E_∞-algebras and Mandell's theorem

Alexander Berglund

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The following are notes for a PhD course given in Stockholm during the spring of 2016. The goal is to understand the statement and proof of the following theorem.

Theorem 0.1 (Mandell [10, 11]). *Two nilpotent p-complete spaces X and Y of finite p-type are weakly homotopy equivalent if and only if their singular cochain complexes $C^*(X;\mathbb{F}_p)$ and $C^*(Y;\mathbb{F}_p)$ are quasi-isomorphic as E_∞-algebras.*

Contents

1 Preliminaries

In this section we review some basics on simplicial sets and chain complexes, mainly to establish notation and terminology. Standard references are [7] or [12].

1.1 Simplicial sets

1.1.1 Simplicial objects

Let Δ denote the category with objects

$$[n] = \{0, 1, \ldots, n\}, \quad n \geq 0,$$

and morphisms $\varphi\colon [m] \to [n]$ all non-decreasing functions. A simplicial object in a category $\mathscr{C}$ is a functor

$$X\colon \Delta^{op} \to \mathscr{C}.$$

We will use the notation $X_n = X([n])$ and $\varphi^* = X(\varphi)\colon X_n \to X_m$ for morphisms $\varphi\colon [m] \to [n]$ in Δ. A morphism of simplicial sets $f\colon X \to Y$, or a *simplicial map*, is a natural transformation, i.e., a sequence of morphisms $f_n\colon X_n \to Y_n$ in $\mathscr{C}$ such that $\varphi^* \circ f_n = f_m \circ \varphi^*$ for every morphism $\varphi\colon [m] \to [n]$. The simplicial objects in $\mathscr{C}$ together with the simplicial maps form a category, which we will denote by $s\mathscr{C}$.

1.1.2 Simplicial sets

If X is a simplicial set, i.e., a simplicial object in the category of sets **Set**, then elements of X_n are called *n-simplices of X*. If $\varphi\colon [m] \to [n]$ is a morphism in Δ we will write

$$\varphi^*(\sigma) = \sigma(\varphi(0) \cdots \varphi(m))$$

for $\sigma \in X_n$. Thus, for $0 \leq i \leq n$, the i^{th} *face* of σ is the $(n-1)$-simplex

$$d_i(\sigma) = \sigma(0 \cdots \hat{i} \cdots n),$$

where $\hat{i}$ means "omit i", and the i^{th} *degeneracy* of σ is the $(n+1)$-simplex

$$s_i(\sigma) = \sigma(0 \cdots i\ i \cdots n).$$

An n-simplex $\sigma \in X_n$ is called *degenerate* if $\sigma = s_j(\tau)$ for some $\tau \in X_{n-1}$ and some $0 \leq j \leq n-1$. A simplex is called *non-degenerate* if it is not degenerate.

We will let $\Delta[n]$ denote the simplicial set $\mathrm{Hom}_\Delta(-,[n])\colon \Delta^{op} \to \mathbf{Set}$.

1.1.3 The singular set

Let Δ^n denote the standard topological n-simplex. Thus,

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1,\ 0 \leq t_i \leq 1 \right\}.$$

The *singular set* of a topological space T is the simplicial set $S(T)$ whose n-simplices are all continuous maps $\sigma\colon \Delta^n \to T$. The map $\varphi^*\colon S(T)_n \to S(T)_m$ associated to a morphism $\varphi\colon [m] \to [n]$ in Δ is given by

$$\varphi^*(\sigma)(t_0, \ldots, t_m) = \sigma(s_0, \ldots, s_n), \quad \text{where} \quad s_i = \sum_{j \in \varphi^{-1}(i)} t_j,$$

for $\sigma \in S(T)_n$. The singular set defines a functor

$$S\colon \mathbf{Top} \to s\mathbf{Set}$$

from the category of topological spaces to the category of simplicial sets; if $f\colon T \to T'$ is a continuous map, then $S(f)\colon S(T) \to S(T')$ is the simplicial map that sends an n-simplex $\sigma\colon \Delta^n \to T$ to $f \circ \sigma\colon \Delta^n \to T'$.

1.2 Chain complexes

1.2.1 Chain complexes

For a commutative ring $\Bbbk$ we let $\mathbf{Ch}(\Bbbk)$ denote the category of $\mathbb{Z}$-graded chain complexes over $\Bbbk$. Thus, an object C of $\mathbf{Ch}(\Bbbk)$ is a sequence of $\Bbbk$-modules and homomorphisms,

$$\cdots \to C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \to \cdots,$$

such that $d^2 = 0$. The homomorphism d is called the *differential.* A morphism, or a *chain map*, $f\colon C \to D$ is a sequence of homomorphisms $f\colon C_n \to D_n$ such that $fd = df$. When we say that x is an element of C or write $x \in C$, we mean that $x \in C_n$ for some n. This n is called the degree of x and we write $|x| = n$ to indicate that x has degree n.

The category $\mathbf{Ch}(\Bbbk)$ admits a symmetric monoidal structure. The tensor product $C \otimes D$ is defined by

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q,$$

with differential $d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$. The unit is the module $\Bbbk$, viewed as a chain complex concentrated in degree 0. The symmetry isomorphism $T\colon C \otimes D \to D \otimes C$ is given by $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$. Moreover, the symmetric monoidal category $\mathbf{Ch}(\Bbbk)$ is *closed*, meaning that for a fixed chain complex C, the functor

$$- \otimes C\colon \mathbf{Ch}(\Bbbk) \to \mathbf{Ch}(\Bbbk)$$

admits a right adjoint $\mathrm{Hom}_\Bbbk(C, -)\colon \mathbf{Ch}(\Bbbk) \to \mathbf{Ch}(\Bbbk)$. Explicitly,

$$\mathrm{Hom}_\Bbbk(C, D)_n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_\Bbbk(C_i, D_{i+n}),$$

with differential

$$\partial(f) = d_D \circ f - (-1)^{|f|} f \circ d_C.$$

Elements of the chain complex $\mathrm{Hom}_\Bbbk(C, D)$ will be referred to as *maps*. Note that a map need not commute with differentials. On the other hand, a *chain map*, or a *morphism of chain complexes*, $f\colon C \to D$ is the same thing as a map of degree 0 such that $\partial(f) = 0$.

We let $\mathbf{Ch}_{\geq 0}(\Bbbk)$ denote the subcategory of non-negatively graded chain complexes. Its objects are chain complexes C such that $C_n = 0$ for $n < 0$. The inclusion functor $i\colon \mathbf{Ch}_{\geq 0}(\Bbbk) \to \mathbf{Ch}(\Bbbk)$ admits a right adjoint

$$\tau_{\geq 0}\colon \mathbf{Ch}(\Bbbk) \to \mathbf{Ch}_{\geq 0}(\Bbbk)$$

given by

$$(\tau_{\geq 0} C)_n = \begin{cases} C_n, & n > 0, \\ \ker(C_0 \to C_{-1}), & n = 0, \\ 0, & n < 0. \end{cases}$$

1.2.2 The normalized chain complex

To a simplicial set X we can associate a chain complex $\Bbbk X$, where $\Bbbk X_n$ is the free $\Bbbk$-module on the set X_n and the differential $d\colon \Bbbk X_n \to \Bbbk X_{n-1}$ is defined on basis elements $\sigma \in X_n$ by

$$d(\sigma) = \sum_{i=0}^{n} (-1)^i d_i(\sigma).$$

The submodule $D_n \subseteq \Bbbk X_n$ spanned by degenerate simplices is closed under the differential, more precisely $d(D_n) \subseteq D_{n-1}$, and the *normalized chain complex* of X is the quotient chain complex

$$C_*(X; \Bbbk) = \Bbbk X / D.$$

The $\Bbbk$-module $C_n(X; \Bbbk)$ is free with basis the non-degenerate n-simplices of X (or, more precisely, their images under the projection $\Bbbk X_n \to C_n(X; \Bbbk)$). We shall omit $\Bbbk$ from the notation when it is understood from the context and write $C_* X = C_*(X; \Bbbk)$. The normalized chain complex defines a functor

$$C_*\colon s\mathbf{Set} \to \mathbf{Ch}_{\geq 0}(\Bbbk).$$

We let $f_*\colon C_* X \to C_* Y$ denote the chain map associated to a simplicial map $f\colon X \to Y$.

The normalized singular chain complex of a topological space T is the chain complex $C_*(T) = C_*(S(T))$ associated to the singular set $S(T)$.

1.2.3 Generalized Eilenberg-Mac Lane spaces

The normalized chain complex functor admits a right adjoint

$$K\colon \mathbf{Ch}_{\geq 0}(\Bbbk) \to s\mathbf{Set}.$$

In fact, since the zero chain complex 0 is an initial object and $K(0) = *$, the simplicial set $K(C)$ is canonically pointed, and there is a natural isomorphism of abelian groups

$$\pi_n(K(C), *) \cong H_n(C)$$

for every $n \geq 0$. In particular, if $C = \Bbbk x$ with x an element of degree $n \geq 0$, then $K(C)$ is an Eilenberg-Mac Lane space $K(\Bbbk, n)$. In general, $K(C)$ is a generalized Eilenberg-Mac Lane space

$$\prod_{n \geq 0} K(H_n(C), n).$$

1.2.4 The Eilenberg-Zilber theorem

The product of two simplicial sets $X \times Y$ has n-simplices $(X \times Y)_n = X_n \times Y_n$. In particular, $C_n(X \times Y)$ is spanned by pairs of n-simplices $(\sigma, \tau) \in X_n \times Y_n$. On the other hand, the degree n component of the tensor product $C_*X \otimes C_*Y$ is spanned by elements of the form $\alpha \otimes \beta$, where $\alpha \in X_p$, $\beta \in Y_q$ and $p + q = n$. It should be clear that $C_*(X \times Y)$ and $C_*X \otimes C_*Y$ are not isomorphic except in trivial cases. But we have the next best thing.

Theorem 1.1 (Eilenberg-Zilber)**.** *There is a natural strong deformation retract of chain complexes*

$$\begin{tikzcd} C_*(X \times Y) \arrow[loop left, "\Phi"] \arrow[r, shift left, "AW"] & C_*X \otimes C_*Y \arrow[l, shift left, "EZ"] \end{tikzcd}$$

In other words, AW and EZ are natural chain maps and Φ is a natural chain homotopy such that

$$AW \circ EZ = 1, \quad EZ \circ AW = 1 + d\Phi + \Phi d.$$

Explicitly, for $(\sigma, \tau) \in X_n \times Y_n$,

$$AW(\sigma, \tau) = \sum_{p=0}^{n} \sigma(0 \cdots p) \otimes \tau(p \cdots n).$$

For $\alpha \in X_p$ and $\beta \in Y_q$,

$$EZ(\alpha \otimes \beta) = \sum_{\omega} (-1)^{\omega} (s_{\omega_{p+q-1}} \cdots s_{\omega_p} \sigma, s_{\omega_{p-1}} \cdots s_{\omega_0} \tau),$$

where the sum is over all permutations ω of the set $\{0, 1, \ldots, p + q - 1\}$ such that $\omega_0 < \cdots < \omega_{p-1}$ and $\omega_p < \cdots < \omega_{p+q-1}$.

The map AW is called the *Alexander-Whitney map* and EZ the *Eilenberg-Zilber map*. No one seems to have bothered to write down an explicit formula for, or give a name to, a chain homotopy Φ between $EZ \circ AW$ and the identity map as in the theorem, but a standard acyclic models argument proves its existence. We refer to [9, VIII.8] for proofs.

Both AW and EZ are associative in the sense that the diagram

$$\begin{tikzcd}
C_*(X \times Y \times Z) \arrow[r, "AW_{X\times Y,Z}"] \arrow[d, "AW_{X,Y\times Z}"'] & C_*(X \times Y) \otimes C_*Z \arrow[d, "AW\otimes 1"] \\
C_*X \otimes C_*(Y \times Z) \arrow[r, "1\otimes AW"] & C_*X \otimes C_*Y \otimes C_*Z,
\end{tikzcd}$$

and the similar diagram for EZ, commute. Formally, this means that C_* together with the map $AW\colon C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$ and the natural isomorphism $C_*(*) \cong \Bbbk$ is a comonoidal functor from the monoidal category $\big(s\mathbf{Set}, \times, *\big)$ to the monoidal category $\big(\mathbf{Ch}_{\geq 0}(\Bbbk), \otimes, \Bbbk\big)$[1]. Similarly, C_* together with EZ and $C_*(*) \cong \Bbbk$ is a monoidal functor.

The map EZ is also symmetric in the sense that the diagram

$$\begin{tikzcd}
C_*X \otimes C_*Y \arrow[r, "EZ"] \arrow[d, "T"'] & C_*(X \times Y) \arrow[d, "t_*"] \\
C_*Y \otimes C_*X \arrow[r, "EZ"] & C_*(Y \times X)
\end{tikzcd}$$

commutes, where $t\colon X \times Y \to Y \times X$ is the symmetry $t(\sigma, \tau) = (\tau, \sigma)$ and

$$T(\alpha \otimes \beta) = (-1)^{|\alpha||\beta|}\beta \otimes \alpha.$$

In effect, (C_*, EZ) is a *symmetric* monoidal functor between the symmetric monoidal categories $\big(s\mathbf{Set}, \times, *, t\big)$ and $\big(\mathbf{Ch}_{\geq 0}(\Bbbk), \otimes, \Bbbk, T\big)$

However, AW is not symmetric. In fact, this what is responsible for the existence of Steenrod operations and it is the reason we need to work with E_∞-algebras instead of commutative differential graded algebras.

2 The E_∞-algebra structure on C^*X

In this section we will review the E_∞-algebra structure on the normalized cochain complex C^*X of a simplicial set X, following McClure-Smith [16] and Berger-Fresse [1].

