

FROM 12 TO 6: SHARPENING THE THREE-CHARGE BOUND IN MAXWELL'S PROBLEM

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ABSTRACT. In [4] we proved that, for every $\alpha > 0$, the potential of three positive point charges has at most 12 nondegenerate equilibrium points. We also observed that the same method would give the sharper bound 6 if a certain auxiliary polynomial system $Q = R = 0$ had at least four solutions, counted with multiplicity, in each open quadrant of the (f, g) -plane. Here we prove this four-solution statement. The main new ingredient is a separation argument at the unique saddle point of a separated-variable first integral. Consequently, the upper bound for three charges improves from 12 to 6.

1. INTRODUCTION

In § 113 of his treatise [6], J. C. Maxwell discussed the maximal number of isolated equilibrium points generated by point charges in $\mathbb{R}^3$ and asserted the bound $(k-1)^2$ for k charges. The argument supporting this assertion is incomplete; see the appendix to [4].¹ We refer to the problem of bounding the number of equilibrium points of a finite configuration of point charges as *Maxwell's problem*.

A general finiteness bound for a natural generalization of Maxwell's problem was established in Theorem 1.5(a) of [4] and subsequently improved, for example, in [9]. Very recently Arathoon, Ball, and Kvalheim [1], building on ideas from [3], subsequently constructed five positive charges in $\mathbb{R}^3$ whose Coulomb potential has at least 24 nondegenerate critical points. Thus Maxwell's proposed bound is false in general.

For three positive charges, Theorem 1.5(b) of [4] gives the upper bound 12, whereas Maxwell's proposed bound is 4. Our goal is to improve 12 to 6. For three equal charges, the bound 4 is known [7].

Our set-up is as follows. We may assume that the three charges are noncollinear since the collinear situation is trivial. In the noncollinear case, place three positive charges $\zeta_1, \zeta_2, \zeta_3 = 1$ at $(0, 0)$, $(1, 0)$, and (a, b) , respectively, with $b \neq 0$. After normalizing the third charge, the potential has the form

$$V_\alpha = \sum_{i=1}^3 \zeta_i \rho_i^{-\alpha}, \quad \rho_i = r_i^2, \quad \alpha > 0. \quad (1.1)$$

Every critical point lies in the affine plane of the charges; in the noncollinear case it lies in the open triangle that they span. As recalled in [4], Morse-theoretic

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¹For historical discussion of contemporary criticism of inconsistencies and incomplete arguments in Maxwell's treatise, including comments by J. J. Thomson, J. C. McConnel, H. Poincaré, and O. Heaviside, see [8, Chapter 6].

considerations give the parity relation

$$\#\{\text{equilibria}\} = 2m_0 + 2, \quad (1.2)$$

where m_0 is the number of planar local minima. In particular, the number of nondegenerate equilibria is even. The main result of this note is as follows.

Theorem 1.1. *For every $n \geq 2$ and every $\alpha > 0$, a potential generated by three positive point charges in $\mathbb{R}^n$ has at most 6 nondegenerate equilibrium points.*

The proof retains the two Rolle–Khovanskii reductions and the mixed-volume count from [4]. The only new step is a four-contact lemma for a compact component of the auxiliary contact curve. We first recall the reduction and isolate the statement that must be proved.

2. REDUCTION OF THE THREE-CHARGE PROBLEM

Following [4], introduce

$$f = \left(\frac{\rho_1}{\rho_2}\right)^{\alpha+1}, \quad g = \left(\frac{\rho_1}{\rho_3}\right)^{\alpha+1}. \quad (2.1)$$

Critical points of V_α correspond bijectively to intersections in the positive quadrant $\mathbb{R}_{>0}^2$ of the curves

$$\gamma_1 = \left\{ f^{1/(\alpha+1)} \frac{\xi_2}{\xi_1} = 1 \right\}, \quad (2.2)$$

$$\gamma_2 = \left\{ g^{-1/(\alpha+1)} \xi_2 = f^{-1/(\alpha+1)} \xi_3 \right\}, \quad (2.3)$$

where

$$\xi_1(f, g) = (ag + \zeta_2 f)^2 + b^2 g^2, \quad (2.4)$$

$$\xi_2(g) = ((a-1)g - \zeta_1)^2 + b^2 g^2, \quad (2.5)$$

$$\xi_3(f) = ((a-1)\zeta_2 f + a\zeta_1)^2 + b^2 (\zeta_2 f + \zeta_1)^2. \quad (2.6)$$

The form ξ_1 is positive definite, and ξ_2, ξ_3 are strictly positive quadratic polynomials on the real line.

