INEQUALITIES FOR HILBERT FUNCTIONS

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Let \( S_n = \mathbb{C}[x_1, \ldots, x_n] \), \( T_n = S_n/(x_1^2, x_2^2, \ldots, x_n^2) \), \( E_n = \mathbb{C}(x_1, \ldots, x_n)/(x_ix_j + x_ix_j, x_i^2) \).

1. Homogeneous ideals in \( S_n \)

Let \( R = S_n/I \), \( I \) homogeneous, let \( h_i = \dim_{\mathbb{C}} R_i \). Given \( h_d \), we want an upper bound for \( h_{d+1} \).

Let \( m, d > 0 \). Write \( m = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1} \) with \( k_d > k_{d-1} > \cdots > k_1 \geq 0 \). There is a unique way to do this.

**Example** \( d = 3, m = 28 \). \( 28 = \binom{6}{3} + \binom{4}{2} + \binom{2}{1} \).

For \( m = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1} \), let \( m^{(d)} = \binom{k_d+1}{d+1} + \binom{k_{d-1}+1}{d-1+1} + \cdots + \binom{k_1+1}{1+1} \), so \( 28^{(3)} = \binom{7}{4} + \binom{5}{3} + \binom{3}{2} = 48 \).

**Theorem 1** (Macaulay). \( h_{d+1} \leq h_d^{(d)} \).

Let \( M \) be a set of monomials of degree \( d \). Order the monomials lexicographically with \( x_1 < x_2 < \cdots < x_n \). \( M \) is called a lexsegment if \( v \in M, u > v \implies u \in M \). It is easy to see that an ideal generated by a lexsegment in degree \( d \) generates a lexsegment in degree \( d + 1 \).

It is natural that if we want an ideal generated by monomials of degree \( d \) which generates as little as possible, we should choose a lexsegment ideal. For a lexsegment ideal of degree \( d \) we have equality in the theorem.

If \( M \) is a lexsegment of degree 3 in \( S_5 \) such that \( h_3(S_5/(M)) = 28 \), then \( \#M = \binom{7}{3} - 28 = 7 \), so \( M = \{ x_3^5, x_3^2x_4, x_3^2x_2, x_3^2x_1, x_3x_4, x_3x_4x_5 \} \). The nonzero monomials of degree 3 in \( \mathbb{C}[x_1, x_2, x_3, x_4] \) are \( \binom{9}{3} \). The nonzero monomials in \( x_5\mathbb{C}[x_1, x_2, x_3] \) are \( \frac{1}{2} \binom{9}{3} \). The nonzero ideals in \( x_5x_4\mathbb{C}[x_1, x_2] \) are \( \binom{7}{3} \). \( M \) generates the lexsegment \( \{ x_3^5, \ldots, x_3x_4x_5x_1 \} \) in degree 4. The number of nonzero monomials of degree 4 in \( \mathbb{C}[x_1, \ldots, x_4] \) are \( \binom{7}{4} \), choose in \( x_5\mathbb{C}[x_1, x_2, x_3] \) are \( \frac{1}{2} \binom{9}{3} \), choose in \( x_5x_4\mathbb{C}[x_1, x_2] \) are \( \frac{1}{2} \binom{7}{3} \).

In this way one can prove that there is equality for lexsegment ideals. The theorem is proved in general in [1]. They use a theorem by Green. For \( m = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1} \) let \( m_{(d)} = \binom{k_d-1}{d-1} + \binom{k_{d-1}-1}{d-2} + \cdots + \binom{k_1-1}{0} \).

**Theorem 2** (Green). Let \( R \) be a homogeneous \( \mathbb{C} \)-algebra, and let \( d \geq 1 \) be an integer. Then \( h_d(R/hR) \leq h_d(R)_{(d)} \) for a general linear form \( h \).
Macaulay’s theorem is extended in Gotzmann’s persistence theorem.

**Theorem 3.** If for $R = S_n/I$ we have equality in Macaulay’s theorem between degrees $d$ and $d + 1$, and if $I$ has no generator of degree $> d + 1$, then we have equality for all higher degrees.

2. **Homogeneous ideals in $E_n$**

Let $\Delta$ be a simplicial complex, and let $f_i$ be the number of $i$-dimensional faces in $\Delta$.

If $\Delta$ is the 2-dimensional simplicial complex with maximal faces $\{x_1, x_2, x_3\}$ and $\{x_3, x_4\}$, then $f_1 = 1$, $f_0 = 4$, $f_1 = 4$, $f_2 = 1$. Consider the ideal $I_\Delta$ in $E_n$ generated by all non-faces of $\Delta$. In the example $I_{\Delta} = (x_1x_4, x_2x_4)$. It is clear that the Hilbert series of $E_n/I_\Delta$ (called the indicator algebra) is $\sum f_it^{i+1}$. (This resembles the Stanley-Reisner algebra of $\Delta$, which is defined as $\mathbb{C}[\Delta] = S_n/I_\Delta$. If the Hilbert series of $\mathbb{C}[\Delta]$ is $g(t)$ and the Hilbert series of the indicator algebra of $\Delta$ is $h(t)$, then $k(t) = h(t/(1-t))$.