2.1 Normalized cochains

Let $\Bbbk$ be a commutative ring. The normalized cochain complex of a simplicial set X is defined by

$$C^*X = \mathrm{Hom}_\Bbbk(C_*X, \Bbbk).$$

[1]Strictly speaking, we should include the data of natural isomorphisms $(X \times Y) \times Z \cong X \times (Y \times Z)$, $* \times X \cong X$ and $X \times * \cong X$ as part of the monoidal structure, and similarly for $\otimes$, but as usual we tacitly treat these isomorphisms as identities, as justified by Mac Lane's coherence theorem.

We may identify C^pX with the set of functions

$$x\colon X_p \to \Bbbk$$

that vanish on degenerate simplices. Such a function is called a (normalized) p-cochain on X. The $\Bbbk$-module structure is defined pointwise: for p-cochains x, y and scalars $a, b \in \Bbbk$ the p-cochain $ax + by$ is given by

$$\big(ax + by\big)(\sigma) = ax(\sigma) + by(\sigma),$$

for $\sigma \in X_p$.

The differential $\delta\colon C^{p-1}X \to C^pX$ is given by

$$\delta(x)(\sigma) = \sum_{i=0}^{p}(-1)^{i+p}x(\sigma(0\cdots\hat{\imath}\cdots p)).$$

for $x \in C^{p-1}X$ and $\sigma \in X_p$. A cochain x is called a *cocycle* if $\delta(x) = 0$.

2.1.1 Cup product

The cup product $\smile\colon C^pX \otimes C^qX \to C^{p+q}X$ is given by

$$(x \smile y)(\sigma) = (-1)^{pq}x(\sigma(0,\dots,p)) \cdot y(\sigma(p,\dots,p+q)).$$

The cup product is associative, $(x \smile y) \smile z = x \smile (y \smile z)$ and the map $\eta\colon \Bbbk \cong C^0(*) \to C^0(X)$ induced by the unique map $X \to *$ furnishes C^*X with a unit element 1 such that $x \smile 1 = 1 \smile x = x$. Moreover, the Leibniz rule holds:

$$\delta(x \smile y) = \delta(x) \smile y + (-1)^p x \smile \delta(y).$$

Thus, the cochains C^*X is a differential graded algebra. However, the cup product is not graded commutative; in general

$$x \smile y \neq (-1)^{pq} y \smile x.$$

2.1.2 Steenrod's cup-i-products

The cup product is not commutative, but it is commutative up to homotopy. Steenrod [20] defined an operation $\smile_1\colon C^pX \otimes C^qX \to C^{p+q-1}X$ by the formula

$$(x \smile_1 y)(\sigma) = \sum_{j}^{p-1}(-1)^{(p-j)(q+1)}x(\sigma(0,\dots,j,j+q,\dots,p+q-1))\cdot y(\sigma(j,\dots,j+q)),$$

and he showed that the coboundary formula is satisfied:

$$\delta(x \smile_1 y) = (-1)^{p+q-1}x \smile y + (-1)^{pq+p+q}y \smile x + \delta x \smile_1 y + (-1)^p x \smile_1 \delta y.$$

In particular, if x and y are cocycles, then $x \smile y - (-1)^{pq} y \smile x$ is a coboundary. This shows that the cup product induces a commutative operation in cohomology.

More generally, he defined the cup-i-product $\smile_i\colon C^pX \otimes C^qX \to C^{p+q-i}X$ by the formula

$$(x \smile_i y)(\sigma) = \sum_{0 \le r_1 < r_2 < \cdots < r_{i+1} \le r} \pm x\big(\sigma(0\_\_r_1 r_2\_\_r_3\cdots)\big)\cdot y\big(\sigma(r_1\_\_r_2 r_3\_\_r_4\cdots)\big)$$

for a p-cochain x, a q-cochain y and an r-simplex σ, where $r = p + q - i$. To make sense of the formula, we use the convention that a p-cochain evaluates to zero on a simplex of dimension $\neq p$. We note that the ordinary cup product $\smile$ may be identified with $\smile_0$. The coboundary formula holds:

$$\delta(x \smile_i y) = (-1)^{p+q-i} x \smile_{i-1} y + (-1)^{pq+p+q} y \smile_{i-1} x + \delta x \smile_i y + (-1)^p x \smile_i \delta y.$$

We remark that the coboundary formula may be expressed succinctly as an equality in the chain complex $\mathrm{Hom}_{\Bbbk}(C^*X^{\otimes 2}, C^*X)$:

$$\partial(\smile_i) = \smile_{i-1} - \tau \smile_{i-1}.$$

Here τ is the generator of the symmetric group Σ_2. Since $\partial(\smile_i) \neq 0$ for $i > 0$, the cup-i-product will in general not induce a binary operation in cohomology. However, if $2 = 0$ in $\Bbbk$, then the coboundary formula shows that $x \smile_i x$ is a cycle whenever x is a cycle.

The *Steenrod operation* $Sq^i \colon H^p(X; \mathbb{F}_2) \to H^{p+i}(X; \mathbb{F}_2)$ is defined by

$$Sq^i([x]) = [x \smile_{p-i} x].$$

2.1.3 McClure-Smith's multivariable operations

The pattern in the formula for the cup-i-product can be generalized to define operations of higher arity. To keep track of the combinatorics involved we need to introduce some notation. For a positive integer n, let $\overline{n}$ denote the set $\{1, 2, \ldots, n\}$.

For integers $i \geq 0$ and $r \geq 2$ and a surjective function $u \colon \overline{i+r} \to \overline{r}$ we will define an operation

$$\langle u \rangle \colon C^*X^{\otimes r} \to C^*X$$

of arity r cohomological degree $-i$. We will say that the surjection u has *degree* i and *arity* r in this situation.

As a notational device, given a sequence $0 = \nu_0 \leq \nu_1 < \ldots < \nu_{i+1} \leq \nu_{i+r} = d$ write

$$A_j = \nu_{j-1} \_\_ \nu_j,$$

for $j = 1, 2, \ldots, i + r$, and write $A_i \coprod A_j$ for the concatenated string.

Given cochains $x_1, \ldots, x_r$ and a d-simplex σ, we define

$$\langle u \rangle (x_1 \otimes \cdots \otimes x_r)(\sigma) = \sum_{0 \leq \nu_1 < \ldots < \nu_{i+r} \leq d} \pm x_1\big(\sigma(\coprod_{u(j)=1} A_j)\big) \cdot \ldots \cdot x_r\big(\sigma(\coprod_{u(j)=r} A_j)\big).$$

If we represent a surjection $u \colon \overline{i+r} \to \overline{r}$ by the sequence of its values $u(0)u(1)\ldots u(i+r)$, then we have

$$\langle 12 \rangle = \smile,$$
$$\langle 121 \rangle = \smile_1,$$
$$\langle 1212 \rangle = \smile_2,$$

and so on.

The action of $\omega \in \Sigma_r$ on $\langle u \rangle \in \mathrm{Hom}_{\Bbbk}(C^*X^{\otimes r}, C^*X)$ admits a simple description:

$$\omega \langle u \rangle = \pm \langle \omega \circ u \rangle.$$

In particular, $\tau\langle 12\rangle = \langle 21\rangle$ and so on. Steenrod's coboundary formula assumes the form

$$\partial\langle 121\rangle = \langle 12\rangle \pm \langle 21\rangle$$

More generally, we have the following description

Proposition 2.1.

$$\partial\langle u_1 \dots u_{i+r}\rangle = \sum_{j=1}^{i+r} \pm\langle u_1 \dots \widehat{u_j} \dots u_{i+r}\rangle.$$

The composition of two operations $\langle u\rangle \circ_i \langle v\rangle$ is also a linear combination of such operations.

Proposition 2.2. *Let u be a surjection of degree i and arity r and let v be a surjection of degree j and arity s. The composite operation*

$$\langle u\rangle \circ_k \langle v\rangle =$$

For every simplicial map $f\colon X \to Y$ and every surjection u of arity r, the diagram

$$\begin{tikzcd} C^*Y^{\otimes r} \arrow[r, "\langle u\rangle"] \arrow[d, "C^*f^{\otimes r}"'] & C^*Y \arrow[d, "C^*f"] \\ C^*X^{\otimes r} \arrow[r, "\langle r\rangle"] & C^*X \end{tikzcd}$$

commutes. In other words, each surjection defines a natural operation on cochains.

2.2 Review of operads

Operads were introduced by May [14].

Given a chain complex C and maps $f \in \mathrm{Hom}_{\Bbbk}(C^{\otimes r}, C)$, $g \in \mathrm{Hom}_{\Bbbk}(C^{\otimes s}, C)$, we may form a new map

$$f \circ_i g \in \mathrm{Hom}_{\Bbbk}(C^{\otimes r+s-1}, C)$$

by plugging g into the ith input of f. Explicitly,

$$(f \circ_i g)(x_1, \dots, x_r) = \pm f(x_1, \dots, g(x_i, \dots, x_{i+s-1}), \dots, x_r).$$

More generally, given r maps $g_1, \dots, g_r$, with $g_j \in \mathrm{Hom}_{\Bbbk}(C^{\otimes i_j}, C)$, we may form the map $\gamma(f; g_1, \dots, g_r) \in \mathrm{Hom}_{\Bbbk}(C^{\otimes i}, C)$, where $i = i_1 + \dots + i_r$, by

$$\gamma(f; g_1, \dots, g_r) = f \circ \big(g_1 \otimes \dots \otimes g_r\big).$$

We furthermore have an action of the symmetric group Σ_r on $\mathrm{Hom}_{\Bbbk}(C^{\otimes r}, C)$: if $\omega \in \Sigma_r$

$$(\omega f)(x_1, \dots, x_r) = \pm f(x_{\omega_1}, \dots, x_{\omega_r}).$$

Operads provide an axiomatization for this kind of structure.

Definition 2.3. An operad in a symmetric monoidal category $(\mathscr{C}, \otimes, \mathbb{I})$ is a collection of objects $\mathcal{O} = \{\mathcal{O}(r)\}_{r\geq 0}$ together with

- composition $\gamma\colon \mathcal{O}(r)\otimes\mathcal{O}(i_1)\otimes\cdots\otimes\mathcal{O}(i_r)\to\mathcal{O}(i_1+\cdots+i_r)$,
- a unit $\eta\colon \mathbb{I}\to\mathcal{O}(1)$,
- an action of the symmetric group Σ_r on $\mathcal{O}(r)$,

subject to associativity, unit and equivariance axioms.

Remark 2.4. Operads can also be axiomatized using the partial composition products

$$\circ_k\colon \mathcal{O}(r)\otimes\mathcal{O}(s)\to\mathcal{O}(r+s-1),\quad 1\le k\le r.$$

Given a composition γ as in Definition 2.3, one may define $\circ_k$ as the composite

$$\begin{aligned}\mathcal{O}(r)\otimes\mathcal{O}(s)&\cong\mathcal{O}(r)\otimes\mathbb{I}^{\otimes k-1}\mathcal{O}(s)\otimes\mathbb{I}^{\otimes r-k}\\&\to\mathcal{O}(r)\otimes\mathcal{O}(1)^{\otimes k-1}\mathcal{O}(s)\otimes\mathcal{O}(1)^{\otimes r-k}\\&\xrightarrow{\gamma}\mathcal{O}(r+s-1),\end{aligned}$$

where the first map is given by applying $\eta\colon\mathbb{I}\to\mathcal{O}(1)$ at the appropriate places.

2.2.1 The surjection operad χ

The surjection operad (or sequence operad) was introduced in [16, 1] is the dg-operad χ where

$$\chi(r)_i=\Bbbk Surj$$

The differential $\delta\colon\chi(r)_i\to\chi(r)_{i-1}$ is given on generators by

$$\delta(u_1,\dots,u_{i+r})=\sum_{j=1}^{i+r}\pm(u_1,\dots,\widehat{u_j},\dots,u_{i+r}).$$

The partial composition $\circ_k\colon\chi(r)_i\otimes\chi(s)_j\to\chi(r+s-1)_{i+j}$ for $1\le k\le r$. [... to be completed]

3 Homotopy theory of algebras over an operad

3.1 Review of homotopical algebra

Homotopical algebra was invented by Quillen [18]. For a very good introduction to the theory, see Dwyer-Spalinski [5].

Recall that a *model structure* on a category $\mathscr{C}$ is the data of three classes of morphisms in $\mathscr{C}$,

- weak equivalences $\xrightarrow{\sim}$,
- fibrations $\twoheadrightarrow$,
- cofibrations $\rightarrowtail$,

such that each class is closed under compositions and contains all identity morphisms, and the axioms below hold.

Some terminology:

A *trivial (co)fibration* is a morphism that is simultaneously a (co)fibration and a weak equivalence.

We say that a morphism $i\colon A \to B$ has the *LLP (left lifting property)* with respect to a morphism $p\colon X \to Y$, and that p has the *RLP (right lifting property)* with respect to i, if for all morphisms $f\colon A \to X$ and $g\colon B \to Y$ such that $pf = gi$, there is a morphism $\lambda\colon B \to X$ such that $g = p\lambda$ and $f = \lambda i$;

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "\lambda"] & Y. \end{tikzcd}$$

Write $i \perp p$ in this situation.

Here are the axioms:

MC1 Finite limits and colimits exist in $\mathscr{C}$.

MC2 If two out of f, g and $g \circ f$ are weak equivalences, then so is the third.

MC3 Each of the three classes is closed under retracts.

MC4 (i) Every trivial cofibration has the LLP with respect to every fibration and (ii) every cofibration has the LLP with respect to every trivial fibration.

MC5 Every morphism can be factored as (i) $\overset{\sim}{\rightarrowtail}\twoheadrightarrow$ and as (ii) $\rightarrowtail\overset{\sim}{\twoheadrightarrow}$.

A *model category* is a category together with a model structure. We recall the following, see [5, Proposition 3.13].

Proposition 3.1. *In a model category, a morphism is a*

- *cofibration if and only if it has the LLP with respect to all trivial fibrations,*
- *trivial cofibration if and only if it has the LLP with respect to all fibrations,*
- *fibration if and only if it has the RLP with respect to all trivial cofibrations,*
- *trivial fibration if and only if it has the RLP with respect to all cofibrations.*

This implies that a model structure is determined by, e.g., the weak equivalences and the fibrations: the cofibrations are then determined as the class of morphisms that have the LLP with respect to all trivial cofibrations.

