Higher Inductive Types in Cubical Type Theories

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EUTypes, October 8, 2018

Univalent Type Theory (UTT)

Aims at providing a foundation for mathematics built on type theory

Founded by Vladimir Voevodsky around 2006–2009 and actively developed in various proof assistants (Agda, Coq, Lean, ...) extended with the **univalence axiom** (which implies both functional and propositional extensionality)

Justified by semantics in spaces (Kan simplicial sets), inherently classical

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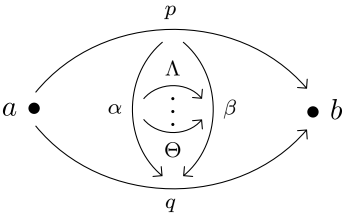
Theorem (Cohen, Coquand, Huber, M. 2015)

Univalent Type Theory has a constructive model in Kan cubical sets

Based on this we developed a **cubical type theory** in which we can prove and compute with the **univalence theorem**

Homotopy (Type Theory) vs. (Homotopy Type) Theory

Type theory	Homotopy theory
A Type	A Space
a, b : A p, q : $a = b$ α, β : $p = q$ Λ, Θ : $\alpha = \beta$ $\vdots$	

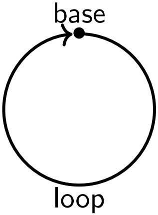


Higher Inductive Types

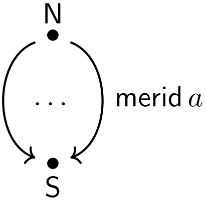
HITs (~ 2011) let us directly represent topological spaces in type theory and do *synthetic* homotopy theory

They also allow us to define quotient types, truncations... that should be useful for computer science applications

$\mathbb{S}^1$:



Susp(A):



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Many impressive developments with HITs using "book HoTT":

- Blakers-Massey
- Homotopy groups of (higher) spheres
- Serre spectral sequence
- ...

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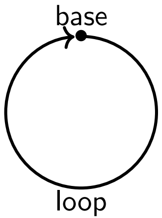

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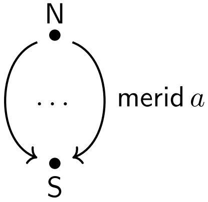

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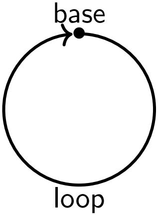
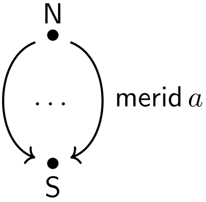


Want: type theories and proof assistants with native support for HITs, while ensuring consistency!

Susp(A):



Cubical Agda demo

https://github.com/Saizan/cubical-demo/tree/hits-transp

Cubical Type Theory

What makes a type theory “cubical”?

Cubical Type Theory

What makes a type theory "cubical"?

Add a formal interval $\mathbb{I}$:

$$r, s \quad ::= \quad 0 \mid 1 \mid i$$

Extend the contexts to include interval variables:

$$\Gamma \quad ::= \quad \bullet \mid \Gamma, x : A \mid \Gamma, i : \mathbb{I}$$

Interval variables

$i : \mathbb{I} \vdash A$ corresponds to a line:

$$A(0/i) \xrightarrow{\quad A \quad} A(1/i)$$

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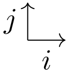
$i : \mathbb{I}, j : \mathbb{I} \vdash A$ corresponds to a square:

$$\begin{tikzcd}
A(0/i)(1/j) \arrow[r, "A(1/j)"] & A(1/i)(1/j) \\
A(0/i)(0/j) \arrow[u, "A(0/i)"] \arrow[r, "A(0/j)"'] \arrow[ur, phantom, "A"] & A(1/i)(0/j) \arrow[u, "A(1/i)"']
\end{tikzcd}$$



and so on...

