A)$:



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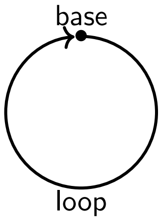
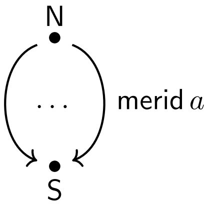


Many impressive developments with HITs:

- Blakers-Massey
- Homotopy groups of (higher) spheres
- Serre spectral sequence
- ...

Susp(A):



At least 4 papers here at LICS 2018

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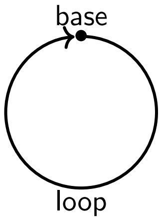

Problem: why is this consistent?

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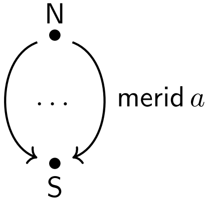

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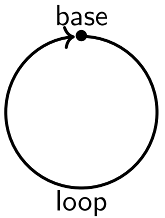

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Recent partial solution (very general, but no universes):

Semantics of higher inductive types,

P. Lumsdaine, M. Shulman, Preprint 2017

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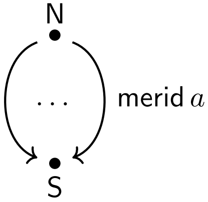

Our paper: constructive semantics of HITs

We describe a constructive semantics with good[1] properties of:

- The circle and spheres ($\mathbb{S}^n$),
- suspensions ($\mathsf{Susp}(A)$),
- two versions of the torus ($\mathbb{T}$, $\mathbb{T}_{\mathsf{F}}$),
- propositional truncation ($\|A\|$), and
- pushouts

These illustrate many of the difficulties that one encounters when trying to give a general semantics of HITs (we sketch a schema)

Corollary: Homotopy Type Theory with these HITs is consistent

[1] To be clarified later...

Constructive semantics of HITs

Expressed in the internal language of a presheaf topos $\widehat{\mathcal{C}}$: "extensional" Martin-Löf type theory (equality reflection, UIP...) extended with:

- Formal interval type $\mathbb{I}$
- Type of cofibrant propositions $\mathbb{F} \hookrightarrow \Omega$

Standard example: $\mathcal{C}$ = category of CCHM cubes

$$
\begin{array}{lllllll}
\mathbb{I}: & 0 & 1 & i & r \wedge s & r \vee s & \neg r \\
\mathbb{F}: & 0_{\mathbb{F}} & 1_{\mathbb{F}} & (i = 0) & (i = 1) & \varphi \wedge \psi & \varphi \vee \psi
\end{array}
$$

Axioms for Modelling Cubical Type Theory in a Topos,
I. Orton and A. Pitts, CSL 2016

Constructive semantics of HITs

A dependent type A over Γ is given by $A : \Gamma \to \mathcal{U}_n$, together with α : $\mathsf{Fib}(\Gamma, A)$, where:

$$
\begin{aligned}
\mathsf{Fib}(\Gamma, A) &= \text{Set of } \textbf{fibration structures} \text{ on A} \\
&= (\gamma : \mathbb{I} \to \Gamma)\,(\varphi : \mathbb{F})\,(u : [\varphi] \to \Pi(i : \mathbb{I})A\gamma(i)) \\
&\quad\; (u_0 : A\gamma(0)[\varphi \mapsto u \; \mathsf{tt} \; 0])(i : \mathbb{I}) \to \\
&\quad A\gamma(i)[\varphi \vee (i = 0) \mapsto u \vee u_0]
\end{aligned}
$$

One gets a (constructive) model of Univalent Type Theory, with universes

Internal Universes in Models of Homotopy Type Theory,
D. Licata, I. Orton, A. Pitts, B. Spitters, FSCD 2018

Semantics of HITs

For every HIT that we consider we need to:

1. Prove the dependent elimination principle
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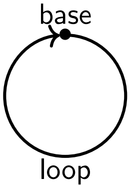

Circle example: We define a notion of S^1-algebra and construct the initial such algebra (externally):

$$(\mathbb{S}^1, \mathsf{base}, \mathsf{loop})$$

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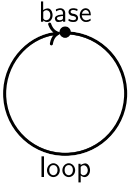

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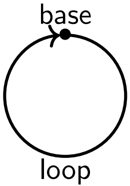

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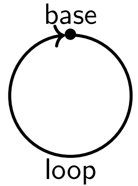

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Fibrancy in general

For an arbitrary HIT A we add homogeneous composition operations:

$$\mathsf{hcomp} : \Pi(\rho : \Gamma) \to \mathsf{Fib}(1, A\rho)$$

This can be seen as “fiberwise fibrant replacement”, but having this structure does **not** generally imply that we have $\mathsf{Fib}(\Gamma, A)$!