3.2 Homotopy theory of $\mathcal{O}$-algebras

Throughout this section we fix a dg-operad $\mathcal{O}$, i.e., an operad in the category $\mathbf{Ch}(\Bbbk)$ of unbounded chain complexes of modules over a commutative ring $\Bbbk$. The goal of the section is to prove the following theorem.

Theorem 3.2. *Let $\mathcal{O}$ be a dg-operad. The following are equivalent:*

1. *The category of $\mathcal{O}$-algebras admits a model structure where the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections.*
2. *Every $\mathcal{O}$-algebra A admits a good path object, i.e., a factorization of the diagonal map $\Delta\colon A \to A \times A$ as a quasi-isomorphism followed by a surjection,*

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[dr, "\Delta"'] & A^I \arrow[d, two heads] \\ & A \times A. \end{tikzcd}$$

3. *The morphism $A \to A[x, dx]$ is a quasi-isomorphism for every $\mathcal{O}$-algebra A.*

Here $A[x, dx]$ denotes the free product of A with the free $\mathcal{O}$-algebra on the chain complex $\Bbbk x \oplus \Bbbk dx$.

3.2.1 The free $\mathcal{O}$-algebra functor

By definition, an $\mathcal{O}$-algebra is a chain complex with extra structure. The forgetful functor $(-)^\natural\colon \mathcal{O}\text{-alg} \to \mathbf{Ch}(\Bbbk)$ admits a left adjoint, the free $\mathcal{O}$-algebra functor.

$$\begin{tikzcd} \mathbf{Ch}(\Bbbk) \arrow[r, shift left, "\mathcal{O}[-]"] & \mathcal{O}\text{-alg} \arrow[l, shift left, "(-)^\natural"] \end{tikzcd}$$

Explicitly,

$$\mathcal{O}[V] = \bigoplus_{r \geq 0} \mathcal{O}(r) \otimes_{\Sigma_r} V^{\otimes r}.$$

It is well known that the category $\mathbf{Ch}(\Bbbk)$ admits a model structure where the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections (see, e.g., [8, Theorem 2.3.11]).

Definition 3.3. We declare a morphism of $\mathcal{O}$-algebras to be a

- *weak equivalence* if it is a quasi-isomorphism,
- *fibration* if it is surjective,
- *cofibration* if it has the LLP with respect to all trivial fibrations (= surjective quasi-isomorphisms of $\mathcal{O}$-algebras).

We do not claim that this makes $\mathcal{O}$-alg into a model category in general, but we will nevertheless use the terminology.

Proposition 3.4. *The free $\mathcal{O}$-algebra functor* $\mathbf{Ch}(\Bbbk) \to \mathcal{O}$-alg *preserves cofibrations.*

Proof. This is a formal consequence of the fact that the forgetful functor preserves and reflects weak equivalences and fibrations. For a morphism of $\mathcal{O}$-algebras $f\colon \mathcal{O}[V] \to A$, let $f^\flat\colon V \to A^\natural$ denote the adjoint morphism in $\mathbf{Ch}(\Bbbk)$.

Let $i\colon V \to W$ be a cofibration of chain complexes. To show that $\mathcal{O}[i]$ is a cofibration in $\mathcal{O}$-alg, we need to solve lifting problems in $\mathcal{O}$-alg of the form

$$\begin{tikzcd} \mathcal{O}[V] \arrow[r, "f"] \arrow[d, "\mathcal{O}[i]"'] & A \arrow[d, "p"] \\ \mathcal{O}[W] \arrow[r, "g"'] \arrow[ur, dashed, "\lambda"] & B, \end{tikzcd}$$

where p is a surjective quasi-isomorphism. By adjunction yoga, such a lifting problem is equivalent to the following lifting problem in $\mathbf{Ch}(\Bbbk)$:

$$\begin{tikzcd} V \arrow[r, "f^\flat"] \arrow[d, "i"'] & A^\natural \arrow[d, "p^\natural"] \\ W \arrow[r, "g^\flat"'] \arrow[ur, dashed, "\lambda^\flat"] & B^\natural. \end{tikzcd}$$

By definition, $p^\natural$ is a trivial fibration in $\mathbf{Ch}(\Bbbk)$. We know that the cofibrations in $\mathbf{Ch}(\Bbbk)$ have the left lifting property with respect to all trivial fibrations, so the latter can always be solved. $\square$

3.2.2 Relative cell algebras

Relative cell algebras will play the role of CW-complexes in the category of $\mathcal{O}$-algebras.

Let A be an $\mathcal{O}$-algebra and let $a \in A$ be a cycle. We can freely add a generator x to kill the cycle a and form the $\mathcal{O}$-algebra $A[x, dx = a]$. Formally, this is constructed as a pushout in the category of $\mathcal{O}$-algebras,

$$\begin{tikzcd} \mathcal{O}[e] \arrow[r] \arrow[d] & \big(\mathcal{O}[e,x], dx = e\big) \arrow[d] \\ A \arrow[r] & A[x, dx = a]. \end{tikzcd}$$

Here, $\mathcal{O}[e]$ denotes the free $\mathcal{O}$-algebra on the chain complex with zero differential and a generator e in degree $|a|$ and $\big(\mathcal{O}[e,x], dx = e\big)$ denotes the free $\mathcal{O}$-algebra on the chain complex $\Bbbk x \to \Bbbk e$, $dx = e$. The left vertical morphism in the pushout diagram is the unique morphism of $\mathcal{O}$-algebras that sends e to a, and the top horizontal map is induced by the inclusion of $\Bbbk e$ into $\Bbbk x \to \Bbbk e$.

More generally, we say that an $\mathcal{O}$-algebra B is obtained from A by attaching cells if there is a pushout diagram of $\mathcal{O}$-algebras,

$$\begin{tikzcd} \mathcal{O}[V] \arrow[r] \arrow[d, "a"'] & \mathcal{O}[CV] \arrow[d] \\ A \arrow[r] & B, \end{tikzcd}$$

where V is a free graded $\Bbbk$-module V, viewed as a chain complex with trivial differential, and the top horizontal morphism is induced by the inclusion of V

into its *cone* CV, i.e., the chain complex $CV = V \oplus sV$ with differential $dv = 0$ and $dsv = v$.

If e_i is a basis for V, then specifying a morphism of $\mathcal{O}$-algebras $\mathcal{O}[V] \to A$ is the same as specifying cycles a_i in A, and B can be thought of as being obtained by adding generators $x_i = se_i$ to kill the cycles a_i.

We can iterate this construction ad infinitum. A morphism of $\mathcal{O}$-algebras $f\colon A \to B$ will be called a *relative cell algebra* is there is a sequence over B,

$$A = A_0 \to A_1 \to A_2 \to \cdots ,$$

such that A_{i+1} is obtained from A_i by attaching cells and $B \cong \operatorname{colim} A_i$ as objects under A.

An $\mathcal{O}$-algebra B is called a *cell algebra* if the unit morphism $\Bbbk \to B$ is a relative cell algebra.

Proposition 3.5. *Every relative cell algebra inclusion is a cofibration.*

Proof. It is an exercise in category theory to show that a class of morphisms defined by a LLP is automatically closed under pushouts and sequential colimits, so to show that a relative cell algebra is a cofibration it is enough to verify that $\mathcal{O}[V] \to \mathcal{O}[CV]$ is a cofibration for every inclusion $V \to CV$ of a free graded $\Bbbk$-module into its cone. But $V \to CV$ is clearly a cofibration in $\mathbf{Ch}(\Bbbk)$ so this follows from Proposition 3.4. □

Clearly, cofibrations are closed under retracts, so it follows that every retract of a relative cell algebra inclusion is a cofibration. We will see in Proposition 3.9 below that the converse also holds: every cofibration is a retract of a relative cell algebra inclusion.

3.2.3 Factorizations

In this section we will see that the factorization axiom MC5(ii) always holds in $\mathcal{O}$-alg. However, the axiom MC(i) need not hold without further hypotheses.

Proposition 3.6. *Every morphism of $\mathcal{O}$-algebras $f\colon A \to B$ can be factored as*

$$A \xrightarrow{j} C \xrightarrow{q} B,$$

where

- *the morphism j is a relative cell algebra inclusion and has the LLP with respect to all fibrations,*
- *the morphism q is a fibration.*

Proof. The factorization may be constructed as follows:

$$A \xrightarrow{j} A[e_b, x_b, dx_b = e_b]_{b \in B} \xrightarrow{q} B.$$

The middle term is the $\mathcal{O}$-algebra obtained from A by adding, for every element $b \in B$, a generator e_b of degree $|b| - 1$ with $de_b = 0$ and a generator x_b of degree $|b|$ with $dx_b = e_b$. The morphism j is the canonical inclusion and the morphism q is defined by $q|_A = f$ and $q(x_b) = b$, $q(e_b) = db$. Clearly, q is a surjective morphism of $\mathcal{O}$-algebras and j is a relative cell algebra inclusion. To show that j has the LLP with respect to all fibrations, we argue as in Proposition 3.4: Let V denote the chain complex spanned by x_b, e_b with differential $dx_b = e_b$. The chain map $0 \to V$ is a trivial cofibration in $\mathbf{Ch}(\Bbbk)$, so it has the LLP with respect to all fibrations. It follows that $\Bbbk = \mathcal{O}[0] \to \mathcal{O}[V]$ has the LLP with respect to all fibrations in $\mathcal{O}$-alg, and hence so does j, because there is a pushout

$$\begin{tikzcd} \Bbbk \arrow[r] \arrow[d] & \mathcal{O}[V] \arrow[d] \\ A \arrow[r, "j"] & A[e_b, x_b, dx_b = e_b]_{b \in B}. \end{tikzcd}$$

□

This almost verifies the factorization axiom MC5(i) for $\mathcal{O}$-alg. In a model category, every morphism that has the LLP with respect to all fibrations is necessarily a weak equivalence. This is however not true in $\mathcal{O}$-alg in general.

Proposition 3.7. *Every morphism of $\mathcal{O}$-algebras $f\colon A \to B$ can be factored as*

$$A \xrightarrow{k} C \xrightarrow{p} B,$$

where

- *the morphism k is a relative cell algebra inclusion,*
- *the morphism p is a quasi-isomorphism.*

Proof. We construct a sequence $A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ and morphisms $f_i\colon A_i \to B$ by induction. Set $A_0 = A$ and $f_0 = f$. Assuming $f_{i-1}\colon A_{i-1} \to B$ has been constructed, define A_i by

$$A_i = A_{i-1}[x^i_{a,b}, dx^i_{a,b} = a]_{(a,b)\in S(f_{i-1})},$$

where $S(f_{i-1})$ is the set of pairs (a, b) such that $a \in A_{i-1}$ is a cycle and $b \in B$ is an element such that $db = f_{i-1}(a)$. Define $f_i\colon A_i \to B$ to be the unique extension of f_{i-1} that satisfies $f_i(x^i_{a,b}) = b$. Finally, let $A_\infty = \cup_i A_i$, let $p\colon A_\infty \to B$ be the unique morphism that restricts to f_i on A_i and let $k\colon A \to A_\infty$ be the inclusion. Clearly, $f = pk$ and k a relative cell algebra inclusion. We need to show that p is a quasi-isomorphism. Given a cycle $b \in B$, we have that $(0, b) \in S(f_0)$, so $x^1_{0,b}$ is a cycle in $A_1 \subseteq A_\infty$ with $p(x^1_{0,b}) = b$. This shows that p is surjective in homology. To show it is injective, let $a \in A_\infty$ be a cycle such that $p(a) = db$ for some $b \in B$. Say $a \in A_i$. Then $p(a) = f_i(a)$ and so $(a, b) \in S(f_i)$. But then $dx^{i+1}_{a,b} = a$ in A_{i+1} and hence in A_∞. This shows that p is injective in homology. □

The next corollary together with the fact, verified in Proposition 3.5 above, that relative cell algebra inclusions are cofibrations implies that the factorization axiom MC5(ii) holds in $\mathcal{O}$-alg in general.

Corollary 3.8. *Every morphism of $\mathcal{O}$-algebras $f\colon A \to B$ can be factored as*

$$A \xrightarrow{i} C \xrightarrow{p} B,$$

where

- *the morphism i is a relative cell algebra inclusion,*
- *the morphism p is a surjective quasi-isomorphism.*

Proof. By Proposition 3.6 we may write $f = qj$, where j is, in particular, a relative cell algebra inclusion and q is a surjection. Next, use Proposition 3.7 to factor q as $q = pk$, where k is a relative cell algebra inclusion and p is a quasi-isomorphism. Since q is surjective, p is necessarily surjective as well. The composite $i = kj$ is a relative cell algebra inclusion, so $f = pi$ is a factorization of the required type. □

Proposition 3.9. *A morphism of $\mathcal{O}$-algebras is a cofibration if and only if it is a retract of a relative cell algebra inclusion.*

Proof. One direction follows from Proposition 3.5 together with the observation that a class defined by a LLP is automatically closed under retracts. For the other direction, suppose that $f\colon A \to B$ is a cofibration of $\mathcal{O}$-algebras. By Corollary 3.8, we may factor f as $A \xrightarrow{i} C \xrightarrow{p} B$, where i is a relative cell algebra inclusion and p is a trivial fibration. Then f has the LLP with respect to p, so we may find a lift in the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dashed, "\lambda"] & B. \end{tikzcd}$$

But this means that f is a retract of i. Indeed, the above diagram may be rewritten as

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "f"] \\ B \arrow[r, "\lambda"] & C \arrow[r, "p"] & B. \end{tikzcd}$$

□

3.2.4 Proof of Theorem 3.2

Before we can finish the proof of Theorem 3.2 we need the following lemma.