Cubical Type Theory: structural rules

Proof theory

Semantics

$$\frac{\Gamma \vdash \mathcal{J}}{\Gamma, i : \mathbb{I} \vdash \mathcal{J}} \text{ WEAKENING}$$

$$\Gamma, i : \mathbb{I} \xrightarrow{\texttt{deg}} \Gamma$$

$$\frac{\Gamma, i : \mathbb{I}, j : \mathbb{I} \vdash \mathcal{J}}{\Gamma, j : \mathbb{I}, i : \mathbb{I} \vdash \mathcal{J}} \text{ EXCHANGE}$$

$$\Gamma, j : \mathbb{I}, i : \mathbb{I} \xrightarrow{\texttt{symm}} \Gamma, i : \mathbb{I}, j : \mathbb{I}$$

$$\frac{\Gamma, i : \mathbb{I}, j : \mathbb{I} \vdash \mathcal{J}}{\Gamma, i : \mathbb{I} \vdash \mathcal{J}(j/i)} \text{ CONTRACTION}$$

$$\Gamma, i : \mathbb{I} \xrightarrow{\texttt{diag}} \Gamma, i : \mathbb{I}, j : \mathbb{I}$$

Cubical Type Theory: additional structure on $\mathbb{I}$

We can also consider additional structure on $\mathbb{I}$:

$$r, s \quad ::= \quad 0 \mid 1 \mid i \mid r \wedge s \mid r \vee s \mid \neg r$$

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$$r, s \quad ::= \quad 0 \mid 1 \mid i \mid r \wedge s \mid r \vee s \mid \neg r$$

Given $i : \mathbb{I} \vdash A$

$$\begin{tikzcd} A(0/i) \arrow[r, "A(i/i)"] & A(1/i) \\ A(0/i) \arrow[u, "A(0/i)"] \arrow[r, "A(0/j)"'] \arrow[ur, phantom, "A(i \wedge j/i)"] & A(0/i) \arrow[u, "A(j/i)"'] \end{tikzcd} \qquad A(1/i) \xrightarrow{A(\neg i/i)} A(0/i)$$

Axioms: distributive lattice, de Morgan algebra, Boolean algebra...

"Varieties of Cubical Sets" - Buchholtz, Morehouse (2017)

Cubical Type Theory: Path types

Fix $\mathbb{I}$ and define

$$\mathsf{Path}(A) := (i : \mathbb{I}) \to A$$

We can also consider paths with fixed end-points (and more generally cubes with fixed boundaries, "extension types"):

$$\mathsf{Path}^i\ A\ a\ b\ := (i : \mathbb{I}) \to A[(i = 0) \mapsto a, (i = 1) \mapsto b]$$

Path types are great!

Given $f : A \to B$ and $p : \mathsf{Path}\ A\ a\ b$ we can define:

$$\mathtt{ap}\ f\ p := \lambda(i : \mathbb{I}).\ f\ (p\ i) : \mathsf{Path}\ B\ (f\ a)\ (f\ b)$$

satisfying definitionally:

$$\begin{array}{lllll} \mathtt{ap} & \mathrm{id} & p & = & p \\ \mathtt{ap} & (f \circ g) & p & = & \mathtt{ap}\ f\ (\mathtt{ap}\ g\ p) \end{array}$$

This way we get new ways for reasoning about equality: inline ap, funext, symmetry... with new definitional equalities

Cubical Type Theory: fibrant types

We also need to equip all types with **transport operations**:[1]

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash a : A(0/i)}{\Gamma \vdash \mathsf{transport}^i\ A\ a : A(1/i)}$$

There are **many** different possibilities for how to do this! Which of these give models of Univalent Type Theory with HITs?

[1] For technical reasons we need to consider more general "composition" operations...

Cubical Type Theory: fibrant types

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There are **many** different possibilities for how to do this! Which of these give models of Univalent Type Theory with HITs?

1. BCH: substructural $\mathbb{I}$ (no contraction), transport for open boxes
2. CCHM: all structural rules and de Morgan (or distributive lattice) structure on $\mathbb{I}$, transport for generalized open boxes
3. Cartesian: all structural rules, but no additional structure, on $\mathbb{I}$ and generalized transport for generalized open boxes

[1] For technical reasons we need to consider more general "composition" operations...