The curves γ_1 and γ_2 are integral curves of closed logarithmic one-forms

$$\eta_1 = d \log \Xi, \quad \eta_2 = d \log \Psi, \quad (2.7)$$

where Ξ and Ψ are defining functions for (2.2) and (2.3). They are separating solutions in Khovanskii's sense, so the Rolle–Khovanskii theorem applies twice; see [4, 5].

Let

$$N = \#(\gamma_1 \cap \gamma_2 \cap \mathbb{R}_{>0}^2). \quad (2.8)$$

For generic parameters, all intersections considered below are isolated and are counted with their local intersection multiplicities. The first Rolle–Khovanskii step gives

$$N \leq N_1 + N_2, \quad (2.9)$$

where $N_1 = 2$ is the number of unbounded components of γ_2 in $\mathbb{R}_{>0}^2$ and

$$N_2 = \#(\gamma_2 \cap \Gamma \cap \mathbb{R}_{>0}^2). \quad (2.10)$$

Here $\Gamma = \{Q = 0\}$ is the contact curve defined by

$$Q \, df \wedge dg = fg \, \xi_1 \xi_2 \xi_3 \, \eta_1 \wedge \eta_2. \quad (2.11)$$

Its Newton polygon is

$$\text{NP}(Q) = \{(p, q) : 2 \leq p + q \leq 6, 0 \leq p, q \leq 4\}. \quad (2.12)$$

The second Rolle–Khovanskii step gives

$$N_2 \leq N_3 + N_4, \quad (2.13)$$

where $N_3 = 0$ and

$$N_4 = \#(\Gamma \cap \{R = 0\} \cap \mathbb{R}_{>0}^2). \quad (2.14)$$

The equality $N_3 = 0$ follows because Γ has no unbounded component and its only point on the coordinate axes is the origin, which is isolated in Γ . The polynomial R is defined by

$$R \, df \wedge dg = fg \, \xi_2 \xi_3 \, dQ \wedge \eta_2. \quad (2.15)$$

After replacing R by a reduction $\tilde{R} = R - qQ$, the Bernstein–Kushnirenko theorem gives

$$\#\{Q = \tilde{R} = 0 \text{ in } (\mathbb{C}^*)^2\} \leq 2 \text{Vol}(\text{NP}(Q), \text{NP}(\tilde{R})) = 28. \quad (2.16)$$

The count is taken with intersection multiplicity; see [2, 4]. Two nonreal conjugate common zeros of ξ_1, ξ_2, ξ_3 contribute multiplicity at least 6 each. The proof in [4] also finds at least two real solutions of $Q = R = 0$ in each of the three open quadrants different from $\mathbb{R}_{>0}^2$. Hence

$$N_4 \leq 28 - 12 - 6 = 10, \quad N \leq 2 + 0 + 10 = 12. \quad (2.17)$$

The improvement rests on the following statement.

Proposition 2.1 (Four-solution proposition). *Fix an open quadrant*

$$\mathcal{Q}_\sigma = I \times J, \quad I, J \in \{(0, \infty), (-\infty, 0)\}.$$

Let p be the unique zero of η_2 in this quadrant, whose existence is established in Lemma 3.1, and let O be the component of $\Gamma \cap \mathcal{Q}_\sigma$ containing p . Assume that $Q = R = 0$ has finitely many solutions in $(\mathbb{C}^)^2$ and that Γ is nonsingular along O . Then the system $Q = R = 0$ has total intersection multiplicity at least 4 in $\mathcal{Q}_\sigma$.*

Assuming Proposition 2.1 for generic parameters, the three quadrants outside $\mathbb{R}_{>0}^2$ contribute at least 12 to the left-hand side of (2.16). Therefore

$$N_4 \leq 28 - 12 - 12 = 4, \quad (2.18)$$

and (2.9)–(2.13) give

$$N \leq 2 + 0 + 4 = 6. \quad (2.19)$$

Sections 3 and 4 prove Proposition 2.1; Section 5 justifies the genericity assumptions and completes the proof of Theorem 1.1.