Lemma 3.10. *Suppose that $i\colon A \to B$ is a morphism of $\mathcal{O}$-algebras that has the left lifting property with respect to all fibrations. If B admits a good path object, then i is a weak equivalence.*

Proof. The morphism $A \to 0$ to the zero $\mathcal{O}$-algebra is evidently a fibration, so we may find a morphism $r\colon B \to A$ such that $ri = 1$ by finding a lift in

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "i"'] & A \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, dashed, "r"] & 0. \end{tikzcd}$$

That B admits a good path object means that there is a factorization of the diagonal morphism $\Delta\colon B \to B \times B$ as a weak equivalence followed by a fibration,

$$B \xrightarrow[\sim]{s_0} B^I \xrightarrow{(d_0,d_1)} B \times B.$$

Thus, $d_0 s_0 = d_1 s_0 = 1$, and this implies that d_0 and d_1 are weak equivalences as well. Now find a lift h in the diagram

$$\begin{tikzcd} A \arrow[r, "s_0 i"] \arrow[d, "i"'] & B^I \arrow[d, "(d_0,d_1)"] \\ B \arrow[r, "(ir,1)"'] \arrow[ur, dashed, "h"] & B \times B. \end{tikzcd}$$

(In effect, this shows that A is a strong deformation retract of B.) Since $d_1 h = 1$ and d_1 is a weak equivalence, it follows that h is a weak equivalence. Since $d_0 h = ir$ and d_0 is a weak equivalence, it then follows that ir is a weak equivalence. But since $ri = 1$, we have that i is a retract of ir:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "i"'] & B \arrow[r, "r"] \arrow[d, "ir"'] & A \arrow[d, "i"'] \\ B \arrow[r, equal] & B \arrow[r, equal] & B. \end{tikzcd}$$

It follows that i is a weak equivalence. $\square$

Proof of Theorem 3.2. (1) $\Rightarrow$ (2): It follows directly from MC5(i) that every object in a model category admits a good path object.

(2) $\Rightarrow$ (3): Just as j in the proof of Proposition 3.6, the morphism $A \to A[x, dx]$ has the left lifting property with respect to all fibrations. Lemma 3.10 then implies that it is a weak equivalence.

(3) $\Rightarrow$ (1): MC1,MC2,MC3 are clear. MC4(ii) holds by definition of cofibrations. MC5(ii) holds by Corollary 3.8. The hypothesis (3) implies that the map j constructed in the proof of Proposition 3.6 is a weak equivalence, showing MC5(i). The only axiom left to verify is MC4(i). Thus, let $f\colon A \to B$ be a trivial cofibration. We will show that f has the LLP with respect to all fibrations by displaying it as a retract of a morphism that has this property. By Proposition 3.6 and the argument for MC5(i) just given, we may factor f as $A \xrightarrow{j} C \xrightarrow{q} B$, where q is a fibration and j has the LLP with respect to all fibrations and is a weak equivalence. Since both f and j are weak equivalences it follows that q is a weak equivalence and hence a trivial fibration. But then f, being in particular a cofibration, has the left lifting property with respect to q, so we may find a lift μ in the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "f"'] & C \arrow[d, "q"] \\ B \arrow[r, equal] \arrow[ur, dashed, "\mu"] & B. \end{tikzcd}$$

This means that f is a retract of j:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "j"'] & A \arrow[d, "f"'] \\ B \arrow[r, "\mu"] & C \arrow[r, "q"] & B \end{tikzcd}$$

Since j has the LLP with respect to all fibrations, it follows that so does f. $\square$

3.3 Examples

3.3.1 Commutative differential graded algebras

The commutative operad $\mathcal{C}om$ is the unique operad with $\mathcal{C}om(r) = \Bbbk$, the trivial Σ_r-module. A $\mathcal{C}om$-algebra is the same thing as a commutative differential graded algebra.

Proposition 3.11. *Let $\Bbbk$ be a commutative ring. The category of commutative differential graded algebras over $\Bbbk$ admits a model structure where the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections if and only if $\mathbb{Q} \subseteq \Bbbk$.*

Proof. Consider the free graded commutative algebra on the contractible chain complex $\Bbbk x \oplus \Bbbk dx$. Here is a picture of the chain complex $\mathcal{C}om[x, dx]$:

$$\begin{tikzcd}
1 & x \arrow[d, "\cdot 1"] & x^2 \arrow[d, "\cdot 2"] & x^3 \arrow[d, "\cdot 3"] & x^4 \arrow[d, "\cdot 4"] & \cdots \\
 & dx & xdx & x^2dx & x^3dx & \cdots
\end{tikzcd}$$

This shows that the unit map $\Bbbk \to \mathcal{C}om[x, dx]$ is a quasi-isomorphism if and only if n is invertible in $\Bbbk$ for every $n > 0$, i.e., $\mathbb{Q} \subseteq \Bbbk$. The claim now follows from Theorem 3.2. $\square$

In particular, the category of commutative differential graded algebras over $\mathbb{F}_p$ does not admit a model structure where the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections.

3.3.2 The Barratt-Eccles operad

The collection of symmetric groups $\Sigma = \{\Sigma_r\}_{r \geq 0}$ forms an operad in $(\mathbf{Set}, \times, *)$, algebras over which are associative monoids.

There is a functor

$$W \colon \mathbf{Set} \to s\mathbf{Set}$$

defined by $W_n X = X^{n+1}$, with face and degeneracy maps given by

$$d_i(x_0, \dots, x_n) = (x_0, \dots, \widehat{x_i}, \dots, x_n),$$

$$s_i(x_0, \dots, x_n) = (x_0, \dots, x_i, x_i, \dots, x_n),$$

for $0 \leq i \leq n$. There is a natural isomorphism

$$W(X \times Y) \cong WX \times WY,$$

making W into a symmetric monoidal functor. Hence, the collection $W\Sigma = \{W\Sigma_r\}_{r \geq 0}$ forms an operad in simplicial sets. Moreover, $W\Sigma$ is an E_∞-operad: The action of Σ_r on $W\Sigma_r$ is clearly free, and $W\Sigma_r$ is contractible because it admits an "extra degeneracy

$$s_{-1}(x_0, \dots, x_n) = (1, x_0, \dots, x_r),$$

cf. [7, Lemma III.5.1]. Here 1 denotes the identity element of the group Σ_r. Next, we can apply the symmetric monoidal functor $C_* \colon s\mathbf{Set} \to \mathbf{Ch}_{\geq 0}(\Bbbk)$ to obtain a dg-operad.

Definition 3.12. The Barratt-Eccles operad is the dg-operad

$$\mathcal{E} = C_* W\Sigma.$$

Since $W\Sigma$ is an E_∞-operad in simplicial sets, it follows that $\mathcal{E}$ is an E_∞-operad in chain complexes. Note that the chain complex $\mathcal{E}(r)$ is exactly the normalized (homogeneous) bar resolution of the trivial module $\Bbbk$ over $\Bbbk\Sigma_r$, cf. [9, IV.5].

One of the main features of the Barratt-Eccles operad $\mathcal{E}$ is that it is a *Hopf operad*, meaning that there is a morphism of operads

$$\Delta\colon \mathcal{E} \to \mathcal{E} \otimes \mathcal{E}.$$

Explicitly, for a generator $(x_0, \dots, x_n) \in \mathcal{E}(r)_n$, we have

$$\Delta(x_0, \dots, x_n) = \sum_{i=0}^{n} (x_0, \dots, x_i) \otimes (x_i, \dots, x_n).$$

In particular, this implies that the tensor product of two $\mathcal{E}$-algebras A, B can be given an $\mathcal{E}$-algebra structure via

$$\begin{aligned}
\mathcal{E}(r) \otimes (A \otimes B)^{\otimes r} &\xrightarrow{\Delta\otimes 1} \mathcal{E}(r) \otimes \mathcal{E}(r) \otimes (A \otimes B)^{\otimes r} \\
&\cong \mathcal{E}(r) \otimes A^{\otimes r} \otimes \mathcal{E}(r) \otimes B^{\otimes r} \\
&\xrightarrow{\rho_A \otimes \rho_B} A \otimes B.
\end{aligned}$$

Here, $\rho_A\colon \mathcal{E}(r) \otimes A^{\otimes r} \to A$ is the structure map $\rho_A(\mu \otimes a) = \mu_A(a)$.

Theorem 3.13 (Berger-Fresse [1])**.** *There is a surjective morphism from the Barratt-Eccles operad to the surjection operad,*

$$TR\colon \mathcal{E} \to \chi.$$

The morphism TR is called the "table reduction morphism". By 'restriction of scalars' it follows that $\mathcal{E}$ also acts naturally on cochains:

$$s\mathbf{Set}^{op} \xrightarrow{C^*} \chi - alg \xrightarrow{TR^*} \mathcal{E} - alg.$$

We will prefer to work with the Barratt-Eccles operad, because the category of $\mathcal{E}$-algebras admits a model structure, as we will see next.

Proposition 3.14. *Every algebra over the Barratt-Eccles operad admits a good path object.*

Proof. Consider the diagram simplicial sets

$$\Delta[0] \sqcup \Delta[0] \xrightarrow{(d^0, d^1)} \Delta[1] \xrightarrow{s^0} \Delta[0].$$

Applying the contravariant functor $C^*\colon s\mathbf{Set} \to \mathcal{E} - alg$ and observing that $C^*(\Delta[0]) \cong \Bbbk$, we get a diagram in $\mathcal{E}$-alg,

$$\Bbbk \xrightarrow{s_0} C^*(\Delta[1]) \xrightarrow{(d_0, d_1)} \Bbbk \times \Bbbk.$$

It is plain that s_0 is a chain homotopy equivalence and that (d_0, d_1) is surjective. Since $\mathcal{E}$ is a Hopf operad, tensoring with a fixed $\mathcal{E}$-algebra gives a functor $-\otimes A\colon \mathcal{E} - alg \to \mathcal{E} - alg$. Tensoring with a fixed chain complex also preserves chain homotopy equivalences, surjections and finite products, so tensoring the the above diagram with an $\mathcal{E}$-algebra A we get a diagram of $\mathcal{E}$-algebras

$$A \xrightarrow{s_0} C^*(\Delta[1]) \otimes A \xrightarrow{(d_0,d_1)} A \times A,$$

where s_0 is a chain homotopy equivalence and (d_0, d_1) is a surjection, showing A admits a good path object. □

Corollary 3.15. *The category of algebras over the Barratt-Eccles operad admits a model structure where the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections.*

Proof. This follows immediately from Proposition 3.14 and Theorem 3.2. □

4 Comparison between spaces and E_∞-algebras

4.1 The spatial realization of an E_∞-algebra

4.1.1 Adjunctions with simplicial sets

Let $\mathscr{C}$ be a category with all colimits.

Proposition 4.1. *Every functor $F\colon s\mathbf{Set} \to \mathscr{C}$ that preserves colimits admits a right adjoint. Explicitly, a right adjoint $G\colon \mathscr{C} \to s\mathbf{Set}$ may be defined by*

$$G(A)_\bullet = \mathrm{Hom}_{\mathscr{C}}(X^\bullet, A),$$

where $X^\bullet$ is the cosimplicial object $F(\Delta[\bullet])$.

Proof. Recall that every simplicial set K admits a canonical decomposition as a colimit of simplices,

$$K \cong \operatorname*{colim}_{\Delta[n]\to K} \Delta[n].$$

The colimit is over the simplex category $(\Delta \downarrow K)$, whose objects are simplicial maps $\Delta[n] \to K$ and whose morphisms are commutative diagrams over K, cf. [7, Lemma I.2.1].

Now let K be a simplicial set and let A be an object of $\mathscr{C}$. By the Yoneda lemma, $\mathrm{Hom}_{s\mathbf{Set}}(\Delta[n], G(A)) \cong G(A)_n = \mathrm{Hom}_{\mathscr{C}}(F(\Delta[n]), A)$. Using this observation together with the fact that F preserves colimits, and remembering that every Hom-functor $\mathrm{Hom}(-, X)$ takes colimits to limits, we get a string of natural isomorphisms,

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}}(F(K), A) &\cong \mathrm{Hom}_{\mathscr{C}}\big(\operatorname*{colim}_{\Delta[n]\to K} F(\Delta[n]), A\big) \\
&\cong \lim_{\Delta[n]\to K} \mathrm{Hom}_{\mathscr{C}}(F(\Delta[n]), A) \\
&\cong \lim_{\Delta[n]\to K} \mathrm{Hom}_{s\mathbf{Set}}(\Delta[n], G(A)) \\
&\cong \mathrm{Hom}_{s\mathbf{Set}}\big(\operatorname*{colim}_{\Delta[n]\to K} \Delta[n], G(A)\big) \\
&\cong \mathrm{Hom}_{s\mathbf{Set}}(K, G(A)),
\end{aligned}$$

showing G is right adjoint to F. $\square$

Remark 4.2. If F preserves colimits, it follows that $F(K) \cong K \otimes_\Delta X^\bullet$, so we see that every adjunction between $s\mathbf{Set}$ and $\mathscr{C}$ is governed by a cosimplicial object $X^\bullet$ in $\mathscr{C}$, in the sense that it is isomorphic to the "Hom-$\otimes$"-adjunction

$$s\mathbf{Set} \underset{\mathrm{Hom}_{\mathscr{C}}(X^\bullet,-)}{\overset{-\otimes_\Delta X^\bullet}{\rightleftarrows}} \mathscr{C}\ .$$

This is analogous to the fact that every adjunction between the category of abelian groups $\mathbf{Ab}$ and the category $\mathrm{Mod}(R)$ of modules over a ring R is governed by an R-module M, in the sense that it is isomorphic to the Hom-$\otimes$-adjunction

$$\mathbf{Ab} \underset{\mathrm{Hom}_R(M,-)}{\overset{-\otimes M}{\rightleftarrows}} \mathrm{Mod}(R).$$

As a digression, we mention some familiar examples.

- The singular set and geometric realization adjunction

$$s\mathbf{Set} \underset{S}{\overset{|-|}{\rightleftarrows}} \mathbf{Top}$$

  is governed by the cosimplicial space $\Delta^\bullet$, where Δ^n is the standard topological n-simplex.

- The normalized chains and generalized Eilenberg-Mac Lane space adjunction

$$s\mathbf{Set} \underset{K}{\overset{C_*}{\rightleftarrows}} \mathbf{Ch}_{\geq 0}(\Bbbk)$$

  is governed by the cosimplicial chain complex $C_*(\Delta[\bullet])$.