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Theorem (CHM)

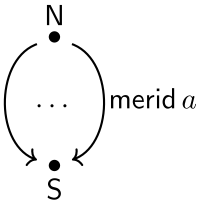
$\mathsf{Fib}(\Gamma, A)$ *is inhabited iff we have* $\mathsf{hcomp} : (\rho : \Gamma) \to \mathsf{Fib}(1, A\rho)$ *and*

$$\begin{aligned}\mathrm{Trans}(\Gamma, A) = {} & (\varphi : \mathbb{F})\,(\gamma : \{p : \mathbb{I} \to \Gamma \mid \varphi \Rightarrow \forall (i : \mathbb{I}).p\,i = p\,0\}) \\ & (u_0 : A\,\gamma(0)) \to A\,\gamma(1)[\varphi \mapsto u_0]\end{aligned}$$

Semantics of suspension: $\mathsf{Susp}(A)$

For $\mathsf{Susp}(A)$ we assume existence of initial algebra

$$(\mathsf{Susp}(A), \mathsf{N}, \mathsf{S}, \mathsf{merid}, \mathsf{hcomp})$$



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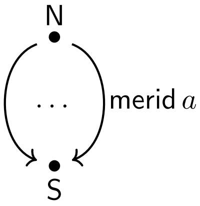

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$$\mathrm{Trans}(\Gamma, \mathsf{Susp}(A))$$

so that we get $\mathsf{Fib}(\Gamma, \mathsf{Susp}(A))$ and $\mathsf{Susp} : \mathcal{U}_n \to \mathcal{U}_n$

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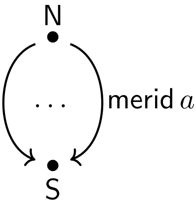

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Key idea: decompose Fib into pointwise fibrancy and transport

Semantics of general HITs

For every HIT A that we consider we:

1. Define A-algebra structure
2. Assume the existence of an initial A-algebra α and
   1. Prove dependent elimination principle
   2. Construct $\mathsf{Fib}(\Gamma, \alpha)$
3. Construct the initial A-algebra structure α (externally)

The final step does **not** involve quotients

Summary: semantics of HITs

We have a semantics of a large class of HITs with **good** properties:

1. Closure under universe levels:

$$\mathsf{Susp} : \mathcal{U}_n \to \mathcal{U}_n$$

2. Commute strictly with substitution:

$$(\mathsf{Susp}(A))\sigma = \mathsf{Susp}(A\sigma)$$

3. Satisfy judgmental/strict computation rules for **all** constructors

This hence proves consistency of HoTT as used in many proof assistants

HITs in cubicaltt

```
-------------------------------------------------------------------------------
-- B.6 The 3-sphere and the join of two circles

data join (A B : U) = inl (a : A)
                    | inr (b : B)
                    | push (a : A) (b : B) <i> [ (i = 0) -> inl a
                                               , (i = 1) -> inr b ]

pushP (A B : U) (a : A) (b : B) : Path (join A B) (inl a) (inr b) =
  <i> push {join A B} a b @ i

joinpt (A : ptType) (B : U) : ptType = (join A.1 B,inl (pt A))

-- B.6.1 Join and associativity

r2lInr (A B C : U) : join B C -> join (join A B) C = split
  inl b -> inl (inr b)
  inr c -> inr c
  push b c @ i -> pushP (join A B) C (inr b) c @ i

r2lPushInl (A B C : U) (a : A) (b : B) :
  Path (join (join A B) C) (inl (inl a)) (inl (inr b)) =
  <i> inl (pushP A B a b @ i)

r2lSquare (A B C : U) (a : A) (b : B) (c : C) :
        PathP (<i> Path (join (join A B) C) (inl (pushP A B a b @ i)) (inr c))
            (pushP (join A B) C (inl a) c) (pushP (join A B) C (inr b) c)
-:---  brunerie.ctt    7% (224,0)    Git:master  (ctt)
```

```
    Checking: x2
    Checking: x3
    Checking: res
  Checking: prealpha
  Checking: alpha
  Checking: f0
  Checking: f1
  Checking: f2
  Checking: f3
  Checking: f31
  Checking: f32
  Checking: f33
  Checking: f4
  Checking: test0To1
  Checking: test0To2
  Checking: test0To31
  Checking: test0To32
  Checking: test0To33
  Checking: test0To3
  Checking: test0To4
  File loaded.
  >
U:**-  *cubical*      Bot (494,2)
```

https://github.com/mortberg/cubicaltt/tree/hcomptrans

Future work

- Formulate a general schema
- Canonicity and normalization for cubicaltt with HITs[2]
- Semantics in simplicial sets?
- Relationship to homotopy theory of spaces

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