3. A SEPARATED SADDLE IN EVERY QUADRANT

The one-form η_2 separates in the variables f and g :

$$\eta_2 = P(f) \, df + S(g) \, dg, \quad (3.1)$$

where

$$P(f) = -\frac{1}{(\alpha+1)f} + \frac{\xi'_3(f)}{\xi_3(f)}, \quad S(g) = \frac{1}{(\alpha+1)g} - \frac{\xi'_2(g)}{\xi_2(g)}. \quad (3.2)$$

Write

$$\xi_3(f) = Af^2 + Bf + C, \quad \xi_2(g) = \hat{A}g^2 + \hat{B}g + \hat{C},$$

where $A, C, \hat{A}, \hat{C} > 0$. Clearing denominators in (3.2) gives

$$P(f) = \frac{\pi(f)}{(\alpha+1)f\xi_3(f)}, \quad S(g) = -\frac{\sigma(g)}{(\alpha+1)g\xi_2(g)}, \quad (3.3)$$

with

$$\pi(f) = (\alpha+1)f\xi_3'(f) - \xi_3(f) = (2\alpha+1)Af^2 + \alpha Bf - C, \quad (3.4)$$

$$\sigma(g) = (\alpha+1)g\xi_2'(g) - \xi_2(g) = (2\alpha+1)\hat{A}g^2 + \alpha\hat{B}g - \hat{C}. \quad (3.5)$$

The product of the two roots of π is $-C/((2\alpha+1)A) < 0$. Thus π has one simple positive root and one simple negative root. The same argument applies to σ .

Lemma 3.1. *On every open quadrant $I \times J$, the form η_2 has a single-valued real-analytic primitive*

$$\Phi(f, g) = u(f) + s(g). \quad (3.6)$$

This primitive has exactly one critical point $p = (f_0, g_0)$ in the quadrant, and p is a nondegenerate saddle. Moreover, u is a one-well function: it decreases strictly to f_0 , increases strictly after f_0 , and tends to $+\infty$ at both ends of I . Similarly, s is a one-hill function and tends to $-\infty$ at both ends of J .

Proof. The functions $f\xi_3(f)$ and $g\xi_2(g)$ do not vanish on I and J . Hence P and S admit single-valued real-analytic primitives on these intervals.

Let $f_- < 0 < f_+$ be the roots of π . On $(0, \infty)$, the denominator of P is positive, so $P < 0$ on $(0, f_+)$ and $P > 0$ on (f_+, ∞) . On $(-\infty, 0)$, the sign of the factor f reverses the sign of the denominator; because π is positive on $(-\infty, f_-)$ and negative on $(f_-, 0)$, one again has $P < 0$ before the root and $P > 0$ after it. Consequently, u decreases and then increases on either interval. Furthermore,

$$\begin{aligned} P(f) &\sim -\frac{1}{(\alpha+1)f} && \text{as } f \rightarrow 0, \\ P(f) &\sim \frac{2\alpha+1}{(\alpha+1)f} && \text{as } |f| \rightarrow \infty. \end{aligned}$$

After integration, these asymptotics show that $u(f) \rightarrow +\infty$ at both ends of I .

The same sign analysis, applied to the second expression in (3.3), shows that s first increases and then decreases; the analogous asymptotics give $s(g) \rightarrow -\infty$ at both ends of J . Hence the unique zeros f_0 of P and g_0 of S give the unique critical point of Φ in the quadrant.

At that point,

$$u''(f_0) = \frac{\pi'(f_0)}{(\alpha+1)f_0\xi_3(f_0)} > 0, \quad s''(g_0) < 0. \quad (3.7)$$

Indeed, $\pi'(f_0)$ and f_0 have the same sign, and the corresponding statement for σ gives the second inequality. Therefore

$$\text{Hess}_p \Phi = \begin{pmatrix} u''(f_0) & 0 \\ 0 & s''(g_0) \end{pmatrix} \quad (3.8)$$

has signature $(+, -)$. $\square$

Set

$$c_0 = \Phi(p) = u(f_0) + s(g_0), \quad H = \text{diag}(u''(f_0), s''(g_0)). \quad (3.9)$$

Lemma 3.2 (Separation lemma). *The set $\{\Phi > c_0\} \cap (I \times J)$ has exactly two connected components,*

$$L = \{f < f_0\} \cap \{\Phi > c_0\}, \quad \mathcal{R} = \{f > f_0\} \cap \{\Phi > c_0\}. \quad (3.10)$$

In particular, no connected subset of $\{\Phi > c_0\}$ meets both sides of the line $\{f = f_0\}$. Symmetrically, $\{\Phi < c_0\}$ has exactly two connected components separated by $\{g = g_0\}$.

Proof. On the line $f = f_0$,

$$\Phi(f_0, g) = u(f_0) + s(g) \leq u(f_0) + s(g_0) = c_0, \quad (3.11)$$

with equality only at $g = g_0$. Thus $\{\Phi > c_0\}$ does not meet $\{f = f_0\}$, and the sets in (3.10) are disjoint open sets whose union is $\{\Phi > c_0\}$. Both are nonempty because u tends to $+\infty$ on either side of f_0 .