- The nerve and the fundamental category adjunction

$$s\mathbf{Set} \underset{N}{\overset{\tau_1}{\rightleftarrows}} \mathscr{C}at$$

  is governed by the cosimplicial category $[\bullet]$, where $[n]$ is the ordered set $\{0, 1, \ldots, n\}$ thought of as a category.

- The nerve and fundamental groupoid adjunction

$$s\mathbf{Set} \underset{N}{\overset{\pi_1}{\rightleftarrows}} \mathscr{G}rp$$

  is governed by the cosimplicial groupoid $[\bullet]'$, where $[n]'$ is the groupoid with objects $0, 1, \ldots, n$ and exactly one morphism between any two objects ($[n]'$ is the groupoid completion of the category $[n]$.)

4.1.2 The definition of spatial realization

Lemma 4.3. *The cochain functor*

$$C^*\colon s\mathbf{Set} \to \mathbf{Ch}(\Bbbk)^{op}$$

admits a right adjoint $\mathbf{Ch}(\Bbbk)^{op} \to s\mathbf{Set}$ *given by*

$$V \mapsto K(\tau_{\geq 0} V^\vee).$$

Proof. The functor $C^*\colon s\mathbf{Set} \to \mathbf{Ch}(\Bbbk)^{op}$ is the composite of the left adjoints in the following diagram of adjunctions (left adjoints on top):

$$s\mathbf{Set} \underset{K}{\overset{C_*}{\rightleftarrows}} \mathbf{Ch}_{\geq 0}(\Bbbk) \underset{\tau_{\geq 0}}{\overset{i}{\rightleftarrows}} \mathbf{Ch}(\Bbbk) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathbf{Ch}(\Bbbk)^{op}.$$

Here, $V^\vee = \mathrm{Hom}_\Bbbk(V, \Bbbk)$ denotes the dual chain complex. It follows that the composite of the right adjoints is right adjoint to C^*. $\square$

Proposition 4.4. *The cochain functor $C^*\colon s\mathbf{Set} \to \mathcal{E} - alg^{op}$ admits a right adjoint*

$$\langle - \rangle\colon \mathcal{E} - alg^{op} \to s\mathbf{Set},$$

which we will call "spatial realization". Explicitly, the spatial realization of an $\mathcal{E}$-algebra A is the simplicial set

$$\langle A \rangle = \mathrm{Hom}_{\mathcal{E}-alg}(A, C^*_\bullet),$$

where $C^*_\bullet$ is the simplicial $\mathcal{E}$-algebra $C^*_n = C^*(\Delta[n])$.

Proof. We will apply Proposition 4.1. To see that C^* takes colimits of simplicial sets to colimits in $\mathcal{E} - alg^{op}$, i.e., to limits in $\mathcal{E} - alg$, one observes that limits in $\mathcal{E}$-alg are created by the forgetful functor $\mathcal{E} - alg \to \mathbf{Ch}(\Bbbk)$, so it is enough to verify that the functor $C^*\colon s\mathbf{Set} \to \mathbf{Ch}(\Bbbk)^{op}$ preserves colimits. But by Lemma 4.3, this functor admits a right adjoint, so in particular it preserves colimits. $\square$

Spatial realizations of free $\mathcal{E}$-algebras are easily computed through playing with adjunctions.

Proposition 4.5. *The spatial realization of a free $\mathcal{E}$-algebra is a generalized Eilenberg-Mac Lane space: for every chain complex V there is a natural isomorphism of simplicial sets*

$$\langle \mathcal{E}[V] \rangle \cong K(\tau_{\geq 0} V^\vee).$$

Proof. This follows by considering the string of adjunctions,

$$s\mathbf{Set} \underset{\langle - \rangle}{\overset{C^*}{\rightleftarrows}} \mathcal{E} - alg^{op} \underset{\mathcal{E}[-]^{op}}{\overset{forget^{op}}{\rightleftarrows}} \mathbf{Ch}(\Bbbk)^{op}$$

The composite of the left adjoints is the functor $C^*\colon s\mathbf{Set} \to \mathbf{Ch}(\Bbbk)^{op}$, whose right adjoint is given by $V \mapsto K(\tau_{\geq 0} V^\vee)$ by Lemma 4.3. $\square$

4.1.3 Quillen adjunctions

[Recollection on Quillen adjunctions ... to be added]

Proposition 4.6. *Cochains and spatial realization form a Quillen adjunction between simplicial sets and $\mathcal{E}$-algebras,*

$$s\mathbf{Set} \underset{\langle - \rangle}{\overset{C^*}{\rightleftarrows}} \mathcal{E} - alg^{op}.$$

Proof. It is classical that the cochain functor C^* takes inclusions of simplicial sets to surjections of cochain complexes and weak equivalences to quasi-isomorphisms. □

Thus, we get a derived adjunction

$$\mathrm{Ho}\, s\mathbf{Set} \underset{R\langle - \rangle}{\overset{C^*}{\rightleftarrows}} \mathrm{Ho}\, \mathcal{E} - alg^{op}$$

Since C^* preserves weak equivalences it plainly descends to a functor on the homotopy categories, so it is equal to its own left derived functor $LC^* = C^*$. The right derived functor of spatial realization is calculated by finding a cofibrant resolution $A \to B$; then $R\langle B \rangle \cong \langle A \rangle$.

4.2 Resolvable spaces

Lemma 4.7. *Every adjunction*

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

restricts to an equivalence of categories

$$\overline{\mathscr{C}} \underset{G}{\overset{F}{\rightleftarrows}} \overline{\mathscr{D}},$$

where $\overline{\mathscr{C}} \subseteq \mathscr{C}$ is the full subcategory of objects X such that the unit,

$$\eta_X \colon X \to GFX,$$

is an isomorphism, and $\overline{\mathscr{D}} \subseteq \mathscr{D}$ is the full subcategory of objects Y such that the counit,

$$\epsilon_Y \colon FGY \to Y,$$

is an isomorphism.

Proof. It suffices to check that $F(\overline{\mathscr{C}}) \subseteq \overline{\mathscr{D}}$ and $G(\overline{\mathscr{D}}) \subseteq \overline{\mathscr{C}}$. We check the first inclusion, the other follows by duality. Thus, let X be an object of $\overline{\mathscr{C}}$. There is a retraction

$$FX \xrightarrow{F(\eta_X)} FGFX \xrightarrow{\epsilon_{FX}} FX, \quad \epsilon_{FX} \circ F(\eta_X) = 1_{FX}.$$

Since η_X is an isomorphism, so is $F(\eta_X)$, and hence so is ϵ_{FX}, which means that FX is an object of $\overline{\mathscr{D}}$. □

Definition 4.8. 1. A simplicial set X is called *resolvable over* $\Bbbk$ if it belongs to $\overline{\mathrm{Ho}\, s\mathbf{Set}}$.

2. An $\mathcal{E}$-algebra A is called *coresolvable* if it belongs to $\overline{\mathrm{Ho}\, \mathcal{E} - alg}$.

Thus, by Lemma 4.7, cochains and derived spatial realization restricts to an equivalence between full subcategories of the homotopy categories

$$\{\text{Resolvable spaces}\} \sim \{\text{Coresolvable } E_\infty\text{-algebras}\}\,.$$

A little more workable characterization is given by the following.

Proposition 4.9. *1. A space X is resolvable if and only if whenever $A \xrightarrow{\sim} C^*X$ is a quasi-isomorphism of E_∞-algebras with A cofibrant, the adjoint map $X \xrightarrow{\sim} \langle A\rangle$ is a weak equivalence of simplicial sets.*

2. A cofibrant E_∞-algebra A is coresolvable if and only if the counit of the adjunction $A \to C^*\langle A\rangle$ is a quasi-isomorphism.

Proposition 4.10. *Suppose X and Y are resolvable spaces. Then $X \sim Y$ in the homotopy category of spaces if and only if $C^*X \sim C^*Y$ in the homotopy category of E_∞-algebras.*

Proof. One implication is clear. For the other implication, assume that C^*X and C^*Y are isomorphic in the homotopy category of E_∞-algebras. Then there exists a cofibrant E_∞-algebra A and quasi-isomorphisms $C^*X \xleftarrow{\sim} A \xrightarrow{\sim} C^*Y$. Since X and Y are resolvable, the adjoints of these morphisms give weak equivalences $X \xrightarrow{\sim} \langle A\rangle \xleftarrow{\sim} Y$, showing X and Y are isomorphic in the homotopy category. $\square$

This is as far as abstract nonsense will take us. The real work will consist in showing that the classes of resolvable spaces and coresolvable E_∞-algebras are non-trivial and interesting.

4.3 Localization and completion of nilpotent spaces

This section will provide an 'executive summary' on localization and completion of nilpotent spaces. For proofs, we refer to [15].

4.3.1 Nilpotent spaces

We remind the reader that a connected space X of finite type is nilpotent if and only if it admits a principally refined Postnikov tower with fibers $K(\mathbb{Z}, n)$ or $K(\mathbb{Z}/p\mathbb{Z}, n)$ for some prime p. In detail, this means that there is a weak homotopy equivalence

$$X \to \lim\big(\cdots \to X_{i+1} \to X_i \to \cdots X_1 \to X_0 = *\big),$$

where $\{X_i\}_i$ is a tower of principal fibrations such that $X_{i+1} \to X_i$ is a pullback

$$\begin{tikzcd} X_{i+1} \arrow[r] \arrow[d] & PK(A_i, n_i+1) \simeq * \arrow[d] \\ X_i \arrow[r] & K(A_i, n_i+1) \end{tikzcd}\,,$$

of the path space fibration along some map $k_i\colon X_i \to K(A_i, n_i+1)$. Furthermore, the tower can be arranged so that

- $1 \leq n_1 \leq n_2 \leq \cdots$
- For every $n \geq 1$, there are at most finitely many i such that $n_i = n$.
- The group A_i is $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$ for some prime p.

4.3.2 Bousfield localization

Let $\mathbb{k}$ be a commutative ring. A map $f\colon X \to Y$ is a $\mathbb{k}$-*equivalence* if the induced map $f_*\colon H_*(X;\mathbb{k}) \to H_*(Y;\mathbb{k})$ is an isomorphism.

Theorem 4.11 (Bousfield [3])**.** *The category of simplicial sets admits a model category where the weak equivalences are the* $\mathbb{k}$*-equivalences and the cofibrations are the inclusions.*

We will refer to this as the $\mathbb{k}$-local model structure on $s\mathbf{Set}$, and we will write $s\mathbf{Set}_{\mathbb{k}}$ when we want to emphasize that we consider the $\mathbb{k}$-local model structure.

The fibrations are what they have to be: they are the maps that have the right lifting property with respect to all trivial cofibrations. A simplicial set X is fibrant in the $\mathbb{k}$-local model structure if and only if it is $\mathbb{k}$-local. By definition, a simplicial set X is $\mathbb{k}$-local if it is a Kan complex and the induced map

$$f^*\colon [Z,X] \to [Y,X] \tag{1}$$

is a bijection for every $\mathbb{k}$-equivalence $f\colon Y \to Z$. Here $[Z,X]$ denotes the morphisms from Z to X in the homotopy category of simplicial sets.

The $\mathbb{k}$-local spaces determine and are determined by the $\mathbb{k}$-equivalences: a map $f\colon Y \to Z$ is a $\mathbb{k}$-equivalence if and only if (1) is a bijection for every $\mathbb{k}$-local space X. In particular, a map between $\mathbb{k}$-local spaces is a homotopy equivalence if and only if it is a $\mathbb{k}$-equivalence.

Fibrant replacement is given by the $\mathbb{k}$-localization functor $X \to L_{\mathbb{k}}X$. The $\mathbb{k}$-localization is characterized up to homotopy by the following two properties:

- The map $X \to L_{\mathbb{k}}X$ is a $\mathbb{k}$-equivalence.
- The space $L_{\mathbb{k}}X$ is $\mathbb{k}$-local.

In the homotopy category, $L_{\mathbb{k}}X$ is the initial $\mathbb{k}$-local object that admits a $\mathbb{k}$-equivalence from X. Clearly, X is $\mathbb{k}$-local if and only if $X \to L_{\mathbb{k}}X$ is a homotopy equivalence.

When X is "$\mathbb{k}$-good", then $L_{\mathbb{k}}X$ may be constructed as Bousfield-Kan's $\mathbb{k}$-completion $\mathbb{k}_\infty X$, cf. [4]

4.3.3 Rationalization, p-localization and p-completion

Classically, the rings $\mathbb{Q}$, $\mathbb{Z}_{(p)}$ and $\mathbb{F}_p$, p a prime, are the ones most studied.

We will say that a space X is of *finite* $\mathbb{k}$*-type* if $H_n(X;\mathbb{k})$ is a finitely generated $\mathbb{k}$-module for every n.

Theorem 4.12. *The following are equivalent for a nilpotent space X of finite* $\mathbb{Q}$*-type.*

1. *The space X is* $\mathbb{Q}$*-local or "rational".*
2. *The homotopy group $\pi_n X$ is uniquely divisible for every $n \geq 1$.*

3. *The space X admits a principally refined Postnikov tower with fibers $K(\mathbb{Q}, n)$, $n \geq 1$.*

The $\mathbb{Q}$-localization $L_{\mathbb{Q}}X$, or "rationalization", of a nilpotent space X is commonly denoted $X_{\mathbb{Q}}$ or $X_{(0)}$. Furthermore,

$$\pi_n(X_{\mathbb{Q}}) \cong \pi_n(X) \otimes \mathbb{Q}.$$

Theorem 4.13. *The following are equivalent for a nilpotent space X of finite $\mathbb{Z}_{(p)}$-type.*

1. *The space X is $\mathbb{Z}_{(p)}$-local or "p-local".*

2. *The homotopy group $\pi_n X$ is uniquely ℓ-divisible for every $p \not| \ell$ and every $n \geq 1$.*

3. *The space X admits a principally refined Postnikov tower with fibers $K(\mathbb{Z}_{(p)}, n)$ or $K(\mathbb{Z}/p\mathbb{Z}, n)$, $n \geq 1$.*

The $\mathbb{Z}_{(p)}$-localization $L_{\mathbb{Z}_{(p)}}X$, or "p-localization", of a nilpotent space X is commonly denoted $X_{(p)}$. Furthermore,

$$\pi_n(X_{(p)}) \cong \pi_n(X) \otimes \mathbb{Z}_{(p)}.$$

Recall that the p-adic completion of an abelian group A is defined by

$$A_p^\wedge = \lim\left(\cdots \to A/p^2A \to A/pA\right).$$

In particular, $\mathbb{Z}_p^\wedge$ is the ring of p-adic integers.

