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POLYNOMIAL INTERPOLATION IN SEVERAL VARIABLES

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Abstract

By treating the outstanding case $d = 3$, we determine the dimension of the space of polynomials in n variables, of degree at most d, vanishing with their first derivatives at r general points.

Introduction

Fix an infinite field k and let $V_{n,d}$ (resp., $V_{n,d,r}$) denote the vector space of polynomials in n variables of degree at most d (resp., and having singularities at r general points of $\mathbf{A}^n$).

In a series of articles [1, 2, 3, 7], the authors have studied the codimension of $V_{n,d,r}$ in $V_{n,d}$ for $d \neq 3$. A simplified proof of the case $d \geq 5$ can be found in [4]. The present article deals with the remaining case $d = 3$. We prove

Theorem 1. *Except for $n = 4$, $r = 7$, the subspace $V_{n,3,r}$ has the expected codimension*

$$\min\left(r(n+1), \binom{n+3}{3}\right)$$

in $V_{n,3}$.

Combining Theorem 1 with [1, (0.1), (4.1.1), (5.2)] and [3, (0.1)] we obtain

Theorem 2. *The subspace $V_{n,d,r}$ has the expected codimension*

$$\min\left(r(n+1), \binom{n+d}{d}\right)$$

except for the following cases

(i) $d = 2$; $2 \leq r \leq n$, $\operatorname{codim} V_{n,2,r} = r(n+1) - r(r-1)/2$;
(ii) $d = 3$; $n = 4$, $r = 7$, $\dim V_{4,3,7} = 1$;

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(iii) $d = 4$; $(n, r) = (2, 5), (3, 9), (4, 14)$, $\dim V_{n,4,r} = 1$.

We can rephrase Theorem 1 by saying that, apart from the indicated exceptions, the $r(n+1)$ linear conditions imposed on the coefficients of a polynomial f by the vanishing of f with its first derivatives at the points $x_1, \cdots, x_r$ are independent. The equivalence of this result with the classical Waring problem (determine the minimum s such that a general degree d homogeneous polynomial f can be expressed as a sum of length s of dth powers of linear forms) was pointed out to us by A. Iarrobino (see [8] for details and an extensive bibliography concerning work on this problem).

In a projective setting, letting Y denote the first infinitesimal neighbourhood of the union of the points $x_1, \cdots, x_r$ in the projective completion $\mathbb{P}^n$ of $\mathbb{A}^n$, $V_{n,d,r}$ is canonically identified with the cohomology group $H^0(\mathcal{I}_Y(d))$ via the exact sequence

$$0 \to H^0(\mathcal{I}_Y(d)) \to H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_Y(d)) \to H^1(\mathcal{I}_Y(d)) \to 0$$

where $\mathcal{I}_Y$ is the ideal sheaf of Y in $\mathbb{P}^n$. In this form, $V_{n,d,r}$ has the expected codimension if and only if one or other of the integers $h^0(\mathcal{I}_Y(d))$, $h^0(\mathcal{I}_Y(d))$ is zero. Using standard techniques (see [3, §1]) we can essentially reduce the proof of Theorem 1 to the case $\chi(X, \mathcal{I}_Y(d)) = 0$, which motivates

Definition 3. If X is a projective variety and $\mathcal{L}$ is an invertible sheaf on X, we say that a closed subscheme Y of X is *numerically $\mathcal{L}$-adjusted* (resp. *$\mathcal{L}$-adjusted*) if $\chi(X, \mathcal{I}_Y \otimes \mathcal{L}) = 0$ (resp. $h^i(X, \mathcal{I}_Y \otimes \mathcal{L}) = 0$) for $i \geq 0$. In [3] (Definition 1.2) these notions are called respectively numériquement $\mathcal{L}$-rangé and $\mathcal{L}$-rangé.

Definition 4. Following the notation in [3], if X is a smooth quasiprojective variety and x is a closed point of X, we say that the first infinitesimal neighbourhood of x in X is a *double point* in X. When the ambient variety is understood, we will simply say "a double point". If L is a smooth quasiprojective subvariety of X of dimension m, passing through x, we will say that the first infinitesimal neighbourhood N_x of x in L is an *$(m+1)$-point* in X. (Note that N_x only depends on the Zariski tangent space to L at x.) One should note that a 2-point is not a double point unless $\dim X = 1$. We simply say "point" for a 1-point, which is just a closed point of X.

Definition 5. Let $Y(n)$ $(n \geq 1)$ be the generic union in $\mathbb{P}^n$ of a_n double points with a b_n-point, where the integers a_n and b_n are defined

by

$$(n+1)a_n + b$$

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1. Outline

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(1.1.2) $AW(n-7) + A$

by

$$(n+1)a_n + b_n = \binom{n+3}{3}, \qquad 0 \le b_n \le n.$$

The expression $AY(n)$ stands for " $Y(n)$ is $\mathscr{O}(3)$-adjusted".

Modulo the case $n = 4$ which was dealt with in [1], and further considered in [5], Theorem 1 results by a standard argument (see [3, §1]) from

Proposition 6. *We have $AY(n)$ for $n \neq 4$.*

1. Outline of the proof of Proposition 6

We prove Proposition 3 by induction on the dimension n using the *La méthode d'Horace éclatée et differentielle* as outlined in [2] and [3]. We use the differential lemma in §4 to treat the initial cases, while the blown-up version is used for the induction in §5.

The main difficulty occurring in the present application of the method is that polynomials of degree three, vanishing on a union of double points, vanish on the lines joining these points two by two. This makes it quite difficult to exploit a divisor which inevitably meets all these lines. For this reason, our proof begins by exploiting a lower-dimensional linear subspace (of codimension two or four). We then exploit one or more divisors, either in the original $\mathbb{P}^n$ or in the corresponding blow-up.

Definition 1.1. (i) For $n = 3l$ or $n = 3l-1$ $(l \ge 1)$, let S be a linear subspace of codimension two in $\mathbb{P}^n$. We denote by $W(n)$ the generic union of a_{n-2} double points with support in S, $a_n - a_{n-2}$ double points, and a b_n-point.

(ii) For $n = 3l+1$ $(l \ge 2)$, let H be a hyperplane in $\mathbb{P}^n$ and let G be the generic linear subspace of codimension four. We denote by $W(n)$ the generic union of $a_n - a_{n-4}$ double points with support in H, with a_{n-4} double points in G, of which $a_{n-6} - 1$ have support in $G \cap H$. (Note that $b(n) = 0$ in this case.)

We let $AW(n)$ denote " $W(n)$ is $\mathscr{O}(3)$-adjusted".

We then have

Lemma 1.2. $AW(n) \Rightarrow AY(n)$.

Proof. $W(n)$ is a specialisation of $Y(n)$. q.e.d.

In §5 we will essentially prove the following implications:

(1.1.1) $AW(n-2) \Rightarrow AW(n)$ for $n = 3l$ or $n = 3l-1$ $(l \ge 1)$;

(1.1.2) $AW(n-7) + AW(n-4) + AW(n-3) \Rightarrow AW(n)$ for $n = 3l+1$ $(n \ge 2)$,

which combined with 1.2 and some initial cases gives Proposition 3.

An essential element in the proof of these implications is the introduction of the following subscheme:

Definition 1.3. Let G be the generic codimension four linear subspace of $\mathbb{P}^n$. For $n = 3l$ $(l \geq 2)$, let $V(n)$ be the generic union of $e_n = a_{n+1} - a_{n-3}$ double points of $\mathbb{P}^n$ with G and $f_n = a_{n-5} - 1$ double points of $\mathbb{P}^n$ with support in G.

Let $AV(n)$ be the statement "$V(n)$ is $\mathscr{O}(3)$-adjusted".

We then decompose (1.1.2) into the following two implications:

$$(1.1.3)\quad AV(n-1) + AY(n-4) \Rightarrow AW(n) \quad \text{for } n = 3l+1 \ (l \geq 2);$$

$$(1.1.4)\quad AY(n-6) + AW(n-2) \Rightarrow AV(n) \quad \text{for } n = 3l \ (l \geq 3),$$

using Lemma 1.2 (i.e., (1.1.3) and (1.1.4) imply (1.1.2)).

Remark 1.4. The scheme $V(n)$ $(n = 3l)$ arises naturally in our proof when we consider $W(n+1)$. If we consider $AW(n-3)$ to be proven, then the trace of $W(n+1) \subset \mathbb{P}^{n+1}$ on the codimension-four linear subspace $G = \mathbb{P}^{n-3}$ is $\mathscr{O}(3)$-adjusted because it specialises to $W(n-3)$. The statement $AW(n+1)$ is then equivalent to "$W(n+1) \cup G$ is $\mathscr{O}(3)$-adjusted" (by [3, 1.6]). The trace of this union on the hyperplane H is just $V(n)$, so that $AV(n)$ allows us to exploit the divisor H.