HITs in Cubical Type Theory: the circle

One of the things that make the cubical setting so natural for HITs is that we can directly use interval variables:[2]

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathsf{base} : \mathbb{S}^1} \qquad \frac{\Gamma \vdash r : \mathbb{I}}{\Gamma \vdash \mathsf{loop}\, r : \mathbb{S}^1}$$

$$\frac{}{\Gamma \vdash \mathsf{loop}\, 0 = \mathsf{base} : \mathbb{S}^1} \qquad \frac{}{\Gamma \vdash \mathsf{loop}\, 1 = \mathsf{base} : \mathbb{S}^1}$$

Note: loop introduces an element of $\mathbb{S}^1$, not its Id-type!

[2] We also need "homogeneous compositions", more about this later...

HITs in Cubical Type Theory: the circle

Elimination:

$$\frac{\begin{array}{c}\Gamma, x : \mathbb{S}^1 \vdash C \\ \Gamma \vdash b : C(\mathsf{base}) \qquad \Gamma \vdash l : \mathsf{Path}^i\, C(\mathsf{loop}\, i)\, b\, b \qquad \Gamma \vdash u : \mathbb{S}^1\end{array}}{\Gamma \vdash \mathbb{S}^1\text{-}\mathsf{elim}_{x.C}\ b\, l\, u : C(u)}$$

The judgmental computation rules for $\mathbb{S}^1\text{-}\mathsf{elim}_{x.C}\ b\, l\, u$ by cases on u

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The judgmental computation rules for $\mathbb{S}^1\text{-}\mathsf{elim}_{x.C}\ b\, l\, u$ by cases on u

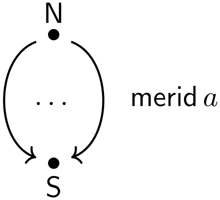
$$\Gamma \vdash \mathbb{S}^1\text{-}\mathsf{elim}_{x.C}\ b\, l\, \mathsf{base} = b : C(\mathsf{base})$$
$$\Gamma \vdash \mathbb{S}^1\text{-}\mathsf{elim}_{x.C}\ b\, l\, (\mathsf{loop}\, r) = l\, r : C(\mathsf{loop}\, r)$$

The last equation is simpler than in book HoTT because of the builtin "path-over" types (i.e. no need for $\mathsf{apd}_{\mathbb{S}^1\text{-}\mathsf{elim}}\,(\mathsf{loop}) = l$)

HITs in Cubical Type Theory: suspensions

$$\frac{\Gamma \vdash A}{\Gamma \vdash \mathsf{N} : \mathsf{Susp}(A)} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \mathsf{S} : \mathsf{Susp}(A)} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash \mathsf{merid}\, a\, r : \mathsf{Susp}(A)}$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{merid}\, a\, 0 = \mathsf{N} : \mathsf{Susp}(A)} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{merid}\, a\, 1 = \mathsf{S} : \mathsf{Susp}(A)}$$



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Elimination:

$$\frac{\Gamma, x : \mathsf{Susp}(A) \vdash C \qquad \Gamma \vdash n : C(\mathsf{N}) \qquad \Gamma \vdash s : C(\mathsf{S}) \qquad \Gamma \vdash m : (a : A) \to \mathsf{Path}^i\, C(\mathsf{merid}\, a\, i)\, \mathsf{N}\, \mathsf{S} \qquad \Gamma \vdash u : \mathsf{Susp}(A)}{\Gamma \vdash \mathsf{Susp\text{-}elim}^A_{x.C}\ n\, s\, m\, u : C(u)}$$

The judgmental computation rules are defined by cases on u:

$$\mathsf{Susp\text{-}elim}^A_{x.C}\ n\, s\, m\, \mathsf{N} = n$$
$$\mathsf{Susp\text{-}elim}^A_{x.C}\ n\, s\, m\, \mathsf{S} = s$$
$$\mathsf{Susp\text{-}elim}^A_{x.C}\ n\, s\, m\, (\mathsf{merid}\, a\, r) = m\, a\, r$$

HITs in cubicaltt: suspensions, Susp(A)

Transport: Defined by cases:

$$\mathsf{trans}^i\,(\mathsf{Susp}(A))\,\mathsf{N} = \mathsf{N}$$
$$\mathsf{trans}^i\,(\mathsf{Susp}(A))\,\mathsf{S} = \mathsf{S}$$
$$\mathsf{trans}^i\,(\mathsf{Susp}(A))\,(\mathsf{merid}\,a\,r) = \mathsf{merid}\,(\mathsf{trans}^i\,A\,a)\,r$$

Note: directly structurally recursive![3]

[3] Transport for more complicated HITs (e.g. pushouts) is a bit more involved...