Theorem 4.14. *The following are equivalent for a nilpotent space X of finite $\mathbb{F}_p$-type.*

1. *The space X is $\mathbb{F}_p$-local or "p-complete".*

2. *The homotopy group $\pi_n X$ is p-complete in the sense that the canonical map $\pi_n X \to (\pi_n X)_p^\wedge$ is an isomorphism.*

3. *The space X admits a principally refined Postnikov tower with fibers $K(\mathbb{Z}_p^\wedge, n)$ or $K(\mathbb{Z}/p\mathbb{Z}, n)$, $n \geq 1$.*

The $\mathbb{F}_p$-localization $L_{\mathbb{F}_p}X$, or "p-completion", of a nilpotent space X is commonly denoted $X_p^\wedge$. If X is of finite type, then

$$\pi_n(X_p^\wedge) \cong \pi_n(X)_p^\wedge.$$

Remark 4.15. For our purposes, it will suffice to study spaces of finite type, but it would be inappropriate not to remark that p-completions are not as well-behaved for spaces not of finite $\mathbb{F}_p$-type. The reason is that the functor $A \to A_p^\wedge$ is not exact in general, unlike the functor $A \mapsto A[S^{-1}]$ for $S \subseteq \mathbb{Z}$ some multiplicative subset. The functor $(-)_p^\wedge$ admits left derived functors L_0 and L_1. It turns out that

$$L_0A \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, A), \quad L_1A = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, A),$$

where

$$\mathbb{Z}/p^\infty = \operatorname{colim}\left(\mathbb{Z}/p\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}/p^2\mathbb{Z} \to \cdots\right) \cong \mathbb{Z}[p^{-1}]/\mathbb{Z}.$$

If A is a finitely generated abelian group, then $L_1 A = 0$ and $L_0 A \cong A_p^\wedge$. For a nilpotent space X, not necessarily of finite $\mathbb{F}_p$-type, there is a natural short exact sequence

$$0 \to L_0(\pi_n X) \to \pi_n(X_p^\wedge) \to L_1(\pi_{n-1} X) \to 0.$$

The sequence splits, but not naturally in X.

4.3.4 Arithmetic square

If X is a nilpotent space of finite type (= finite $\mathbb{Z}$-type), then the homotopy groups $\pi_n X$ are finitely generated. In particular, for $n \geq 2$, there is a decomposition

$$\pi_n X \cong \mathbb{Z}^r \oplus \bigoplus_p T_p,$$

where T_p denotes the p-torsion subgroup. Thus,

$$T_p \cong \mathbb{Z}/p^{r_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{r_k}\mathbb{Z},$$

for some $r_1, \ldots, r_k$.

Since $\mathbb{Z}/p^m\mathbb{Z} \otimes \mathbb{Q} = 0$ and $\left(\mathbb{Z}/p^m\mathbb{Z}\right)_p^\wedge = \mathbb{Z}/p^m\mathbb{Z}$ and $\left(\mathbb{Z}/p^m\mathbb{Z}\right)_\ell^\wedge = 0$ for $\ell \neq p$, we note that

$$\pi_n(X_\mathbb{Q}) \cong \mathbb{Q}^r$$

and

$$\pi_n(X_p^\wedge) \cong \left(\mathbb{Z}_p^\wedge\right)^r \oplus T_p.$$

Let

$$X^\wedge = \prod_p X_p^\wedge,$$

where the product is over all primes p.

Theorem 4.16. *For every nilpotent space X of finite type there is a homotopy cartesian square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X^\wedge \arrow[d] \\ X_\mathbb{Q} \arrow[r] & \left(X^\wedge\right)_\mathbb{Q}. \end{tikzcd}$$

4.4 The E_∞ Eilenberg-Moore theorem

Theorem 4.17 (E_∞ Eilenberg-Moore)**.** *Let $\Bbbk$ be a field. Consider a pullback square of spaces,*

$$\begin{tikzcd} X \times_B E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B, \end{tikzcd}$$

where

- *The map p is a fibration, and f is any map.*
- *The spaces E, B, X are connected and of finite $\Bbbk$-type.*
- *B is simply connected.*

Then

$$\begin{tikzcd} C^*B \arrow[r] \arrow[d] & C^*E \arrow[d] \\ C^*X \arrow[r] & C^*(X \times_B E) \end{tikzcd}$$

is a homotopy pushout diagram of E_∞-algebras.

In short, the theorem says that, under mild restrictions, the cochain functor C^* takes homotopy pullbacks of spaces to homotopy pushouts of E_∞-algebras. This theorem is due to Mandell [10] and it is one of the main technical ingredients in the proof of the main theorem. The proof requires a careful analysis of homotopy pushouts in the category of E_∞-algebras.

Corollary 4.18. *With hypotheses as in Theorem 4.17, if X, E and B are resolvable over $\Bbbk$, then so is $X \times_B E$.*

Proof. We may construct cofibrant E_∞-algebra resolutions A_X, A_E, A_B, of the cochains on X, E, B, respectively, that fit in a commutative diagram

$$\begin{tikzcd} A_X \arrow[d, two heads, "p_X"', "\sim"] & A_B \arrow[l, tail, "i_X"'] \arrow[r, tail, "i_E"] \arrow[d, two heads, "p_B"', "\sim"] & A_E \arrow[d, two heads, "p_E"', "\sim"] \\ C^*X & C^*B \arrow[l] \arrow[r] & C^*E, \end{tikzcd}$$

where the vertical maps are surjective quasi-isomorphisms and the maps in the top row are cofibrations of E_∞-algebras. By the E_∞-Eilenberg Moore theorem, the induced map

$$A_X \sqcup_{A_B} A_X \to C^*(X \times_B E) \tag{2}$$

is a quasi-isomorphism, so $A_X \sqcup_{A_B} A_X$ is a cofibrant resolution of $C^*(X \times_B E)$. Since it is the right adjoint in a (contravariant) Quillen adjunction, the spatial realization functor takes pushouts of E_∞-algebras to pullbacks of spaces, and it takes cofibrations to fibrations. We therefore get a diagram of spaces,

$$\begin{tikzcd} \langle A_X \rangle \arrow[r, two heads, "\langle i_X \rangle"] \arrow[d, "p_X^\flat"] & \langle A_B \rangle \arrow[d, "p_B^\flat"] & \langle A_E \rangle \arrow[l, two heads, "\langle i_E \rangle"'] \arrow[d, "p_E^\flat"] \\ X \arrow[r, "f"'] & B & E, \arrow[l, two heads, "p"] \end{tikzcd} \tag{3}$$

where the maps in the top row are fibrations, and the induced map on pullbacks,

$$X \times_B E \to \langle A_X \sqcup_{A_B} A_E \rangle, \tag{4}$$

is the adjoint map to (2). The pullbacks are homotopy pullbacks since p and, e.g., $\langle i_E \rangle$ are fibrations. Since X, E, B are resolvable over $\Bbbk$, the vertical maps in (3) are weak equivalences. It follows that the induced map on homotopy pullbacks (4) is a weak equivalence. This shows that $X \times_B E$ is resolvable. $\square$

For later reference, we record here an Eilenberg-Moore type spectral sequence for calculating the cohomology of a homotopy pushout of E_∞-algebras.

Proposition 4.19. *Given a homotopy pushout diagram of E_∞-algebras,*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D, \end{array}$$

there is a spectral sequence

$$\mathrm{Tor}_*^{H^*A}(H^*B, H^*C) \Rightarrow H^*D.$$

Theorem 4.20. *If X admits a principally refined Postnikov tower with resolvable fibers, then X is resolvable.*

Proof. Let $\{X_i\}_i$ be a principally refined Postnikov tower with resolvable fibers. The space $X_0 = *$ is resolvable. Assume by induction that X_i is resolvable. The space $K(A_i, n_i + 1)$ is resolvable by hypothesis and $PK(A_i, n_i + 1)$ is resolvable because every contractible space is resolvable. It follows from Corollary 4.18 that X_{i+1} is resolvable, because X_{i+1} is weakly equivalent to the pullback $X_i \times_{K(A_i,n_i+1)} PK(A_i, n_i + 1)$.

To conclude that $X \sim \lim X_i$ is resolvable, note that we may construct cofibrant resolutions A_i of C^*X_i that fit in a commutative diagram

$$\begin{array}{ccccccc} A_0 & \rightarrowtail & A_1 & \rightarrowtail & A_2 & \rightarrowtail & \cdots \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \\ C^*X_0 & \longrightarrow & C^*X_1 & \longrightarrow & C^*X_2 & \longrightarrow & \cdots . \end{array}$$

The colimit $A = \operatorname{colim} A_i$ is then cofibrant, and the induced morphism of E_∞-algebras,

$$A = \operatorname{colim} A_i \to \operatorname{colim} C^*X_i \to C^*(\lim X_i) \sim C^*(X),$$

is a quasi-isomorphism, so A is a cofibrant resolution of C^*X. Taking adjoints, we get a map of towers of fibrations,

$$\begin{array}{ccccccc} X_0 & \twoheadleftarrow & X_1 & \twoheadleftarrow & X_2 & \twoheadleftarrow & \cdots \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \\ \langle A_0 \rangle & \twoheadleftarrow & \langle A_1 \rangle & \twoheadleftarrow & \langle A_1 \rangle & \twoheadleftarrow & \cdots , \end{array}$$

where the vertical maps are weak equivalences because each X_i is resolvable. It follows that the induced map on inverse limits,

$$X \sim \lim X_i \xrightarrow{\sim} \lim \langle A_i \rangle \cong \langle A \rangle,$$

is a weak equivalence. This implies that X is resolvable. $\square$

4.5 Rational homotopy theory

In this section we focus on the ground ring $\Bbbk = \mathbb{Q}$.

Theorem 4.21. *The space $K(\mathbb{Q}, n)$ is resolvable over $\mathbb{Q}$ for every $n \geq 1$.*

Proof. As a model for $K(\mathbb{Q}, n)$ we may choose the spatial realization of $\mathcal{E}[x]$, the free $\mathcal{E}$-algebra on one generator of cohomological degree n, see Proposition 4.5. Clearly, $\mathcal{E}[x]$ is cofibrant. If we can show that the canonical map $f\colon \mathcal{E}[x] \to C^*(\mathcal{E}[x]; \mathbb{Q})$ is a quasi-isomorphism, then it will follow that $K(\mathbb{Q}, n) = \langle \mathcal{E}[x] \rangle$ is resolvable, because the adjoint, $f^\flat \colon \langle \mathcal{E}[x] \rangle \to \langle \mathcal{E}[x] \rangle$, is the identity map.

Recall that the rational cohomology ring of $K(\mathbb{Q}, n)$ is a free graded commutative algebra on a generator of degree n. It is not difficult to see that the map f sends x to a cocycle representative for the generator of the cohomology ring. So we are done if we can show that the cohomology of $\mathcal{E}[x]$ is a free graded commutative algebra generated by the class of x.

There is a morphism of operads $\mathcal{E} \to \mathcal{C}om$. For each r, the map

$$\mathcal{E}(r) \to \mathcal{C}om(r) = \mathbb{Q}$$

is a quasi-isomorphism, because $\mathcal{E}$ is an E_∞-operad. Since we work over a field of characteristic zero, the functor $- \otimes_{\Sigma_r} V \colon \mathbf{Ch}(\mathbb{Q}\Sigma_r) \to \mathbf{Ch}(\mathbb{Q})$ preserves quasi-isomorphisms for every graded $\mathbb{Q}\Sigma_r$-module V. In particular, $\mathcal{E}(r) \otimes_{\Sigma_r} \otimes(\mathbb{Q}x)^{\otimes r} \to \mathbb{Q} \otimes_{\Sigma_r} (\mathbb{Q}x)^{\otimes r}$ is a quasi-isomorphism for every r. Hence, the map

$$\mathcal{E}[x] \to \mathcal{C}om[x]$$

is a quasi-isomorphism. In particular, $H^*(\mathcal{E}[x]) \cong \mathcal{C}om[x]$, as claimed. □

Remark 4.22. Suppose $\Bbbk$ is a field of characteristic zero. Then $K(Q, n)$ is resolvable over $\Bbbk$ *only if* $\Bbbk = \mathbb{Q}$. To see this, note that we have a quasi-isomorphism $f\colon \mathcal{E}[x] \to C^*(K(\mathbb{Q}, n); \Bbbk)$, defined by sending x to a cocycle representative for the canonical generator for $\mathrm{H}^n(K(\mathbb{Q}, n); \Bbbk) \cong \Bbbk$. But over $\Bbbk$, we have that $\langle \mathcal{E}[x] \rangle = K(\Bbbk, n)$. The adjoint to f is the map $f^\flat \colon K(\mathbb{Q}, n) \to K(\Bbbk, n)$ which on π_n induces the inclusion $\mathbb{Q} \subseteq \Bbbk$. So $f^\flat$ is a weak equivalence only if $\Bbbk = \mathbb{Q}$.

Corollary 4.23. *Every nilpotent rational space of finite $\mathbb{Q}$-type is resolvable over $\mathbb{Q}$.*

Proof. Combine Theorem 4.21, Theorem 4.12 and Theorem 4.20. □

4.6 p-adic homotopy theory

4.6.1 Cohomology operations

Let $\Bbbk$ be a field of characteristic 2. Let A be an E_∞-algebra. We have the coboundary formula

$$\delta(x \smile_i y) + \delta x \smile_i y + x \smile_i \delta y = x \smile_{i-1} y + y \smile_i i - 1x.$$

It shows that if x is a cycle, then so is $x \smile_i x$.