A slight complication arises from $AW(10)$. While the implication $AV(9) + AW(6) \Rightarrow AW(10)$ holds, the statement $AW(6)$ is false because it involves seven double points in $\mathbb{P}^4$, and this is the exception to Theorem 1. To deal with this case we replace the implication (1.1.3) by

$$(1.1.5)\qquad AV(9) + AY(6) \Rightarrow AW(10).$$

We now show how to deduce Proposition 6 from the results of §§4,5.

Proposition 1.5. *The initial cases* $AW(1)$, $AW(2)$, $AY(6)$, *and* $AV(6)$, *with the implications* (1.1.1), (1.1.3), (1.1.4), *and* (1.1.5) *imply Proposition* 6.

Proof. Using the initial cases and the implications we easily obtain $AY(3)$, $AY(5)$, and $AW(7), \cdots, AW(13)$. Now for $n \geq 13$ we obtain $AW(n)$ from $AW(n-7), \cdots, AW(n-1)$ using (1.1.1) in the cases $n = 3l$ and $n = 3l+1$ and using (1.1.2) in the case $n = 3l-1$. Since (1.1.2) is implied by (1.1.3) and (1.1.4), this gives the proposition using 1.2. q.e.d.

The implication (1.1.1) is proven in 5.1 and 5.2. The implications (1.1.3) and (1.1.5) are proven in 5.3, and the implication (1.1.4) is proven in 5.4.

Of the initial cases $AW(1)$, $AW(2)$, $AY(6)$, and $AV(6)$, the first two are trivial and the latter two are proven in §4 using the differential techniques of [2].

2. S-compatibility

In §5 we prove the implications (1.1.2–5) using the proposition [3, 2.6]. To apply this proposition we must verify the property of *S-compatibility* [3, 2.1] for the related subschemes. We remind the reader that a subscheme Y of a smooth quasiprojective variety X is S-compatible with respect to a subscheme S of X if, denoting by $\pi: \tilde{X} \to X$ the blowing-up of X in the ideal of S and by Z the inverse image of Y on $\tilde{X}$, we have $\pi_* \mathscr{I}_Z = \mathscr{I}_Y$ and $R^i \pi_* \mathscr{I}_Z = 0$ for $i > 0$. This condition then assures that, for any invertible $\mathscr{O}_X$-module $\mathscr{L}$,

$$H^i(\tilde{X}, \mathscr{I}_Z \otimes \pi^* \mathscr{L}) = H^i(X, \mathscr{I}_Y \otimes \mathscr{L}).$$

The results we need will follow from

Proposition 2.1. *Let X be a smooth quasiprojective variety, and let S, A be two smooth subvarieties of X such that $S \cap A$ is also smooth. Let Y be a smooth subvariety of $S \cap A$, and let Y_m denote the subscheme of X defined by the ideal $\mathscr{I}_Y^m$, where m is a positive integer and $\mathscr{I}_Y$ is the ideal sheaf of Y in X. Then the scheme-theoretic union $A \cup Y_m$ is S-compatible in the sense of* [3, 2.1].

Proof. We consider the blowing-up $\pi: \tilde{X} \to X$ of X with respect to S. We let D be the exceptional divisor on $\tilde{X}$ over S, and we let B (resp. Z) be the inverse image of A (resp. Y) on $\tilde{X}$. We denote by $\tilde{A}$ the strict transform of A and by Z_m the subscheme of $\tilde{X}$ defined by the ideal $\mathscr{I}_Z^m$, where $\mathscr{I}_Z$ is the ideal of Z in $\tilde{X}$.

We remind the reader that the residual W' with respect to D, of a closed subscheme W of $\tilde{X}$, is defined by the conductor ideal

$$\mathscr{I}_{W'} = (\mathscr{I}_W : \mathscr{I}_D) = (\mathscr{I}_Z : \mathscr{O}(-D))$$

of $\mathscr{I}_D$ into $\mathscr{I}_W$, so that we have the *residual exact sequence*

$$0 \to \mathscr{I}_{W'}(-D) \to \mathscr{I}_W \to \mathscr{I}_{W \cap D, D} \to 0.$$

In particular, $\mathscr{I}_{W'}(-D) = \mathscr{I}_{W \cup D}$. We note that the generally valid identity for ideals

$$(\mathscr{I}_{W_1} \cap \mathscr{I}_{W_2} : \mathscr{I}_D) = (\mathscr{I}_{W_1} : \mathscr{I}_D) \cap (\mathscr{I}_{W_2} : \mathscr{I}_D)$$

implies that the residual of the schematic union $W_1 \cap W_2$ is the schematic union of the residuals.

Now we begin the proof with the following lemma.

Lemma 2.1.1. *In the above notation we have*

(i) $\pi^{-1}(\mathscr{I}_{A\cup Y_m})\cdot\mathscr{O}_{\tilde{X}} = \mathscr{I}_{B\cup Z_m}$, *so that* $B\cup Z_m$ *is the inverse image of* $A\cup Y_m$ *on* $\tilde{X}$.

(ii) $\pi^{-1}(\mathscr{I}_{A\cup Y_m\cup S})\cdot\mathscr{O}_{\tilde{X}} = \mathscr{I}_{B\cup Z_m\cup D} = \mathscr{I}_{B\cup Z_m}(-D)$.

(iii) $(\mathscr{I}_{\tilde{A}}\cap\mathscr{I}_{Z_m})+\mathscr{I}_D = (\mathscr{I}_{\tilde{A}}+\mathscr{I}_D)\cap(\mathscr{I}_{Z_m}+\mathscr{I}_D)$, $(\mathscr{I}_A\cap\mathscr{I}_{Y_m})+\mathscr{I}_S = (\mathscr{I}_A+\mathscr{I}_S)\cap(\mathscr{I}_{Y_m}+\mathscr{I}_S)$, *or equivalently* $(\tilde{A}\cup Z_m)\cap D = (\tilde{A}\cap D)\cup(Z_m\cap D)$, $(A\cup Y_m)\cap S = (A\cap S)\cup(Y_m\cap S)$,

(iv) *The residual of* B *with respect to* D *is the blowing-up of* A *with respect to* $S\cap A$, *and the residual of* $\tilde{A}$ *with respect to* D *is* $\tilde{A}$.

(v) *The residual of* Z_m *with respect to* D *is* Z_{m-1}.

Proof (of the lemma). Parts (iv) and (v) are well known. For the remaining parts, the question is local on both X and $\tilde{X}$ so that we can restrict our attention to points in $S\cap A$. Using the fact that S, A, and $S\cap A$ are all smooth, we can find, in an open neighbourhood of any such point, a regular sequence $x_1,\cdots,x_{r+s+t}$ which generates $\mathscr{I}_{S\cap A}$ such that

$$\mathscr{I}_S = (x_1,\cdots,x_r,x_{r+1},\cdots,x_{r+s}),$$
$$\mathscr{I}_A = (x_1,\cdots,x_r,x_{r+s+1},\cdots,x_{r+s+t}).$$

In a neighbourhood of a closed point of Y, this sequence can be extended to a regular sequence $x_1,\cdots,x_{r+s+t+p}$ generating the ideal $\mathscr{I}_Y$. We can thus assume that X, S, A, Y are all linear spaces.

Using the identification

$$\tilde{X} = \operatorname{Proj}(R[t_1,\cdots,t_r,t_{r+1},\cdots,t_{r+s}]/((x_it_j - x_jt_i);\ 1\le i<j\le r+s))$$

where $R = k[x_1,\cdots,x_n]$, $n = r+s+t+p$, and $X = \operatorname{Spec} R$, the lemma results (by symmetry) from a standard calculation on the open sets of $\tilde{X}$ where t_1 (resp. t_{r+1}) is invertible.

For this calculation we fix the following notation:

$$\mathfrak{a} = (x_1,\cdots,x_r);\qquad \mathfrak{b} = (x_{r+1},\cdots,x_{r+s});$$
$$\mathfrak{c} = (x_{r+s+1},\cdots,x_{r+s+t});\qquad \mathfrak{d} = (x_{r+s+t+1},\cdots,x_{r+s+t+p});$$

so that

$$\mathscr{I}_S = (\mathfrak{a}+\mathfrak{b});\quad \mathscr{I}_A = (\mathfrak{a}+\mathfrak{c});\quad \mathscr{I}_Y = (\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d}).$$

The following basic calculations use the modular law and the following observation: let $\mathfrak{p}, \mathfrak{q} \in k[x_1, \cdots, x_n]$ be two ideals generated by monomials $\mu_1, \cdots, \mu_{n_1}$ and $\nu_1, \cdots, \nu_{n_2}$ respectively in the x_i, such that μ_i and ν_j are relatively prime for all values of i, j (we will say that $\mathfrak{p}, \mathfrak{q}$ are monomially prime). Then we have $\mathfrak{p} \cap \mathfrak{q} = \mathfrak{p}\mathfrak{q}$. Clearly two products of the ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ having no common factor are monomially prime.