Categorical Semantics

On Higher Inductive Types in Cubical Type Theory

Coquand, Huber, M. - LICS 2018

Constructive cubical semantics of HITs

We describe a constructive semantics with good properties of:

- The circle and spheres ($\mathbb{S}^n$),
- suspensions ($\mathsf{Susp}(A)$),
- two versions of the torus ($\mathbb{T}$, $\mathbb{T}_{\mathsf{F}}$),
- propositional truncation ($\|A\|$), and
- pushouts

These illustrate many of the difficulties that one encounters when trying to give a general categorical semantics of HITs (we sketch a schema)

Constructive semantics of HITs

Expressed in the internal language of a presheaf topos $\widehat{\mathcal{C}}$: "extensional" Martin-Löf type theory (equality reflection, UIP...) extended with:

- Formal interval type $\mathbb{I}$
- Type of cofibrant propositions $\mathbb{F} \hookrightarrow \Omega$

Standard example: $\mathcal{C}$ = category of CCHM cubes

$$
\begin{array}{lllllll}
\mathbb{I}: & 0 & 1 & i & r \wedge s & r \vee s & \neg r \\
\mathbb{F}: & 0_{\mathbb{F}} & 1_{\mathbb{F}} & (i = 0) & (i = 1) & \varphi \wedge \psi & \varphi \vee \psi
\end{array}
$$

Axioms for Modelling Cubical Type Theory in a Topos
Orton, Pitts - CSL 2016

Constructive semantics of HITs

A dependent type A over Γ is given by $A : \Gamma \to \mathcal{U}_n$, together with α : Fib(Γ, A), where:

$$\begin{aligned}
\mathsf{Fib}(\Gamma, A) &= \text{Set of } \textbf{fibration structures} \text{ on A} \\
&= (\gamma : \mathbb{I} \to \Gamma)\,(\varphi : \mathbb{F})\,(u : [\varphi] \to \Pi(i : \mathbb{I}) A\gamma(i)) \\
&\quad\ (u_0 : A\gamma(0)[\varphi \mapsto u\ \mathsf{tt}\ 0])(i : \mathbb{I}) \to \\
&\quad\ A\gamma(i)[\varphi \vee (i = 0) \mapsto u \vee u_0]
\end{aligned}$$

One gets a (constructive) model of Univalent Type Theory, with universes

Internal Universes in Models of Homotopy Type Theory
Licata, Orton, Pitts, Spitters - FSCD 2018

Semantics of HITs

For every HIT that we consider we need to:

1. Prove the dependent elimination principle
2. Prove that it is fibrant

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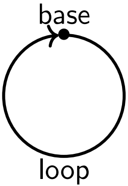

Circle example: We define a notion of S^1-algebra and construct the initial such algebra (externally):

$$(\mathbb{S}^1, \mathsf{base}, \mathsf{loop})$$

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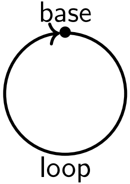

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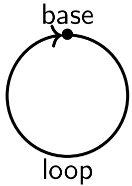

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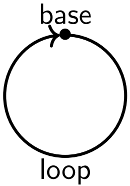

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Fibrancy in general

For an arbitrary HIT A we add homogeneous composition operations:[4]

$$\mathsf{hcomp} : \Pi(\rho : \Gamma) \to \mathsf{Fib}(1, A\rho)$$

This can be seen as "fiberwise fibrant replacement", but having this structure does **not** generally imply that we have $\mathsf{Fib}(\Gamma, A)$!

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Theorem (Coquand, Huber, M.)