Definition 4.24. Let A be an E_∞-algebra. For $i \in \mathbb{Z}$, the Steenrod square

$$Sq^i\colon H^k A \to H^{k+i} A$$

is defined by

$$Sq^i[x] = [x \smile_{k-i} x]$$

for $i \leq k$ and $Sq^i[x] = 0$ for $i > k$.

Theorem 4.25. *Let A be an E_∞-algebra over a field of characteristic 2.*

(Adem relations)
For $a < 2b$ we have

$$Sq^a Sq^b = \sum_k (2k-a, b-k-1) Sq^{b+k} Sq^{a-k}.$$

Here we use the notation $(i,j) = \binom{i+j}{i}$. The sum is finite; the constraints on k are $a \leq 2k < 2b$.

(Instability)
$Sq^i x = 0$ for $i > |x|$ and $Sq^{|x|} x = x \smile x$.

(Cartan formula)
We have

$$Sq^n(x \smile y) = \sum_{i+j=n} Sq^i x \smile Sq^j y,$$

and $Sq^n(1) = \delta_{n0}$.

Definition 4.26. The big Steenrod algebra is the graded algebra

$$\mathcal{B} = \mathbb{F}_2\langle \ldots, Sq^{-1}, Sq^0, Sq^1, \ldots \rangle / (\text{Adem relations}).$$

As we have seen, the cochains of a space is an E_∞-algebra. In particular, the cohomology H^*X is an unstable algebra over $\mathcal{B}$. However, the cohomology of a space satisfies some further relations that are not necessarily satisfied by the cohomology of an arbitrary E_∞-algebra.

Proposition 4.27. *Let $\Bbbk$ be a field of characteristic 2 and let X be a space. For every cochain $x \in C^n X$ we have that*

$$x \smile_n x = \Phi_*(x),$$

where $\Phi_*\colon C^n X \to C^n X$ is the map induced by the Frobenius homomorphism $\Phi\colon \Bbbk \to \Bbbk$, $\Phi(a) = a^2$.

Proof. Let σ be an n-simplex. By definition,

$$\big(x \smile_n x\big)(\sigma) = \sum x\big(\sigma(A_1 \sqcup A_3 \sqcup \cdots)\big) x\big(\sigma(A_2 \sqcup A_4 \sqcup \cdots)\big),$$

where the sum is over all overlapping partitions of $\{0, 1, \ldots, n\}$. Since x vanishes on simplices of dimension less than n, the only possibility of getting a non-zero term in the above sum is if $\sigma(A_1 \sqcup A_3 \sqcup \cdots) = \sigma(A_2 \sqcup A_4 \sqcup \cdots) = \sigma$. This happens for precisely one overlapping partition, namely the one with $A_1 = \{0,1\}$, $A_2 = \{1,2\}$, $A_3 = \{2,3\}$, etc. Hence, the sum simplifies to

$$\big(x \smile_n x\big)(\sigma) = x(\sigma)x(\sigma) = \Phi_*(x(\sigma)).$$

This shows that $x \smile_n x = \Phi_*(x)$. $\square$

Corollary 4.28. *Let $\Bbbk$ be a field of characteristic 2. For every space X and every, the map $Sq^0\colon H^*X \to H^*X$ is the Frobenius homomorphism $Sq^0 = \Phi_*$. In particular, if $\Bbbk = \mathbb{F}_2$, then Sq^0 is the identity map.*

The classical Steenrod algebra $\mathcal{A}$ may be identified with the quotient algebra

$$\mathcal{A} = \mathcal{B}/(Sq^0 - 1).$$

The cohomology of a space is an unstable algebra over the Steenrod algebra.

A good source for unstable modules and algebras over the Steenrod algebra is [19].

Theorem 4.29. *Let $\Bbbk$ be a field of characteristic 2. For every $n \geq 1$,*

$$H^*K(\mathbb{Z}/2\mathbb{Z}, n) \cong \mathcal{A}^{alg}[x],$$

the free unstable algebra over $\mathcal{A}$ on a generator of degree x.

A proof can be found in [17] or [13].

4.7 Cohomology of free E_∞-algebras

In this subsection, we will compute the cohomology of the free E_∞-algebra on one generator.

Fix $\Bbbk = \mathbb{F}_2$ and an E_∞-operad $\mathcal{E}$ in chain complexes over $\mathbb{F}_2$. Let $\mathcal{E}[x]$ denote the free $\mathcal{E}$-algebra on a generator x of (cohomological) degree n. The cohomology $H^*\mathcal{E}[x]$ is an unstable algebra over the big Steenrod algebra $\mathcal{B}$. The generator x is a cycle and represents a class in $H^*\mathcal{E}[x]$, so by abstract nonsense there is a map $\mathcal{B}^{alg}[x] \to H^*\mathcal{E}[x]$ from the free unstable $\mathcal{B}$-algebra on x.

Theorem 4.30. *The map $\mathcal{B}^{alg}[x] \to H^*\mathcal{E}[x]$ is an isomorphism.*

The proof of this result will occupy the rest of the subsection. We begin by showing that the map is surjective. This argument will use the transfer in group homology; we digress to recall the necessary facts.

4.7.1 Digression: Group homology and transfers

Let G be a group and $\Bbbk$ a commutative ring. By the term *G-module* we will understand a $\Bbbk$-module M with a left action of the group G. Equivalently, a G-module is the same thing as a left module over the group algebra $\Bbbk G$. The *coinvariants* of a G-module M is the quotient $\Bbbk$-module

$$M_G = M/\big(gm - m \;\mid\; g \in G,\; m \in M\big).$$

The k^{th} homology group of G with coefficients in M, denoted $H_k(G; M)$, is by definition the value of the k^{th} left derived functor of $(-)_G$ at M. It is a standard exercise to check that it may be expressed as

$$H_k(G; M) = \mathrm{Tor}_k^{\Bbbk G}(\Bbbk, M),$$

and it may be calculated by taking a projective resolution P_* of the trivial $\Bbbk G$-module $\Bbbk$, forming the chain complex $P_* \otimes M$ with diagonal G-action, and then taking the homology of the chain complex coinvariants $P_* \otimes_G M = \big(P_* \otimes M\big)_G$;

$$H_k(G; M) = H_k(P_* \otimes_G M).$$

When $M = \Bbbk$ with trivial G-action, it is customary to denote $H_k(G;\Bbbk)$ by $H_k(G)$.

For a subgroup $H \subseteq G$, recall that the index $[G : H]$ is defined as the cardinality of the set of left cosets $H\backslash G = \{Hg \mid g \in G\}$. Every G-module M may be viewed as an H-module by restriction, and if the index $[G : H]$ is finite, then one may define the transfer homomorphism

$$tr\colon M_G \to M_H, \quad tr[m]_G = \sum_{Hg\in H\backslash G} [gm]_H,$$

where $[m]_G$ denotes the image of $m \in M$ under the quotient map $M \to M_G$. Clearly, the composite

$$M_G \to M_H \to M_G$$

is multiplication by the index $[G : H]$.

If P_* is a $\Bbbk G$-projective resolution of $\Bbbk$, then it is also a $\Bbbk H$-projective resolution of $\Bbbk$. Thus, the transfer homomorphism $tr\colon (P_*)_G \to (P_*)_H$ induces a map in homology, also denoted tr,

$$tr\colon H_*(G) \to H_*(H).$$

As before the composite $H_*(G) \to H_*(H) \to H_*(G)$ is multiplication by $[G : H]$.

We will record an important consequence of this fact for later reference:

Proposition 4.31. *If $H \subseteq G$ is a subgroup of finite index and if $[G : H]$ is invertible in $\Bbbk$, then the induced map in group homology*

$$H_k(H) \to H_k(G)$$

is surjective for all k.

Proof. Precomposing $H_k(H) \to H_k(G)$ with the transfer $tr\colon H_k(G) \to H_k(H)$ yields an isomorphism as multiplication by $[G : H]$ is invertible, so the map must be surjective. □

Embarking now on the proof of surjectivity, we note that the free $\mathcal{E}$-algebra on a generator x admits a decomposition

$$\mathcal{E}[x] = \bigoplus_{k\geq 0} \mathcal{E}^k[x],$$

where $\mathcal{E}^k[x] = \mathcal{E}(k) \otimes_{\Sigma_k} \left(\mathbb{F}_2 x\right)^{\otimes k}$.

Proposition 4.32. *There is an isomorphism*

$$H^i\mathcal{E}^k[x] = H_{nk-i}(\Sigma_k),$$

where the right hand side denotes the homology of the symmetric group Σ_k with trivial coefficients $\mathbb{F}_2$.

Proof. We have that $\mathcal{E}^k[x] = \mathcal{E}(k) \otimes_{\Sigma_k} \left(\mathbb{F}_2 x\right)^{\otimes k}$. Since x has degree n, the Σ_k-module $\left(\mathbb{F}_2 x\right)^{\otimes k}$ may be identified with the trivial Σ_k-module $\mathbb{F}_2$ concentrated in degree $-nk$. Since $\mathcal{E}$ is an E_∞-operad, we have in particular that $\mathcal{E}(k)$ is a $\mathbb{F}_2\Sigma_k$-projective resolution of the trivial module $\mathbb{F}_2$. Hence, the i^{th} cohomology group of

$$\mathcal{E}(k) \otimes_{\Sigma_k} \left(\mathbb{F}_2 x\right)^{\otimes k} \cong \mathcal{E}(k)_{\Sigma_k}[-nk],$$

may be identified with the homology group $H_{nk-i}(\Sigma_k)$. □

The cup product induces a map

$$\mathcal{E}^k[x] \otimes \mathcal{E}^\ell[x] \to \mathcal{E}^{k+\ell}[x]. \tag{5}$$

Lemma 4.33. *If the binomial coefficient* $\binom{k+\ell}{k}$ *is odd, then the map* (5) *is surjective in cohomology.*

Proof. Arguing as in Proposition 4.32, the map induced by (5) in cohomology may, up to a degree shift, be identified with the map in group homology,

$$H_*(\Sigma_k) \otimes H_*(\Sigma_\ell) \cong H_*(\Sigma_k \times \Sigma_\ell) \to H_*(\Sigma_{k+\ell}),$$

induced by the Künneth isomorphism and the standard inclusion of $\Sigma_k \times \Sigma_\ell$ as a subgroup of $\Sigma_{k+\ell}$. The index $[\Sigma_{k+\ell} : \Sigma_k \times \Sigma_\ell] = \binom{k+\ell}{k}$ is invertible in $\mathbb{F}_2$ if it is odd, so the claim follows from Proposition 4.31. $\square$

We note that if r is not a power of 2, then it is possible to find $k, \ell < r$ such that $r = k + \ell$ and $\binom{k+\ell}{k}$ is odd (why?). It follows that $H^*\mathcal{E}[x]$ is generated as an algebra by

$$H^*\mathcal{E}^1[x], H^*\mathcal{E}^2[x], H^*\mathcal{E}^4[x], H^*\mathcal{E}^8[x], \ldots.$$

Next, the structure map $\mathcal{E}[\mathcal{E}[x]] \to \mathcal{E}[x]$ restricts to a map

$$\mathcal{E}(2) \otimes_{\Sigma_2} \mathcal{E}^k[x]^{\otimes 2} \to \mathcal{E}^{2k}[x]. \tag{6}$$

Lemma 4.34. *If k is a power of* 2, *then the map* (6) *is surjective in cohomology.*

Proof. A moment's thought reveals an isomorphism

$$\mathcal{E}(2) \otimes_{\Sigma_2} \mathcal{E}^k[x]^{\otimes 2} \cong \left(\mathcal{E}(2) \otimes \mathcal{E}(k)^{\otimes 2}\right)_{\Sigma_k^2 \rtimes \Sigma_2}[-2k].$$

Up to a degree shift, the map induced in cohomology by (6) may be identified with the map in group homology

$$H_*(\Sigma_k^2 \rtimes \Sigma_2) \to H_*(\Sigma_{2k})$$

induced by the standard inclusion of $\Sigma_k^2 \rtimes \Sigma_2$ into Σ_{2k}. The index,

$$[\Sigma_{2k} : \Sigma_{2k}] = \frac{(2k)!}{(k!)^2 \cdot 2},$$

is odd if k is a power of 2 (why?). The claim then follows from Proposition 4.31. $\square$

If A is an $\mathcal{E}$-algebra, then the image of $\mathcal{E}(2) \otimes_{\Sigma_2} A^{\otimes 2} \to A$ in cohomology is generated by $a \smile b$ and $Sq^i a$ for $a, b \in H^*A$. It follows that $H^*\mathcal{E}^{2^s}[x]$ is generated by $a \smile b$ and $Sq^i a$ for $a, b \in H^*\mathcal{E}^{2^{s-1}}[x]$.

Summing up, we see that $H^*\mathcal{E}[x]$ is generated by $H^*\mathcal{E}^1[x] = \mathbb{F}_2 x$ as an unstable $\mathcal{B}$-algebra. In other words, the map $\mathcal{B}^{alg}[x] \to H^*\mathcal{E}[x]$ is onto. We will now proceed to show injectivity.