Applying this we find

$$\mathscr{I}_{A \cup Y_m} = \mathscr{I}_A \cap (\mathscr{I}_Y)^m = (\mathfrak{a} + \mathfrak{b})(\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d})^{m-1}$$

and

$$\begin{aligned} \mathscr{I}_{A \cup Y_m \cup S} &= \mathscr{I}_A \cap (\mathscr{I}_Y)^m \cap \mathscr{I}_S \\ &= \mathfrak{a}(\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d})^{m-1} + \mathfrak{b}\mathfrak{c}(\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d})^{m-2}. \end{aligned}$$

Now on the open set of $\tilde{X}$ where t_{r+1} is invertible, we have

$$\tilde{X} = \operatorname{Spec} k[x_{r+1}, x_{r+s+1}, \cdots, x_{r+s+t+p}, t_1, \cdots, t_{r+s}]$$

and

$$\begin{aligned} \mathscr{I}_D &= \pi^{-1}(\mathscr{I}_S) = (x_{r+1}), \\ \mathscr{I}_B &= \pi^{-1}(\mathscr{I}_A) = (t_1 x_{r+1}, \cdots, t_r x_{r+1}, x_{r+s+1}, \cdots, x_{r+s+t}), \\ \mathscr{I}_Z &= \pi^{-1}(\mathscr{I}_Y) = (x_{r+1}, x_{r+s+1}, \cdots, x_{r+s+t+p}). \end{aligned}$$

Putting

$$\begin{aligned} \mathfrak{q} &= (t_1, \cdots, t_r), \\ \mathfrak{a}_0 &= \pi^{-1}\mathfrak{a} = x_{r+1}\mathfrak{q}, \\ \mathfrak{b}_0 &= \pi^{-1}\mathfrak{b} = (x_{r+1}), \\ \mathfrak{c}_0 &= \pi^{-1}\mathfrak{c} = (x_{r+s+1}, \cdots, x_{r+s+t}), \\ \mathfrak{d}_0 &= \pi^{-1}\mathfrak{d} = (x_{r+s+t+1}, \cdots, x_{r+s+t+p}) \end{aligned}$$

and noting that π^{-1} commutes with products of ideals, we find

$$\begin{aligned} \mathscr{I}_{B \cup Z_m} &= \mathscr{I}_B \cap (\mathscr{I}_Z)^m \\ &= (\mathfrak{a}_0 + \mathfrak{b}_0)(\mathfrak{a}_0 + \mathfrak{b}_0 + \mathfrak{c}_0 + \mathfrak{d}_0)^{m-1} \\ &= \pi^{-1}(\mathscr{I}_{A \cup Y_m}) \end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_{B\cup Z_m\cup D} &= \mathcal{I}_B \cap (\mathcal{I}_Z)^m \cap \mathcal{I}_D \\ &= \mathfrak{a}_0(\mathfrak{a}_0+\mathfrak{b}_0+\mathfrak{c}_0+\mathfrak{d}_0)^{m-1} + \mathfrak{b}_0\mathfrak{c}_0(\mathfrak{a}_0+\mathfrak{b}_0+\mathfrak{c}_0+\mathfrak{d}_0)^{m-2} \\ &= \pi^{-1}(\mathcal{I}_{A\cup Y_m\cap S}),\end{aligned}$$

giving the first and second parts of the lemma. The third part follows easily by a similar argument. We leave the reader to make corresponding calculation on the open set where t_1 is invertible. q.e.d.

End of proof of Proposition 2.1. By 2.1.1(i) and the definition of S-compatibility, we must show that $R^i\pi_*\mathcal{O}_{B\cup Z_m} = 0$ for $i > 0$ and that the canonical map $\pi_*\mathcal{O}_{B\cup Z_m} \to \mathcal{O}_{A\cup Y_m}$ is an isomorphism. Using the S-compatibility of X [3, 2.3] and considering the canonical diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_{A\cup Y_m} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_{A\cup Y_m} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_*\mathcal{I}_{B\cup Z_m} \arrow[r] & \pi_*\mathcal{O}_{\tilde{X}} \arrow[r] & \pi_*\mathcal{O}_{B\cup Z_m} \arrow[r] & R^1\pi_*\mathcal{O}_{B\cup Z_m} \arrow[r] & 0
\end{tikzcd}$$

one easily concludes that $A\cup Y_m$ is S-compatible if and only if $R^i\pi_*\mathcal{I}_{B\cup Z_m} = 0$ for $i > 0$ and the canonical map $\pi_*\mathcal{I}_{B\cup Z_m} \to \mathcal{I}_{A\cup Y_m}$ is an isomorphism (see [3, 2.6.2)]).

Using 2.1.1(iv), (v) and the above remarks about residual exact sequences, we obtain the following commutative diagram with exact rows from the residual exact sequence of $B\cup Z_{m+1}$ with respect to D:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_{A\cup Y_m\cup S} \arrow[r] \arrow[d, "\alpha'"] & \mathcal{I}_{A\cup Y_{m+1}} \arrow[r] \arrow[d, "\alpha"] & \mathcal{I}_{(A\cup Y_{m+1})\cap S,\, S} \arrow[r] \arrow[d, "\alpha''"] & 0 \\
0 \arrow[r] & \pi_*\mathcal{I}_{\tilde{A}\cup Z_m}(-D) \arrow[r] & \pi_*\mathcal{I}_{B\cup Z_{m+1}} \arrow[r] & \pi_*\mathcal{I}_{(B\cup Z_{m+1})\cap D,\, D} &
\end{tikzcd}$$

where α' is the adjoint of the composed map

$$\pi^*\mathcal{I}_{A\cup Y_m\cup S} \to \mathcal{I}_{B\cup Z_m\cup D} \to \mathcal{I}_{\tilde{A}\cup Z_m\cup D} = \mathcal{I}_{\tilde{A}\cup Z_m}(-D).$$

Now by [3, (3.3.3)], α'' is an isomorphism and $R^i\pi_*\mathcal{I}_{(B\cup Z_{m+1})\cap D,\, D} = 0$ for $i > 0$, so that it is enough to show that

(2.1.1) $\quad \alpha'$ is an isomorphism and $R^i\pi_*\mathcal{I}_{\tilde{A}\cup Z_m}(-D) = 0$ for $i > 0$.

To prove (2.1.1) we first consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I}_{(A\cup Y_m)\cap S,\,S} \arrow[r] \arrow[d, "\beta'"] & \mathscr{I}_{Y_m\cap S,\,S} \arrow[r] \arrow[d, "\beta"] & \mathscr{I}_{(Y_m\cap A\cap S),\,A\cap S} \arrow[r] \arrow[d, "\beta''"] & 0 \\
0 \arrow[r] & \pi_*\mathscr{I}_{(\widetilde{A}\cup Z_m)\cap D,\,D} \arrow[r] & \pi_*\mathscr{I}_{Z_m\cap D,\,D} \arrow[r] & \pi_*\mathscr{I}_{(Z_m\cap B\cap D),\,B\cap D} &
\end{tikzcd}$$

where the exact rows come from 2.1.1(iii), the morphism β' is the adjoint of the map

$$\pi^*\mathscr{I}_{(A\cup Y_m)\cap S,\,S} \to \mathscr{I}_{(B\cup Z_m)\cup D,\,D} \to \mathscr{I}_{(\widetilde{A}\cup Z_m)\cap D,\,D}$$

and β'' is the adjoint of the map

$$\pi^*\mathscr{I}_{(Y_m\cap A\cap S),\,A\cap S} \to \mathscr{I}_{(Z_m\cap B\cap D),\,\widetilde{A}\cap D} \to \mathscr{I}_{(Z_m\cap\widetilde{A}\cap D),\,\widetilde{A}\cap D}.$$

Since $\widetilde{A}\cap D$ is a projective space bundle over $A\cap S$ and $Z_m\cap\widetilde{A}\cap D$ is the corresponding bundle over $Y_m\cap A\cap S$, we conclude by [6, III, Example 8.4], that β'' is an isomorphism and $R^i\pi_*\mathscr{I}_{Z_m\cup\widetilde{A}\cap D,\,\widetilde{A}\cap D}=0$ for $i>0$. We can apply the same argument to β, to conclude that β is an isomorphism and that $R^i\pi_*\mathscr{I}_{Z_m\cup D,\,D}=0$ for $i>0$. This shows that β' is an isomorphism and that $R^i\pi_*\mathscr{I}_{(\widetilde{A}\cup Z_m)\cap D,\,D}=0$ for $i>0$.