We verify that L is connected. For fixed g , its horizontal slice is an interval adjacent to the left endpoint of I :

$$\{f \in I : f < f_0, u(f) > c_0 - s(g)\} = \{f \in I : f < \beta(g)\}, \quad (3.12)$$

where

$$\beta(g) = \left(u|_{I \cap (-\infty, f_0)}\right)^{-1}(c_0 - s(g)). \quad (3.13)$$

The function β is continuous. Given two points of L , the g -coordinates between them form a compact interval. Choose an f -coordinate lying strictly to the left of the minimum of β on that interval. Moving horizontally to this coordinate, then vertically, and finally horizontally to the second point produces a path in L . Thus L is connected. The same argument applies to $\mathcal{R}$. Interchanging the roles of the one-well function u and the one-hill function s proves the assertion for $\{\Phi < c_0\}$. $\square$

4. THE OVAL AND ITS FOUR CONTACTS

The following identity is proved in [4]:

$$Q = -\frac{1+2\alpha}{(\alpha+1)^2} \xi_1 \xi_2 \xi_3 - fgQ_1, \quad (4.1)$$

where $\text{NP}(fgQ_1)$ lies strictly inside $\text{NP}(\xi_1 \xi_2 \xi_3)$. It follows that $\Gamma = \{Q = 0\}$ has no unbounded component, that $Q < 0$ on each coordinate axis away from the origin, and that the origin is an isolated point of Γ . Since $\eta_2(p) = 0$, equation (2.11) also gives $Q(p) = 0$.

Lemma 4.1. *The component O defined in Proposition 2.1 is compact and contained in the open quadrant. Under the assumptions of Proposition 2.1, O is a smooth closed curve, $\Phi|_O$ is nonconstant, and the zeros of R on O are exactly the critical points of $\Phi|_O$, with the same multiplicities. More precisely, if γ is a regular parametrization of O , then*

$$R(\gamma(t)) = \rho(t)(\Phi \circ \gamma)'(t) \quad (4.2)$$

for a nowhere-vanishing analytic function ρ .

Proof. The absence of unbounded components makes O bounded. It cannot accumulate at a nonzero point of a coordinate axis because Q is strictly negative there, and it cannot accumulate at the origin because the origin is an isolated zero of Q . Hence O is compact and lies in the open quadrant. The regularity assumption makes it a compact connected one-dimensional manifold without boundary, and therefore a

smooth closed curve. If $\Phi|_O$ were constant, then (4.2) below would imply that Q and R share the whole curve O , contrary to the finiteness assumption.

Let γ be a regular analytic parametrization of O , and choose a unit normal field N along it. Evaluating (2.15) on $(\gamma'(t), N(t))$ gives

$$R(\gamma(t)) \, df \wedge dg(\gamma'(t), N(t)) = -fg\xi_2\xi_3 \, dQ(N(t)) (\Phi \circ \gamma)'(t), \quad (4.3)$$

up to the fixed orientation sign. The factors $df \wedge dg(\gamma', N) = \pm|\gamma'|$, $dQ(N) = \pm|\nabla Q|$, and $fg\xi_2\xi_3$ are all nonzero. This proves (4.2), including equality of the vanishing orders. $\square$

The saddle $p \in O$ is a critical point of $\Phi|_O$ because $d\Phi(p) = 0$. If v_0 is a nonzero tangent vector to O at p , then p is a simple zero of $(\Phi \circ \gamma)'$ precisely when

$$v_0^\top H v_0 \neq 0. \quad (4.4)$$

Lemma 4.2 (Four-contact lemma). *The total multiplicity of the zeros of R on O is at least 4.*

Proof. The function $h = \Phi|_O$ is a nonconstant real-analytic function on a circle. Its critical points are isolated. A zero of odd order of h' is exactly a point at which h' changes sign; hence the number of odd-order zeros of h' around the circle is even. Moreover, h attains its minimum and maximum at two distinct critical points.

Suppose, for a contradiction, that the total multiplicity of the zeros of h' is at most 3. Since p is one of these zeros and there are at least two distinct critical points, only the following possibilities can occur:

- (i) exactly two critical points, both simple;
- (ii) exactly two critical points, of total multiplicity 3;
- (iii) exactly three critical points, all simple.

Cases (ii) and (iii) are impossible because each gives an odd number of odd-order zeros of h' .