Let $\mathcal{B}^{alg}(n)$ denote the free unstable $\mathcal{B}$-algebra on a generator x_n of degree n, and similarly for $\mathcal{A}^{alg}(n)$. Let $\pi \colon \mathcal{B}^{alg}(n) \to \mathcal{A}^{alg}(n)$ denote the quotient map. There is an isomorphism of commutative rings

$$S \colon \mathcal{B}^{alg}(n) \to \mathcal{B}^{alg}(n+1)$$

determined by

$$Sq^{i_1} \cdots Sq^{i_r} x_n \mapsto Sq^{i_1 + 2^{r-1}} \cdots Sq^{i_r + 1} x_{n+1}.$$

Lemma 4.35. *If $\xi \in \mathcal{B}^{alg}(n)$ is non-zero, then $\pi S^p(\xi) \neq 0$ for large enough p.*

Proof. Writing ξ as a linear combination of elements of the form $Sq^I x_n$, one applies S enough times to make all sequences I appearing in the decomposition non-negative. (Note that admissibility is preserved by S.) $\square$

To prove injectivity of $\mathcal{B}^{alg}(n) \to H^*\mathcal{E}[x_n]$, let $\xi \in \mathcal{B}^{alg}(n)$ be non-zero. Then picking p large enough so that $\pi S^p(\xi) \neq 0$, we may chase ξ in the following diagram:

$$\begin{tikzcd}
\mathcal{B}^{alg}(n) \arrow[r] \arrow[d, "S^p"'] & H^*\mathcal{E}[x_n] \arrow[d] \\
\mathcal{B}^{alg}(n+p) \arrow[r] \arrow[d, "\pi"'] & H^*\mathcal{E}[x_{n+p}] \arrow[d] \\
\mathcal{A}^{alg}(n+p) \arrow[r, "\cong"] & H^*K(\mathbb{Z}/2\mathbb{Z}, n+p)
\end{tikzcd}$$

Going down and then right gives a non-zero element by our choice of p. Hence, the image of ξ in $H^*\mathcal{E}[x_n]$ had better be non-zero. This finishes the proof of injectivity and Theorem 4.30 is proved.

4.8 Cofibrant resolution of $K(\mathbb{Z}/2\mathbb{Z}, n)$.

Fix a field $\Bbbk$ of characteristic 2. In this section, we will let K_n denote the Eilenberg-Mac Lane space $K(\mathbb{Z}/2\mathbb{Z}, n)$.

There is a map of E_∞-algebras

$$\mathcal{E}[x] \to C^*K_n$$

sending the generator x to a cocycle representative ξ of the canonical class ι in H^nK_n. However, the map is not a quasi-isomorphism. In cohomology, it induces the quotient map

$$\mathcal{B}^{alg}[x] \to \mathcal{A}^{alg}[x]$$

from the free unstable algebra over the big Steenrod algebra $\mathcal{B}$ to the free unstable algebra over the ordinary Steenrod algebra $\mathcal{A}$.

However, we know that $Sq^0\iota = \iota$, so we know that $\xi \smile_n \xi - \xi = d\zeta$ for some cochain $\zeta \in C^{n-1}K_n$. Thus, we may define a morphism of E_∞-algebras

$$\mathcal{E}[x, y | dy = x \smile_n x - x] \to C^*K_n \tag{7}$$

by sending x to ξ and y to ζ.

Theorem 4.36. *The map (7) is a quasi-isomorphism. Hence, $\mathcal{E}[x, y | dy = x \smile_n x - x]$ is a cofibrant E_∞-algebra model for $K(\mathbb{Z}/2\mathbb{Z}, n)$.*

Proof. We will only sketch the argument.

By construction, there is a pushout of E_∞-algebras

$$\begin{tikzcd}
\mathcal{E}[e] \arrow[r, "\varphi"] \arrow[d] & \mathcal{E}[x] \arrow[d] \\
\mathcal{E}[e, y | dy = e] \arrow[r] & \mathcal{E}[x, y | dy = x \smile_n x - x],
\end{tikzcd}$$

where $\varphi(e) = x \smile_n x - x$. The vertical maps are cofibrations, so it is a homotopy pushout.

The spectral sequence for calculating the cohomology of a homotopy pushout of E_∞-algebras (see Proposition 4.19) takes the form

$$\mathrm{Tor}_*^{\mathcal{B}^{alg}[e]}(\Bbbk, \mathcal{B}^{alg}[x]) \Rightarrow H^*(\mathcal{E}[x,y], d).$$

To analyze it, we need to understand the structure of $\mathcal{B}^{alg}[x]$ as a module over $\mathcal{B}^{alg}[e]$ via the map $\varphi\colon \mathcal{B}^{alg}[e] \to \mathcal{B}^{alg}[x]$, $\varphi[e] = Sq^0 x - x$.

Proposition 4.37. *There is an isomorphism of left $\mathcal{B}^{alg}[e]$-modules*

$$\mathcal{B}^{alg}[x] \cong \mathcal{B}^{alg}[e] \otimes \mathcal{A}^{alg}[x].$$

In particular, we see that $\mathcal{B}^{alg}[x]$ is free as a left $\mathcal{B}^{alg}[e]$-module, and there is an isomorphism

$$\mathrm{Tor}_*^{\mathcal{B}^{alg}[e]}(\Bbbk, \mathcal{B}^{alg}[x]) \cong \mathcal{A}^{alg}[x].$$

The spectral sequence collapses at E_2, and $H^*\mathcal{E}[x,y] \cong \mathcal{A}^{alg}[x]$. □

Theorem 4.38. *Let $\Bbbk$ be a field of characteristic $p > 0$. The Eilenberg-Mac Lane space $K(\mathbb{Z}/p\mathbb{Z}, n)$ is resolvable over $\Bbbk$ if and only if the map*

$$\Bbbk \xrightarrow{\Phi - 1} \Bbbk, \quad x \mapsto x^p - x,$$

is onto.

Proof. We do the proof in the case $p = 2$. We have the cofibrant resolution

$$\mathcal{E}[x, y | dy = x \smile_n x - x] \to C^* K(\mathbb{Z}/2\mathbb{Z}, n),$$

so $K(\mathbb{Z}/2\mathbb{Z}, n)$ is resolvable over $\Bbbk$ if and only if the adjoint map

$$K(\mathbb{Z}/2\mathbb{Z}, n) \to \langle \mathcal{E}[x,y] \rangle \tag{8}$$

is a weak equivalence. Applying spatial realization to the homotopy pushout diagram defining $\mathcal{E}[x,y]$, we get a homotopy pullback of spaces,

$$\begin{tikzcd} \langle \mathcal{E}[x,y] \rangle \arrow[r] \arrow[d] & \langle \mathcal{E}[x] \rangle \arrow[d] \\ \langle \mathcal{E}[e,y] \rangle \arrow[r] & \langle \mathcal{E}[e] \rangle. \end{tikzcd}$$

It follows from Proposition 4.5 that $\langle \mathcal{E}[e,y] \rangle$ is contractible and that both $\langle \mathcal{E}[x] \rangle$ and $\langle \mathcal{E}[e] \rangle$ are Eilenberg-Mac Lane spaces $K(\Bbbk, n)$. Thus, $\langle \mathcal{E}[x,y] \rangle \to \langle \mathcal{E}[x] \rangle \to \langle \mathcal{E}[e] \rangle$ is a homotopy fibration and the long exact sequence of homotopy groups associated to it shows that $\pi_k \langle \mathcal{E}[x,y] \rangle = 0$ for $k \neq n-1, n$. The non-zero portion of the long exact sequence is

$$0 \to \pi_n \langle \mathcal{E}[x,y] \rangle \to \pi_n \langle \mathcal{E}[x] \rangle \to \pi_n \langle \mathcal{E}[e] \rangle \to \pi_{n-1} \langle \mathcal{E}[x,y] \rangle \to 0.$$

We claim that the map $\langle \mathcal{E}[x] \rangle \to \langle \mathcal{E}[e] \rangle$ induces $\Bbbk \xrightarrow{\Phi - 1} \Bbbk$ on π_n. Granted this, we see can conclude that

$$\pi_n \langle \mathcal{E}[x,y] \rangle \cong \ker(\Phi - 1) = \mathbb{F}_2,$$
$$\pi_{n-1} \langle \mathcal{E}[x,y] \rangle \cong \mathrm{coker}(\Phi - 1).$$

Thus, $\langle\mathcal{E}[x,y]\rangle$ is a $K(\mathbb{Z}/2\mathbb{Z},n)$ if and only if $\operatorname{coker}(\Phi-1)=0$, i.e., $\Phi-1\colon \Bbbk\to\Bbbk$ is onto (and in this case, one shows easily that the adjoint map (8) is a weak equivalence, as one needs to do). $\square$

4.9 Models for $K(\mathbb{Z}/p^m\mathbb{Z},n)$ and $K(\mathbb{Z}_p^\wedge,n)$

In this section we will construct cofibrant E_∞-algebra models for Eilenberg-Mac Lane spaces of the form $K(\mathbb{Z}/p^m\mathbb{Z},n)$ and $K(\mathbb{Z}_p^\wedge,n)$ and show they are resolvable over any field $\Bbbk$ of characteristic p such that $\Phi-1\colon\Bbbk\to\Bbbk$, $x\mapsto x^p-x$ is onto.

We begin by recalling some facts about Bocksteins. Denote $K_{m,n}=K(\mathbb{Z}/p^m\mathbb{Z},n)$. The *Bockstein* $\beta_{m,n}$ is a cohomology class

$$\beta_{m,n}\in H^{n+1}(K_{m,n},\mathbb{F}_p).$$

It may be constructed as the image of the canonical class

$$\iota_{m,n}\in H^n(K_{m,n};\mathbb{Z}/p^m\mathbb{Z})=[K_{m,n},K_{m,n}]$$

under the connecting homomorphism

$$\partial\colon H^n(K_{m,n};\mathbb{Z}/p^m\mathbb{Z})\to H^{n+1}(K_{m,n};\mathbb{Z}/p\mathbb{Z})$$

associated to the short exact sequence of coefficient groups

$$0\to\mathbb{Z}/p\mathbb{Z}\to\mathbb{Z}/p^{m+1}\mathbb{Z}\to\mathbb{Z}/p^m\mathbb{Z}\to 0.$$

The Bockstein corresponds to a homotopy class of maps

$$\beta_{m,n}\colon K_{m,n}\to K_{1,n+1}.$$

One shows easily that the homotopy fiber is an Eilenberg-Mac Lane space $K(\mathbb{Z}/p^{m+1},n)$. Thus, we have a homotopy fiber sequence

$$K_{m+1,n}\to K_{m,n}\xrightarrow{\beta_{m,n}}K_{1,n+1},$$

for all m,n. This may be used to construct a cofibrant model $B_{m,n}$ for $C^*K_{m,n}$ by induction on m:

As we saw earlier, we have a cofibrant model for $C^*K_{1,n}$ of the form

$$B_{1,n}=\mathcal{E}[x,y]\xrightarrow{\sim}C^*K_{1,n}.$$

Assuming we have constructed models $B_{m,n}$ for all n, we construct $B_{m+1,n}$ as follows. Consider the homotopy commutative diagram

$$\begin{tikzcd} B_{1,n+1} \arrow[r, dashed, "\varphi_{m,n}"] \arrow[d] & B_{m,n} \arrow[d] \\ C^*K_{1,n+1} \arrow[r, "C^*\beta_{m,n}"] & C^*K_{m,n}, \end{tikzcd}$$

where the existence of $\varphi_{m,n}$ is ensured by cofibrancy of $B_{1,n+1}$. Define $B_{m+1,n}$ as the (homotopy) pushout in

$$\begin{tikzcd} B_{1,n+1} \arrow[r, "\varphi_{m,n}"] \arrow[d] & B_{m,n} \arrow[d] \\ CB_{1,n+1} \arrow[r] & B_{m+1,n}. \end{tikzcd}$$

By the E_∞ Eilenberg-Moore theorem (Theorem 4.17), the induced map $B_{m+1,n} \to C^*K_{m+1,n}$ is a quasi-isomorphism. This completes the contruction of a cofibrant model for $C^*K_{m,n}$.

Next, by contruction we have a sequence of cofibrations

$$B_{1,n} \to B_{2,n} \to \cdots,$$

modelling the tower of fibrations

$$\cdots \to K_{2,n} \to K_{1,n}.$$

The colimit $B_{\infty,n}$ of the sequence is a cofibrant model for $C^*K(\mathbb{Z}_p^\wedge, n)$.

Using these models, one deduces as in the previous section

Theorem 4.39. *Let $\Bbbk$ be a field of characteristic p such that $\Phi - 1\colon \Bbbk \to \Bbbk$ is onto. Then $K(\mathbb{Z}/p^m\mathbb{Z}, n)$ and $K(\mathbb{Z}_p^\wedge, n)$ are resolvable over $\Bbbk$ for all $m, n \geq 1$.*

Corollary 4.40. *Every p-complete nilpotent space of finite p-type is resolvable over $\overline{\mathbb{F}}_p$.*

Proof. Combine Theorem 4.39 and Theorem 4.14. □

Corollary 4.41. *Let $\Bbbk$ be a field of characteristic $p > 0$ such that $\Phi - 1\colon \Bbbk \to \Bbbk$ is onto. Two nilpotent p-complete spaces of finite p-type are weakly homotopy equivalent if and only if their singular cochain complexes, with $\Bbbk$-coefficients, are weakly equivalent as E_∞-algebras.*

Proof. Combine Proposition 4.10 and Corollary 4.40. □

As we have seen, $K(\mathbb{Z}/p\mathbb{Z}, n)$ is not resolvable over $\mathbb{F}_p$. However, we can deduce the main theorem from the previous corollary by passing to the algebraic closure $\overline{\mathbb{F}}_p$, for which $\Phi - 1$ is cleary surjective.

Theorem 4.42. *Two nilpotent p-complete spaces X, Y of finite p-type are weakly homotopy equivalent if and only if their singular cochain complexes, with $\mathbb{F}_p$-coefficients, are weakly equivalent as E_∞-algebras.*

Proof. One direction is clear. For the other direction, assume that $C^*(X; \mathbb{F}_p)$ and $C^*(Y; \mathbb{F}_p)$ are weakly equivalent as E_∞-algebras over $\mathbb{F}_p$. Since X is of finite p-type, the evident map $C^*(X; \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \to C^*(X; \overline{\mathbb{F}}_p)$ is a weak equivalence, and similarly for Y. It follows that $C^*(X; \overline{\mathbb{F}}_p)$ and $C^*(Y; \overline{\mathbb{F}}_p)$ are weakly equivalent as E_∞-algebras over $\overline{\mathbb{F}}_p$. By the previous corollary, X and Y are weakly homotopy equivalent. □

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