Now we consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I}_{A\cup Y_m\cup D} \arrow[r] \arrow[d, "\alpha"] & \mathscr{I}_{A\cup Y_{m+1}} \arrow[r] \arrow[d, "\gamma"] & \mathscr{I}_{(A\cup Y_{m+1})\cap S,\,S} \arrow[r] \arrow[d, "\beta'"] & 0 \\
0 \arrow[r] & \pi_*\mathscr{I}_{\widetilde{A}\cup Z_m}(-D) \arrow[r] & \pi_*\mathscr{I}_{\widetilde{A}\cup Z_{m+1}} \arrow[r] & \pi_*\mathscr{I}_{(\widetilde{A}\cup Z_{m+1})\cap D,\,D} &
\end{tikzcd}$$

where γ is the adjoint of the map

$$\pi^*\mathscr{I}_{A\cup Y_{m+1}} \to \mathscr{I}_{B\cup Z_{m+1}} \to \mathscr{I}_{\widetilde{A}\cup Z_{m+1}}.$$

We have just seen that β is an isomorphism and that $R^i\pi_*\mathscr{I}_{Z_m\cup D,\,D}=0$ for $i>0$, so that (2.1.1) results from

(2.1.2) $\qquad \gamma$ is an isomorphism and $R^i\pi_*\mathscr{I}_{\widetilde{A}\cup Z_m}=0$ for $i>0$.

Finally to prove (2.1.2) we consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I}_{A\cup Y_m} \arrow[r] \arrow[d, "\gamma"] & \mathscr{I}_{Y_m} \arrow[r] \arrow[d, "\delta"] & \mathscr{I}_{A\cap Y_m, A} \arrow[r] \arrow[d, "\delta''"] & 0 \\
0 \arrow[r] & \pi_*\mathscr{I}_{\widetilde{A}\cup Z_m} \arrow[r] & \pi_*\mathscr{I}_{Z_m} \arrow[r] & \pi_*\mathscr{I}_{\widetilde{A}\cap Z_m, \widetilde{A}} &
\end{tikzcd}$$

We know by [3, (3.3.4)] applied to (X, Y, S) and $(A, Y, A \cap S)$ that δ and δ'' are isomorphisms and that $R^i\pi_*\mathscr{I}_{Z_m} = 0 = R^i\pi_*\mathscr{I}_{\widetilde{A}\cap Z_m, \widetilde{A}}$ for $i > 0$. This gives (2.1.2) and proves the proposition. q.e.d.

Corollary 2.2. *With X, A, S as in 2.1, the union of A with a union Y of infinitesimal neighbourhoods (of diverse orders) of closed points in X is S-compatible.*

Proof. Since the question is local on X, we can suppose that Y is just an mth order infinitesimal neighbourhood of a closed point y of X. By 2.1, A is S-compatible, and this gives the corollary when y is not in S (cover X with $X - S$ and $X - y$). If y is in $S - A$, we can apply [3, (3.3.4)] on $X - A$ and 2.1 on $X - y$. Finally when y is in $A \cap S$ we can conclude directly by 2.1 above. q.e.d.

3. Results and notation on $D = \mathbb{P}^m \times \mathbb{P}^1$

The results of the previous section will be applied in §5 with $X = \mathbb{P}^n$ and $S = \mathbb{P}^{n-2}$. In this section we will prove a number of technical results for certain numerically adjusted closed subschemes of the exceptional divisor $D = \mathbb{P}^{n-2} \times \mathbb{P}^1$. Proposition 3.2 is used in the proof of 5.1 and 5.2 while 3.3 is used in the proof of 5.4. Proposition 3.1 is used in the proofs of 3.2 and 3.3. We begin by fixing the following notation.

Let S be the generic linear, codimension-two subspace of $X = \mathbb{P}^n$, and let $\pi\colon \widetilde{X} \to X$ be the blowing-up of X with respect to S. On $\widetilde{X}$, we let $D = \mathbb{P}^{n-2} \times \mathbb{P}^1$ denote the exceptional divisor over S, and we let $\widetilde{H}$ be the inverse image on $\widetilde{X}$ of the generic hyperplane H of X, so that $\widetilde{H}$ is just the blowing-up of H with respect to $H \cap S$. As is well known, any divisor on $\widetilde{H}$ belongs to a unique linear system of the form $|a\widetilde{H} + b(\widetilde{H} - D)|$ where $a, b \in \mathbb{Z}$. We will say that (a, b) is the *type* or *bidegree* of the divisor, and we will write $\mathscr{O}(a, b)$ for the corresponding invertible sheaf on $\widetilde{X}$. This notation is compatible with the canonical bidegree of a divisor on $D = \mathbb{P}^{n-2} \times \mathbb{P}^1$, in the sense that we have $\mathscr{O}(a, b) \otimes \mathscr{O}_D = \mathscr{O}_D(a, b)$. A divisor of type $(0, 1)$ on $\widetilde{X}$ (resp. D) will be called a *stratum* of $\widetilde{X}$ (resp. D). Such a stratum is isomorphic to $\mathbb{P}^{n-1}$ (resp. $\mathbb{P}^{n-2}$). The

inverse image of a closed point of S will be called a *fiber* of D. Clearly a fiber of D is isomorphic to $\mathbb{P}^1$. We note that the restriction of $\mathcal{O}(a, b)$ to a stratum T is just $\mathcal{O}_T(a)$ and that its restriction to a fiber $f = \mathbb{P}^1$ is $\mathcal{O}_f(b)$. Finally, the union of two distinct closed points in a stratum will be called a *doublet*.

In the following proofs we will repeatedly use *Horace's method* as introduced in [7]. This is equivalent to [3, (2.6)] in the case where one exploits a divisor. In particular there is no blowing-up and there are no conditions of S-compatibility to verify.

Lemma 3.1. *The generic, numerically $\mathcal{O}_D(1, 1)$-adjusted union of any number of closed points and fibers with a linear subspace of the form $\mathbb{P}^s \times \mathbb{P}^1$ is $\mathcal{O}_D(1, 1)$-adjusted.*

Proof. This is obvious if $s = m$. One finishes with an easy induction. q.e.d.

Proposition 3.2. *For $m \geq 2$, let D' be the generic subvariety of D of the form $\mathbb{P}^{m-2} \times \mathbb{P}^1$. We consider numerically $\mathcal{O}_D(2, 1)$-adjusted closed subschemes $V(a, b, c)$ of D, given by the generic unions of D' with a double points of D with support in D', b fibers, and c doublets. We suppose further that the following three conditions are verified:*

(3.1.1) $\qquad a \leq m - 1;$

(3.1.2) $\qquad$ *if $a = 0$ then c is even;*

(3.1.3) $\qquad$ *if $a = 1$ then $c > 0$.*

Then $V(a, b, c)$ is $\mathcal{O}_D(2, 1)$-adjusted.

Proof. Note first that the hypothesis $V(a, b, c)$ *is numerically $\mathcal{O}_D(2, 1)$-adjusted* is equivalent to

(3.1.4) $$a + b + c = 2m + 1.$$

We begin with the following lemma.

Lemma 3.2.1. *If $c \geq 2$ and either*

$$V(a, b + 2, c - 2) \quad \text{or} \quad V(a, b + 1, c - 1)$$

is $\mathcal{O}_D(2, 1)$-adjusted, then so is $V(a, b, c)$.

Proof. Since $c \geq 2$, we can choose two doublets (s_1, s_2), (t_1, t_2) in $V(a, b, c)$. On the one hand, by specialising s_1 (resp. s_2) into the fiber through t_1 (resp. t_2), the trace of the specialisation on both fibers is $\mathcal{O}(1)$-adjusted. By [3, 1.6] we can extend by these two fibers obtaining $V(a, b + 2, c - 2)$. On the other hand, by specialising s_1 into the fiber through t_1, extending by this fiber then specialising the stratum through s_1 to that through t_1 we obtain $V(a, b + 1, c - 1)$. q.e.d.

End of proof of 3.2. By Lemma 3.2.1 we can suppose that $c \leq 1$ which assures us that $b \geq 2$. We begin with the cases $a = 0, 1$ then we treat the case $a \geq 2$.

Case: $a = 0$. In this case, c is even, so we can suppose $c = 0$ by 3.2.1. This leaves us with the union of D' and $2m + 1$ fibers. The trace on any strata is clearly $\mathscr{O}(2)$-adjusted.

Case: $a = 1$. In this case we have $c > 0$, so by 3.2.1, we can suppose $c = 1$. We argue by induction on m.