$\mathsf{Fib}(\Gamma, A)$ *is inhabited iff we have* $\mathsf{hcomp} : (\rho : \Gamma) \to \mathsf{Fib}(1, A\rho)$ *and*

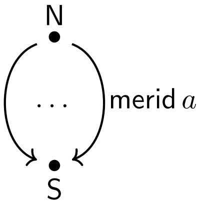
$$\begin{aligned}\mathrm{Trans}(\Gamma, A) = {} & (\varphi : \mathbb{F})\,(\gamma : \{p : \mathbb{I} \to \Gamma \mid \varphi \Rightarrow \forall(i : \mathbb{I}).p\,i = p\,0\}) \\ & (u_0 : A\,\gamma(0)) \to A\,\gamma(1)[\varphi \mapsto u_0]\end{aligned}$$

[4] We need to do this in the type theory as well...

Semantics of suspension: $\mathsf{Susp}(A)$

For $\mathsf{Susp}(A)$ we assume existence of initial algebra

$$(\mathsf{Susp}(A), \mathsf{N}, \mathsf{S}, \mathsf{merid}, \mathsf{hcomp})$$



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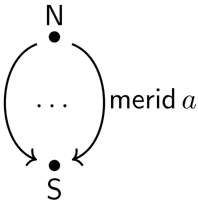

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We then assume that we have $\mathsf{Fib}(\Gamma, A)$ and construct

$$\mathrm{Trans}(\Gamma, \mathsf{Susp}(A))$$

so that we get $\mathsf{Fib}(\Gamma, \mathsf{Susp}(A))$ and $\mathsf{Susp} : \mathcal{U}_n \to \mathcal{U}_n$

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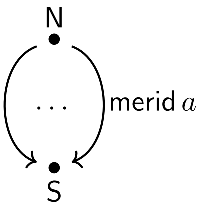

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Key idea: decompose Fib into pointwise fibrancy and transport

Semantics of general HITs

For every HIT A that we consider we:

1. Define A-algebra structure
2. Assume the existence of an initial A-algebra α and
   1. Prove dependent elimination principle
   2. Construct $\mathsf{Fib}(\Gamma, \alpha)$
3. Construct the initial A-algebra structure α (externally)

Summary: semantics of HITs

We have a semantics of a large class of HITs with **good** properties:

1. Closure under universe levels:
$$\mathsf{Susp} : \mathcal{U}_n \to \mathcal{U}_n$$
2. Commute strictly with substitution:
$$(\mathsf{Susp}(A))\sigma = \mathsf{Susp}(A\sigma)$$
3. Satisfy judgmental/strict computation rules for **all** constructors

This hence proves consistency of HoTT as used in many proof assistants

Semantics in spaces?

For synthetic homotopy theory it is very important that we also have a semantics in topological spaces (i.e. Kan simplicial sets)

Recent partial solution (very general, but e.g. Susp not an operation on the same universe):

Semantics of higher inductive types
Lumsdaine, Shulman - Preprint 2017

[5] In various comments on the HoTT Google group, short notes and preprints

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Our arguments in cubical sets generalize to Kan simplicial sets:

Coquand-Sattler-Swan:[5] any provable statement about homotopy groups of spheres in CCHM cubical type theory corresponds to one in Kan simplicial sets

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Meta-theory of Cubical Type Theories with HITs

The categorical semantics gives consistency, but other meta-theoretic properties have also been established:

1. Huber (Ph.D. Thesis, 2016): Canonicity for CCHM Cubical Type Theory with spheres and propositional truncation
2. Cavallo, Harper (Preprint, 2018): Canonicity[6] for Cartesian Cubical Type Theory with general schema for HITs
3. Coquand (Note, last week): Homotopy canonicity for a variation of CCHM Cubical Type Theory

The final result also gives a solution to Voevodsky's computability conjecture for a variation of Cubical Type Theory!

[6] Based on NuPRL style computational type theoretic semantics

Proof assistants based on Cubical Type Theory with HITs

Experimental real proof assistants:

1. Cubical Agda: `https://github.com/Saizan/cubical-demo`
2. redtt: `https://github.com/RedPRL/redtt`
3. RedPRL: `http://www.redprl.org/`

Experimental proof checkers:

1. cubicaltt (hcomptrans/pi4s3 branches): `https://github.com/mortberg/cubicaltt`
2. yacctt: `https://github.com/mortberg/yacctt`

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Thank you for your attention!