For $m = \binom{k_1}{d} + \binom{k_2-1}{d-1} + \cdots + \binom{k_1}{1}$ let $\langle d \rangle m = \binom{k_1}{d} + \binom{k_2-1}{d-1} + \cdots + \binom{k_1}{1}$.

The following is proved in [2].

**Theorem 4.** (1) If $(f_1, f_2, \ldots, f_{d-1})$ is the $f$-vector of a pure (all maximal faces have the same dimension) simplicial complex, then $\langle i \rangle f_i \leq f_{i-1}$ for $1 \leq i \leq d - 1$.

(2) If $\langle i \rangle f_i = f_{i-1}$ for some $i$ in a pure simplicial complex, then $\langle j \rangle f_j = f_{j-1}$ for all $1 \leq j \leq i$.

(3) If $1 + \sum_0^n h_it^i$ is the Hilbert series of a graded $\mathbb{C}$-algebra $E_n/I$, then $h_{i+1} \leq \langle i \rangle h_i$ for $0 < i \leq n - 1$.

(4) If $h_{i+1} = \langle i \rangle h_i$ for some $i$, and $I$ has no generator of degree $> i + 1$, then $h_{j+1} = \langle j \rangle h_j$ if $j \geq i$.

(1) was proved by Kruskal and Katona independently.

3. **Generic forms in $T_n$ and $E_n$**

$T_n$ and $E_n$ are isomorphic as graded vector spaces, but not as rings. (In $E_n$ every odd element has square 0.)

3.1. **One generic form in $T_n$**.

**Theorem 5.** The Hilbert series of $T_n/(f)$, $f$ generic of degree $d$ is $[(1-t^d)(1+t)^n]$. (Take only terms as long as they are positive.)

**Proof** It suffices to get one example since we have an equality and the generic case is the worst. Take the sum of all squarefree monomials of degree $d$. Let the squarefree monomials $\{m_i\}$ of degree $i - d$ denote the rows, and the squarefree monomials $\{n_i\}$ of degree $i$ denote the columns in a matrix. The multiplication matrix then is an incidence matrix, in place $(j, k)$ there is a 1 if $m_j$ divides $n_k$ and 0 otherwise. That this matrix has full rank is well-known.
3.2. One generic form of even degree in $E_n$. It is proved in [3] that the same formula as for $T_n$ is true.

3.3. One generic form of odd degree in $E_n$. We have that

$$0 \rightarrow \text{Ann}(f)(-d) \rightarrow E_n(-d) \xrightarrow{f} E_n \rightarrow E_n/(f) \rightarrow 0$$

is exact. It is clear that $(f) \subseteq \text{Ann}(f)$. Sometimes the inclusion is strict, i.e. $(f) \neq \text{Ann}(f)$.

E.g. when $d = 3$ and $n = 16$ the differens in Hilbert series is $16t^9 + 120t^8 + 559t^7$. If $d > 3$ (and odd) the differens is much smaller. If $n - d = 0, 1, 3, 4, 5$, the differens seems to be 0.

3.4. Several generic forms.

Theorem 6. Consider the exterior algebra $E_5$ over a vector space with basis $\{e_1, ..., e_5\}$ (over any field). Let

$$f_1 = \sum c_{ij} e_i \wedge e_j$$

and

$$f_2 = \sum d_{ij} e_i \wedge e_j$$

be two forms in $E_5$. Then

$$\{e_i \wedge f_j\}$$

is linearly dependent.

Proof It suffices to prove the theorem for generic forms (so we can suppose that the $c_{ij}$’s and $d_{ij}$’s are algebraically independent over the prime field of $k$). I calculated a relation. In [3] they calculate the Hilbert series for two quadratic forms in $n \leq 13$ variables. It differs from the "expected" series $(1 - t^2)^2(1 + t)^n$ with $t^3$ for $n = 5$, with $t^4$ for $n = 7, 8$, with $t^5$ for $n = 9$, with $10^5 t^5$ for $n = 10$, with $t^5 + t^6$ for $n = 11$, with $64t^6$ for $n = 12$, and with $13t^6 + t^7$ for $n = 13$.

Conjectures There is no difference for two forms of degree 2 from the "expected" series $[(1 - t^2)^2(1 + t)^n]$ in degrees $< [(n + 1)/2]$. For two forms of even degree $d > 2$, the series is the "expected" $[(1 - t^d)^2(1 + t)^n]$.

References