If $m = 2$, then $b = 3$. Let T_1 (resp. T_2) be the stratum through the double point (resp. the doublet). The trace of $V(1, 1, 3)$ on T_1 is the generic union of a double point and three closed points, while the trace on T_2 is the generic union of six closed points. Clearly both traces are $\mathscr{O}(2)$-adjusted in $T_i = \mathbb{P}^2$. We apply [3, 2.6] by exploiting $T_1 \cup T_2$. The degue in bidegree $(2, -1)$ is trivial.

If $m > 2$, we can proceed as follows. Specialise the double point, the doublet, and $b - 2$ of the fibers into the generic divisor H' of type $(1, 0)$. Then $H' = \mathbb{P}^{m-1} \times \mathbb{P}^1$ meets D' in $\mathbb{P}^{m-3} \times \mathbb{P}^1$. We apply [3, 2.6] by exploiting H'. The dime is just the induction hypothesis while the degue, which says that the union of D' with two generic fibers is $\mathscr{O}(1, 1)$-adjusted, is a special case of 3.1.

Case: $a \geq 2$. By (3.1.1), if $m = 2$ we have $a = 0$ or 1, so for the rest of the proof we can suppose that $m \geq 3$. Using 3.1.1 we can also suppose $c \leq 1$. Now under these hypotheses we obtain from (3.1.1)

$$b = 2m + 1 - a - c > 2. \tag{3.1.5}$$

Let $H' = \mathbb{P}^{m-3} \times \mathbb{P}^1$ be the generic divisor of type $(1, 0)$ through $a - 1$ of the double points. We specialise $(b - 2)$ fibers and the c doublets into H'. Let T be the stratum through the remaining double point p, and let f_1, f_2 be the two remaining fibers. In T there is a uniquely determined line d_1 (resp. d_2) meeting both f_1 (resp. f_2) and the double point p. The trace of our specialisation on both of these lines is $\mathscr{O}(2)$-adjusted, so by [3, 1.6] we can extend by $d_1 \cup d_2$. Let V_0 be this extension of our specialisation.

Since the union of d_1 and d_2 meets H' in the generic doublet of the stratum $H' \cap T$ and $1 \leq a - 1 \leq m - 2$, we see that the trace of V_0 on H' is just $V(a - 1, b + 2, c + 1)$. Now we apply [3, (2.6)] by exploiting H'. The dime is just the induction hypothesis. The degue says that the generic union of $a - 1$ points and the two fibers f_1, f_2, with D' and the two lines d_1, d_2 is $\mathscr{O}(1, 1)$-adjusted. Noting that the trace of $D' \cup f_1 \cup f_2$

on d_i $(i = 1, 2)$ is $\mathscr{O}(1)$-adjusted, we can apply [3, (1.6)] to conclude that d_1, d_2 can be ignored in the degue which becomes a special case of 3.1. q.e.d.

Proposition 3.3. *In $D = \mathbb{P}^m \times \mathbb{P}^1$ $(m \geq 1)$ the numerically $\mathscr{O}(2, 1)$-adjusted generic union of $a \geq 2$ double points with b fibers is $\mathscr{O}(2, 1)$-adjusted.*

Proof. We argue by induction on m with three cases to consider.

Case: $a = m+1$. In this case, any subset of m double points uniquely determines a divisor of the form $H' = \mathbb{P}^{m-1} \times \mathbb{P}^1$ containing its support. If $m = 1$, then the trace on $H' = \mathbb{P}^1$ is trivially $\mathscr{O}(1)$-adjusted. If $m \geq 2$, then the trace on H' is $\mathscr{O}(2, 1)$-adjusted by the induction hypothesis. We apply [3, (2.6)] by exploiting two such divisors H_1', H_2'. The degue, which says that the generic union of two points in D is $\mathscr{O}(0, 1)$-adjusted, is trivial.

We note that this deals with the case $m = 1$.

Case: $a = m \geq 2$. We apply [3, (2.6)] by exploiting the divisor of the form $H' = \mathbb{P}^{m-1} \times \mathbb{P}^1$ uniquely determined by the support of the m double points. The dime is the preceding case and the degue says that the numerically $\mathscr{O}(1, 1)$-adjusted, generic union of m (necessarily even) points and $(m+2)/2$ fibers is $\mathscr{O}(1, 1)$-adjusted. This is a special case of 3.1.

Case: $2 \leq a \leq m - 1$. We specialise the a double points and $m(m - 1 - a)/2$ fibers into the generic subvariety of the form $D' = \mathbb{P}^{m-2} \times \mathbb{P}^1$. Applying the induction hypothesis and [3, (1.6)] we can extend by D', so that this case results from 3.1. q.e.d.

Proposition 3.4. *Let G' be the generic subvariety of $H' = \mathbb{P}^{m-1} \times \mathbb{P}^1$ $(m \geq 4)$ of the form $\mathbb{P}^{m-4} \times \mathbb{P}^1$. Then the numerically $\mathscr{O}(2, 1)$-adjusted, generic union of G' with four double points, b fibers, and $a \leq m - 3$ double points with support in G' is $\mathscr{O}(2, 1)$-adjusted.*

Proof. We argue by induction on m. If $m = 4$, then we are dealing with either the generic union of four double points, a fiber, and one double point with support in the fiber or with four double points and three fibers. Since the trace of a double point on the fiber through its support is $\mathscr{O}(1)$-adjusted, it follows from [3, (1.6)] that in the first case we can "forget" the fiber through the double point. As such, both cases result from 3.1. Now suppose that $m \geq 5$. Let H' be the generic divisor of type $\mathbb{P}^{m-1} \times \mathbb{P}^1$.

We argue by induction on a.

If $a = 0$, we specialise everything into H' except for two fibers. We apply [3, (2.6)] by exploiting the divisor H'. The dime is the induction hypothesis on m while the degue, which says that the generic union of

G' with four simple points and two fibers is $\mathscr{O}(1, 1)$-adjusted, is a special case of 3.0. If $a > 0$, we specialise everything into H' except for one of the double points with support in H'. We apply [3, (2.6)] by exploiting H'. The dime is the induction hypothesis on m, while the degue says that the generic union of G' with four points and one double point with support in G' is $\mathscr{O}(1, 1)$-adjusted. For the degue we apply [3, (2.6)] once again by exploiting the stratum T through the double point. The new dime and degue are trivial. q.e.d.

4. The initial cases $AY(6)$, $AV(6)$

In this section we will apply Horace's differential lemma [2, (1.3)] in the form given in [2, (1.4)(5)].

Proposition 4.1. *We have* $AY(6)$.

Proof. We will apply, as indicated above, Horace's differential lemma [2, (1.4)(5)]. We begin by fixing the following generic configuration of linear subspaces in $\mathbb{P}^6$:

- a flag $\mathbb{P}^4 \subset \mathbb{P}^5 \subset \mathbb{P}^6$,
- two hyperplanes A, B in $\mathbb{P}^4$ intersecting in $C = A \cap B$,
- two lines D_1, D_2 in C meeting in the point z,
- a line S (resp. T) in A (resp. B). We denote by Δ the unique line in C meeting both S and T.

Now we define the subscheme W_0 (see Figure 1) of $\mathbb{P}^6$ to be the generic union of

- three double points $a_1, a_2, a_3 \in \mathbb{P}^6$,
- two double points with support w_1, w_2 in $\mathbb{P}^5$ (we let R denote the line joining these two points and w denote the point $R \cap \mathbb{P}^4$),
- a double point with support δ in the line D_2,
- the double point with support $D_1 \cap \Delta$.

See Figure 1 for this configuration.

The union of W_0 with the generic double point of $\mathbb{P}^6$ is a specialisation of $Y(6)$, so it suffices to prove that the former union is $\mathscr{O}(3)$-adjusted.

Since the trace of W_0 on R, S, T and its trace on the three lines which join pairwise the three points a_i $(i = 1, 2, 3)$, is $\mathscr{O}(3)$-adjusted, we can extend [3, (1.6)] by these six lines. Now the trace on Δ is equally $\mathscr{O}(3)$-adjusted, and we can extend by Δ. Henceforth we let W be the extension of W_0 by these seven lines. Now we apply [2, (1.4)(5)] with degue2 and dime2. In the notation of [2] we let H be the hyperplane $\mathbb{P}^5$. For Z, we

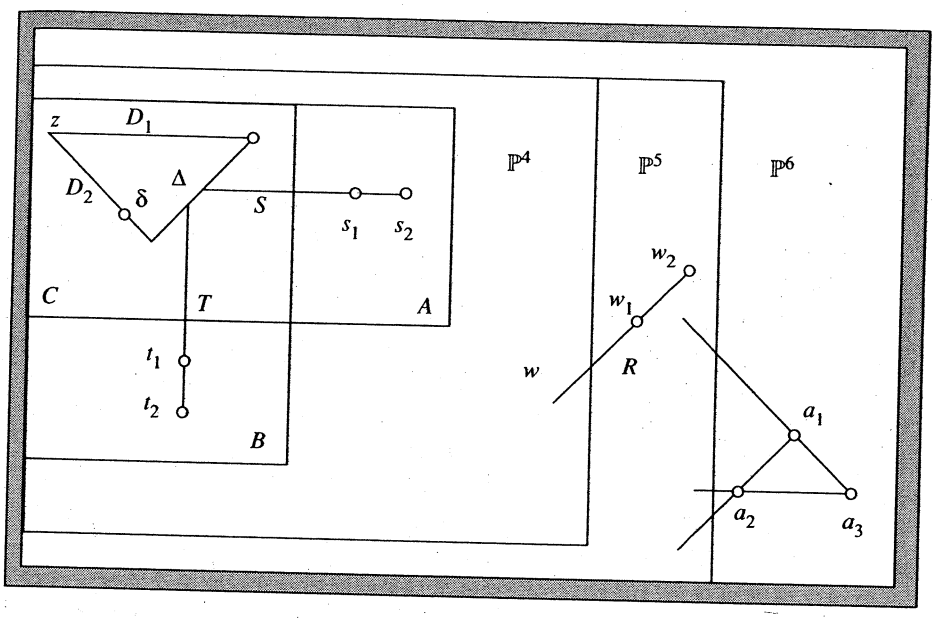


FIGURE 1

take the double point of $\mathbb{P}^5$ with support z. We let B'' be the 2-point in D_2 with support z, and we let B' be the generic 5-point in $\mathbb{P}^5$ meeting C in the 2-point of D_1 with support z. In accordance with the notation established in [2] we let W' (resp. W'') denote the trace of W on H (resp. the residual of W with respect to H). We begin by proving

degue2. *B' is contained in the base of the linear system*

$$|H^0(H, \mathscr{I}_{W''\cup Z', H} \otimes \mathscr{O}_H(3))|$$

and $W' \cup B'$ is $\mathscr{O}(2)$-adjusted $\mathbb{P}^6$.

In fact, the trace of W'' on the line D_2 is the union of the 2-point of D_2 having support δ, with the point of intersection of D_2 and Δ. This means that $W'' \cup Z'$ (note that $Z' = z$) meets D_2 in a subscheme of degree 4. As such, any cubic in $H = \mathbb{P}^5$ which vanishes on $W'' \cup Z'$ vanishes identically on D_2. That is to say, D_2 is in the base of the linear system $|H^0(H, \mathscr{I}_{W''\cup Z', H} \otimes \mathscr{O}_H(3))|$. Since B' is contained in D_2, this gives the first part of degue2.

For the second part we note that W' which is the union of the three double points a_i in $\mathbb{P}^6$ with the lines joining them, together with the points δ, $D_1 \cap \Delta$, s_1, s_2, t_1, t_2, w_1, w_2, is $\mathscr{O}(2)$-adjusted. Clearly the trace of $W' \cup B'$ on A is $\mathscr{O}(2)$-adjusted (one exploits D_2 in C, then C in A,

applying [3, (2.6)]), so by [3, (1.6)] we can extend to the union of A with three double points and the lines through them. This is $\mathcal{O}(2)$-adjusted by [3, 4.9.1].

dime2. *$W'' \cup B''$ is $\mathcal{O}(3)$-adjusted in $\mathbb{P}^5$.*

We apply [3, (2.6)] by exploiting $\mathbb{P}^4$.

The residual V of $W'' \cup B''$ with respect to $\mathbb{P}^4$ is the union of three collinear points (these are the three points of intersection of H with the three lines through the a_i), the two double points with support w_1, w_2, and the points $z, \delta, D_1 \cap \Delta, s_1, s_2, t_1, t_2$. The degue says that V is $\mathcal{O}(2)$-adjusted in $H = \mathbb{P}^5$. We specialise the three collinear points into C and the points t_1, t_2 into A, so that the trace on A is $\mathcal{O}(2)$-adjusted (exploit the three collinear points in C, then C in A). Then using [3, (1.6)] we extend by A leaving the generic union of A, the two double points with support w_1, w_2 and the line through these two points. This is $\mathcal{O}(2)$-adjusted by [3, 4.9.1] so we have the degue. We prove the dime in several steps. The result we are after is step (3).

Step (1). *The trace of $W'' \cup B''$ on C is $\mathcal{O}(3)$-adjusted.*
Since the trace on the three lines D_1, D_2, Δ and on the line through $\delta, D_1 \cap \Delta$, is $\mathcal{O}(3)$-adjusted.

Step (2). *The trace of $W'' \cup B''$ on A (hence on B) is $\mathcal{O}(3)$-adjusted.*
We apply [3, (2.6)] by exploiting C. The dime is step (1) and the degue follows from [3, 4.9.1].

Step (3). *The trace of $W'' \cup B''$ on $\mathbb{P}^4$ is $\mathcal{O}(3)$-adjusted.*
We apply [3, (2.6)] by exploiting $A \cup B$. The dime is step (2) while the degue, which says that the generic union of the five points s_1, s_2, t_1, t_2, w is $\mathcal{O}(1)$-adjusted, is trivial. q.e.d.

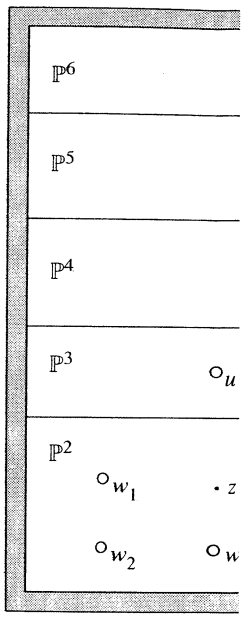
Proposition 4.2. *We have $A(6)$.*

Proof. As in the previous case we prove $A(6)$ using Horace's differential lemma [2, (1.4)(5)]. We begin as before by fixing the following generic configuration of linear subspaces in $\mathbb{P}^6$:

- a flag $\mathbb{P}^2 \subset \mathbb{P}^3 \subset \cdots \subset \mathbb{P}^6$,
- a plane $F \subset \mathbb{P}^5$ meeting $\mathbb{P}^4$ in the line Δ and $\mathbb{P}^3$ in the point δ,
- a point z in $\mathbb{P}^2$.

Now we define the subscheme W_0 (see Figure 2) in $\mathbb{P}^6$ to be the generic union of

- three double points v_1, v_2, v_3 in $\mathbb{P}^6$,
- a double point s (resp. t, resp. u) with support in $\mathbb{P}^5$ (resp. $\mathbb{P}^4$, resp. $\mathbb{P}^3$),



- the plane

See Figure 2 for th

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degue2. *B' is*

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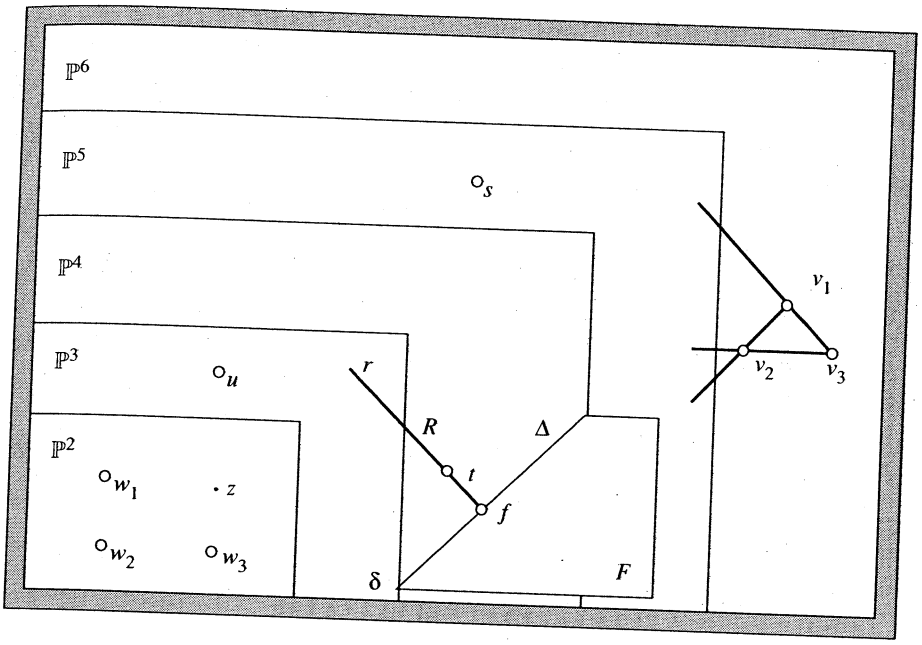

FIGURE 2

- the plane F and a double point with support f in Δ.

See Figure 2 for this configuration.

The union of W_0 with the generic double point of $\mathbb{P}^6$ is a specialisation of $W(6)$, so it is enough to show that this union is $\mathscr{O}(3)$-adjusted. The trace of W_0 on the three lines joining pairwise the three points v_i $(i = 1, 2, 3)$ and on the line R through t and f is $\mathscr{O}(3)$-adjusted, so by [3, (1.6)], we can extend W_0 to the union W of W_0 with these four lines and show that the union of W with the generic double point of $\mathbb{P}^6$ is $\mathscr{O}(3)$-adjusted. We will apply [2, (1.4)(5)] with dime2 and degue2. In the notation of [2] we let $H = \mathbb{P}^5$, and we let Z be the double point with support z. We let B' be the 3-point in $\mathbb{P}^2$ with support z and B'' the generic 4-point of $\mathbb{P}^5$ with support z. Note that the trace of B'' on $\mathbb{P}^i$ $(i = 2, \cdots, 5)$ is the generic $(i+1)$-point in $\mathbb{P}^i$ with support z, and the residual of $B'' \cap \mathbb{P}^{i+1}$ with respect to $\mathbb{P}^i$ $(i = 2, 3, 4)$ is the point z.

degue2. *B' is contained in the base of the linear system*

$$|H^0(H, \mathscr{I}_{W''\cup Z', H} \otimes \mathscr{O}_H(3))|$$

and $W' \cup B'$ is $\mathscr{O}(2)$-adjusted in $\mathbb{P}^6$.

In fact, the trace of $W'' \cup Z'$ (note that $Z' = z$) on $\mathbb{P}^2$ is the generic union of three double points and the point z. This trace is $\mathscr{O}(2)$-adjusted in $\mathbb{P}^2$ by $AY(2)$. This shows that $\mathbb{P}^2$ is contained in the base of the linear system $|H^0(H, \mathscr{I}_{W''\cup Z', H} \otimes \mathscr{O}_H(3))|$ and, since B' is contained in $\mathbb{P}^2$, this gives the first part of degue2.

For the second part, we begin by noting that $W' \cup Z'$ is the generic union of the three double points v_i $(i = 1, 2, 3)$ and the lines which join them pairwise, with the points s, t, u, f, w_i $(i = 1, 2, 3)$ and the 3-point B' with support z. The trace of $W' \cup Z'$ on $\mathbb{P}^2$ is the generic union of three points and one double point which is $\mathscr{O}(2)$-adjusted in $\mathbb{P}^2$ by [3, 4.9.1]. We extend by $\mathbb{P}^2$ and specialise s, t, f into $\mathbb{P}^3$ so that the trace on $\mathbb{P}^3$ is $\mathscr{O}(2)$-adjusted (exploit $\mathbb{P}^2$ in $\mathbb{P}^3$); then we extend by $\mathbb{P}^3$. This leaves us with the generic union of $\mathbb{P}^3$, the three double points v_i $(i = 1, 2, 3)$, and the lines which join them pairwise. This is $\mathscr{O}(2)$-adjusted by [3, 4.9.1].

dime2. *$W'' \cup Z''$ is $\mathscr{O}(3)$-adjusted in $\mathbb{P}^5$.*

We prove this in several steps by showing successively that the trace of $W'' \cup Z''$ on $\mathbb{P}^i$ is $\mathscr{O}(3)$-adjusted in $\mathbb{P}^i$ for $i = 1, \cdots, 5$.

Step (1). *The trace of $W'' \cup Z''$ on $\mathbb{P}^2$ is $\mathscr{O}(3)$-adjusted.*

Since B'' meets $\mathbb{P}^2$ in z, this is just $E(2)$.

Step (2). *The trace of $W'' \cup Z''$ on $\mathbb{P}^3$ is $\mathscr{O}(3)$-adjusted.*

We apply [3, (2.6)] by exploiting $\mathbb{P}^2$. The degue says that the union of the double point with support u with the points z, d, w_i $(i = 1, 2, 3)$ and $r = R \cap \mathbb{P}^3$ is $\mathscr{O}(2)$-adjusted in $\mathbb{P}^3$. Specialising r, d into $\mathbb{P}^2$ we can clearly extend [3, (1.6)] by $\mathbb{P}^2$ and conclude by [3, 4.9.1].

Step (3). *The trace of $W'' \cup Z''$ on $\mathbb{P}^4$ is $\mathscr{O}(3)$-adjusted.*

We apply [3, (2.6)] by exploiting $\mathbb{P}^3$. The degue is the following lemma.

Lemma 4.2.1. *The union of the points u, z, w_i $(i = 1, 2, 3)$ with the double points having support t, f and the lines R, Δ is $\mathscr{O}(2)$-adjusted in $\mathbb{P}^4$.*

Proof. We specialise $\mathbb{P}^2$ to the generic plane containing u and δ. The trace of this specialisation on $\mathbb{P}^2$ is the generic union of six points, hence is $\mathscr{O}(2)$-adjusted. By [3, (1.6)] we can extend by $\mathbb{P}^2$. Now the trace on Δ of the union of $\mathbb{P}^2$ and the double point at f is $\mathscr{O}(2)$-adjusted, so by [3, (1.6)] we can "forget" Δ. This leaves the generic union of $\mathbb{P}^2$ with the double poins at f and r, which is $\mathscr{O}(2)$-adjusted by [3, 4.9.1]. q.e.d.

Step (4). *The trace of $W'' \cup Z''$ on $\mathbb{P}^5$ is $\mathscr{O}(3)$-adjusted.*

We apply [3, (2.6)] by exploiting $\mathbb{P}^4$. The degue is the following lemma.

Lemma 4.2.2. *The generic union S of three collinear points v_i' $(i = 1, 2, 3)$, one double point at s, the plane F (meeting $\mathbb{P}^4$ in Δ and $\mathbb{P}^3$ in δ), the points t in $\mathbb{P}^4$, u in $\mathbb{P}^3$, and z, w_i $(i = 1, 2, 3)$ in $\mathbb{P}^2$ is $\mathscr{O}(2)$-adjusted in $\mathbb{P}^5$.*

Proof. We specialise everything into $\mathbb{P}^4$, except for the double point at s. We do this in such a way that F meets $\mathbb{P}^3$ in Δ and $\mathbb{P}^2$ in δ, while R becomes a generic line in $\mathbb{P}^3$. We specialise v_3' to $\mathbb{P}^3$. Now specialise R into $\mathbb{P}^2$, so that t is in $\mathbb{P}^2$ and R joins t and δ. Let S_0 be this specialisation of S. The trace of S_0 on $\mathbb{P}^2$ is six generic points of $\mathbb{P}^2$ so by [3, (1.6)] we can extend by $\mathbb{P}^2$. Since Δ, u, v_3' are not contained in a plane we can extend by $\mathbb{P}^3$ and finally, since F', v_1', v_2' are not contained in a hyperplane of $\mathbb{P}^4$, we can extend by $\mathbb{P}^4$. This leaves us with the union of $\mathbb{P}^4$ with the double point at s, which is $\mathscr{O}(2)$-adjusted in $\mathbb{P}^5$ by [3, 4.9.1]. q.e.d.

5. The induction

In this section we use the notation introduced at the beginning of §3. Throughout, $S = \mathbb{P}^{n-2}$ is the generic, codimension-two, linear subspace of $X = \mathbb{P}^n$, and $D = \mathbb{P}^{n-2} \times \mathbb{P}^1$ is the exceptional divisor on the blowing-up of X in S.

Proposition 5.1. *For $l \geq 1$ and $n = 3l - 1$, we have $AY(n-2) \Rightarrow AW(n)$.*

Proof. In this case $b_{n-2} = 0$ and the trace of $W(n)$ on S is just $Y(n-2)$. Using the notation introduced at the beginning of §3, we will apply [3, (2.6)] by exploiting S. The dime is just $AY(n-2)$ while the degue is

Lemma 5.1.1. *The generic, numerically $\mathscr{O}(2, 1)$-adjusted union of $c_n = a_n - a_{n-2}$ double points with a b_n-point and a_{n-2} fibers in D in $\mathscr{O}(2, 1)$-adjusted.*

Proof. We specialise $c_n - 2$ of the double points to have their support in D, and we let T_1, T_2 be the two strata through the remaining two double points. By 3.2, the trace on D is $\mathscr{O}(2, 1)$-adjusted, so by [3, (1.6)] we can extend by D. Now the traces on T_1 and T_2 are $\mathscr{O}(2)$-adjusted by [3, 4.9.1]. We now reapply [3, (2.6)] by exploiting the union of D with T_1, T_2. The degue, which says that the generic, numerically $\mathscr{O}(1, 0)$-adjusted union of two points and a b_n-point in $\widetilde{X}$, with c_n points in D, is $\mathscr{O}(1, 0)$-adjusted is trivial. q.e.d.

Proposition 5.2. *For $l \geq 1$ and $n = 3l$, we have $AY(n-2) \Rightarrow AW(n)$.*

Proof. In this case $b_n = 0 = b_{n-2}$. As in the previous case the trace of $W(n)$ on S is just $Y(n)$. We apply [3, (2.6)] by exploiting S. The dime is just $AY(n-2)$ while the degue is the

Lemma 5.2.1. *The generic, numerically $\mathcal{O}(2, 1)$-adjusted union of $c_n = a_n - a_{n-2}$ double points with a_{n-2} fibers in D is $\mathcal{O}(2, 1)$-adjusted.*

Proof. We choose $c_n - 1$ fibers and $c_n - 1$ double points which we put in one-to-one correspondence. (This is possible because $a_{n-2} \geq n - 1 \geq c_n - 1 \geq 2$). For each couple we specialise the double point to the corresponding fiber along the germ of the generic smooth curve meeting the fiber. Let S be this specialisation. The trace of S on D is the generic union of $c_n - 1$ *fibered double points* (i.e., the schematic union of a fiber and a double point with support in the fiber) with $a_{n-2} - c_n + 1$ fibers. One easily verifies that this trace is numerially $\mathcal{O}(2, 1)$-adjusted. We now apply [3, (2.6)] by exploiting D.

Since the trace of a double point in D on a fiber through its support is $\mathcal{O}(1)$-adjusted in the fiber, it follows from [3, (1.6)] that for the dime we can "forget" the fibers through the fibered double points and only consider the union of $c_n - 1$ double points with $a_{n-2} - c_n + 1$ fibers. As we remarked above, $2 \geq c_n - 1 \geq n - 1$ so the dime follows from 3.2. The degue is

Lemma 5.2.2. *The generic, numerically $\mathcal{O}(1, 2)$-adjusted union of a double point of $\tilde{X}$ with $(c_n - 1)$ 3-points, each meeting a fiber of D in a 2-point, is $\mathcal{O}(1, 2)$-adjusted in $\tilde{X}$.*

Proof. We apply [3, (2.6)] by exploiting the stratum through the double point. We must then show that the generic, numerically $\mathcal{O}(1, 1)$-adjusted union of a point of $\tilde{X}$ with $(c_n - 1)$ 3-points, each meeting a fiber of D in a 2-point, is $\mathcal{O}(1, 1)$-adjusted in $\tilde{X}$. By [3, (1.6)], we can extend by the fibers through the support of the 3-points.

We let T_1 and T_2 be two generic strata of D. For $i = 1, 2$, we specialise $(c_n - 1)/2$ of the 3-points to have support in T_i, and we specialise into D, $(c_n - 3)/2$ of the fibered 3-points having support in T_i. Now, it is clear that the traces on the strata T_1 and T_2, each of which is a numerically $\mathcal{O}(1)$-adjusted union of $((c_n - 3)/2)$ 2-points and $(c_n - 1)/2$ points, are $\mathcal{O}(1)$-adjusted. We apply [3, (2.6)] by exploiting D. The dime is easily proven by exploiting T_1 and T_2. The degue, which says that the generic union of one point in $\tilde{X}$ with two points in D is $\mathcal{O}(0, 2)$-adjusted, is trivial. q.e.d.

Proposition 5.3. *For $l \geq 2$ and $n = 3l + 1$, we have $AV(n-1) + AW(n-4) \Rightarrow AW(n)$. Also $AV(9) + AY(6) \Rightarrow AW(10)$.*

Proof. We first look at the special case $AV(9)+AY(6) \Rightarrow AW(10)$. In this case $W(10)\cap G$ is generic in G, since there are only six double points with support in $G\cap H \simeq \mathbb{P}^5$. We conclude from $AY(6)$ that $W(10)\cap G$ is $\mathscr{O}(3)$-adjusted in G.

For $n = 3l+1$ $(l \geq 2)$, the trace $W(n)\cap G$ can be specialised, in G, to $W(n-4)$. In fact it is enough to specialise into the generic hyperplane of $G\cap H$ all the double points in $G\cap H$ and one other double point with support in G. This means that $AW(n-4)$ implies that the trace $W(n)\cap G$ is $\mathscr{O}(3)$-adjusted in G.

Now, in both of the above cases, we can extend by G. We apply [3, (2.6)] by exploiting H. The dime is $AV(n-1)$, and the degue follows from the more general

Lemma 5.3.1. *Let $X = \mathbb{P}^n$ $(n \geq 4)$ and let H and G be the generic linear subspaces of X of codimension one and four respectively. Then the generic, numerically $\mathscr{O}(2)$-adjusted union of G, $1 \leq a \leq (n-3)$ double points with support in G, and s points in H is $\mathscr{O}(2)$-adjusted.*

Proof. If $n = 4$, then we simply have the union of a double point with ten points in H which is $\mathscr{O}(2)$-adjusted by [3, 4.9].

Now suppose that $n > 4$ and that the lemma is true in dimension $n-1$. If $a = n-3$, then the trace on G of the union of the double points with the lines joining them pairwise is $\mathscr{O}(2)$-adjusted by [3, 4.9]. As such, using [3, (1.6)], we can replace the union of G with the double points by the union of the double points with the lines joining them pairwise. In this case we conclude by [3, 4.9]. If $1 \leq a < (n-3)$, then we consider the generic hyperplane H_0 through G into which we specialise $s-4$ of the points in H. We apply [3, (2.6)] by exploiting H_0. The dime is the induction hypothesis while the degue is trivial. q.e.d.

Proposition 5.4. *For $l \geq 3$ and $n = 3l$, we have*

$$AY(n-6) + AW(n-2) \Rightarrow AV(n).$$

Proof. We specialise $a_{n-2} - a_{n-6}$ of the generic double points of $\mathbb{P}^n$ into S. Then we specialise $a_{n-6} = f_n - (n-6)/3$ of the double points with support in G into $G_0 = G\cap H \simeq \mathbb{P}^{n-6}$. We denote this specialisation by $U(n)$. We claim that the trace $U(n)\cap S$ is $\mathscr{O}(3)$-adjusted in S. In fact this trace is the generic union of $a_{n-2} - a_{n-6} = e_{n-4}$ double points in S with G_0 and a_{n-6} double points in G_0. Since a_{n-6} double points in G_0 are $\mathscr{O}(3)$-adjusted by $AY(n-6)$, we conclude by [3, (1.6)] that it is enough to show that the union of $a_{n-2} - a_{n-6}$ double points in S with a_{n-6} double points in G_0 is $\mathscr{O}(3)$-adjusted in S. However, this union

can be specialised to $Y(n-2)$ in $S = \mathbb{P}^{n-2}$, so it is $\mathcal{O}(3)$-adjusted by $AW(n-2)$.

Now we apply [3, (2.6)] to $U(n)$, by exploiting S. The subscheme $U(n)$ is S-compatible by 2.2, and the dime was proven in the preceding paragraph. The degue is the following

Lemma 5.4.1. *Let $\widetilde{X}$ be the blowing-up of $\mathbb{P}^n$ in S. Then in the notation of §3, the generic union of four double points and e_{n-4} fibers, with $\widetilde{G}$ (the blowing-up of G in $S \cap G$) and $(n-6)/3$ double points with support in $\widetilde{G}$ is $\mathcal{O}(2, 1)$-adjusted in $\widetilde{X}$.*

Proof. We specialise the four generic double points into D. Then, of the double points supported in $\widetilde{G}$, we specialise $((n-6)/3)-1$ into $D \cap \widetilde{G}$. Now we apply [3, (2.6)] to this specialisation by exploiting D. The dime is just 3.3, while the degue is

Lemma 5.4.2. *The generic union of four points in D with $\widetilde{G}$ and one double point in $\widetilde{G}$ is $\mathcal{O}(1, 2)$-adjusted in $\widetilde{X}$.*

Proof. We apply [3, (2.6)] by exploiting the stratum through the double point, so we are left with proving that the generic union of $\widetilde{G}$ with four points in D is $\mathcal{O}(1, 1)$-adjusted in $\widetilde{X}$. The trace on D is $\mathcal{O}(1, 1)$-adjusted in D by 3.0, so by applying [3, (2.6)] once again, this time exploiting D, we are left with proving that $\widetilde{G}$ is $\mathcal{O}(0, 2)$-adjusted in $\widetilde{X}$, but this is trivial. q.e.d.

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