It remains to exclude case (i). The two critical points are the unique global minimum and maximum of h . First suppose that p is the minimum. Then

$$\Phi > c_0 \quad \text{on } O \setminus \{p\}. \quad (4.5)$$

The punctured circle $O \setminus \{p\}$ is connected, so the separation lemma implies that it is contained entirely in L or entirely in $\mathcal{R}$.

Because p is a simple critical point of h and a local minimum,

$$v_0^\top H v_0 > 0. \quad (4.6)$$

The positive cone of the quadratic form $H = \text{diag}(+, -)$ contains no vertical vector. Thus the f -component of v_0 is nonzero. The two ends of the arc $O \setminus \{p\}$ approach p with opposite tangent directions v_0 and $-v_0$, and therefore with opposite signs of $f - f_0$. One end lies in L and the other in $\mathcal{R}$, a contradiction.

If p is the maximum, the same argument uses the two components of $\{\Phi < c_0\}$ separated by the line $g = g_0$; in this case the negative cone of H contains no horizontal vector. Hence case (i) is also impossible, and the total multiplicity is at least 4. $\square$

At a smooth point of Γ , the local intersection multiplicity of $\{Q = 0\}$ and $\{R = 0\}$ equals the vanishing order of R along O . Lemmas 4.1 and 4.2 therefore prove Proposition 2.1.

5. GENERICITY AND COMPLETION OF THE PROOF

We now verify that the hypotheses of Proposition 2.1 hold on a dense open set of parameters. Put

$$\theta = (a, b, \zeta_1, \zeta_2, \alpha). \quad (5.1)$$

After multiplication by harmless powers of $\alpha + 1$, the coefficients of Q and R are polynomial functions of θ .

Proposition 5.1. *There is a nonempty Zariski-open subset U of the complex parameter space with the following properties:*

- (G1) *for every $\theta \in U$, the system $Q = R = 0$ has finitely many solutions in $(\mathbb{C}^*)^2$;*
- (G2) *for every admissible real $\theta \in U$, the curve Γ is nonsingular in $(\mathbb{R}^*)^2$.*

Moreover, U meets the admissible real parameter set $b \neq 0$, $\zeta_1, \zeta_2, \alpha > 0$ in a Euclidean-dense subset.

Proof. For (G1), note that, for the fixed supports of Q and R , the pairs having a nonconstant common factor form a Zariski-closed subset of the coefficient space. Indeed, this set is the finite union, over the possible bidegrees of a common factor, of the projective images of the corresponding multiplication maps. At

$$\theta_* = \left(\frac{3}{10}, \frac{7}{5}, \frac{7}{10}, \frac{14}{5}, \frac{1}{2} \right), \quad (5.2)$$

an exact computation over $\mathbb{Q}$ gives $\gcd(Q, R) = 1$ in $\mathbb{Q}[f, g]$. Thus (G1) holds on a nonempty Zariski-open set.

For (G2), set

$$r_2(f) = \text{Res}_g(Q, Q_g), \quad r_4(f) = \text{Res}_g(Q_f, Q_g). \quad (5.3)$$

At a singular point of Γ in $(\mathbb{C}^*)^2$, the f -coordinate is a common zero of r_2 and r_4 . The factor f is a structural common factor, coming from the isolated zero at the origin and the support condition (2.12). At each of the two common zeros of ξ_1, ξ_2, ξ_3 recalled in Section 2, the curve Γ is singular; this is part of the multiplicity-six analysis in [4]. The two corresponding f -coordinates are precisely the roots of ξ_3 . Hence ξ_3 is also a structural common factor. Thus

$$f\xi_3(f) \mid \gcd(r_2, r_4). \quad (5.4)$$

At the witness (5.2), exact arithmetic gives

$$\deg_f r_2 = 24, \quad \deg_f r_4 = 21, \quad \gcd(r_2, r_4) = f\xi_3(f) \quad (5.5)$$

up to a nonzero constant. The leading coefficients responsible for the two displayed degrees are nonzero at the witness. Hence these degrees remain constant on a Zariski-open neighborhood of the witness. The subresultant criterion, together with (5.4), then shows that

$$\gcd(r_2, r_4) = f\xi_3(f) \quad (5.6)$$

throughout a nonempty Zariski-open set.

For admissible real parameters, $\xi_3(f) > 0$ for every real f , and a point of $(\mathbb{R}^*)^2$ has $f \neq 0$. Thus (5.6) rules out real singular points of Γ in the torus, proving (G2). Since the witness (5.2) is admissible and the complement of U is a proper algebraic subset, U intersects the admissible real parameter set in a Euclidean-dense subset. $\square$

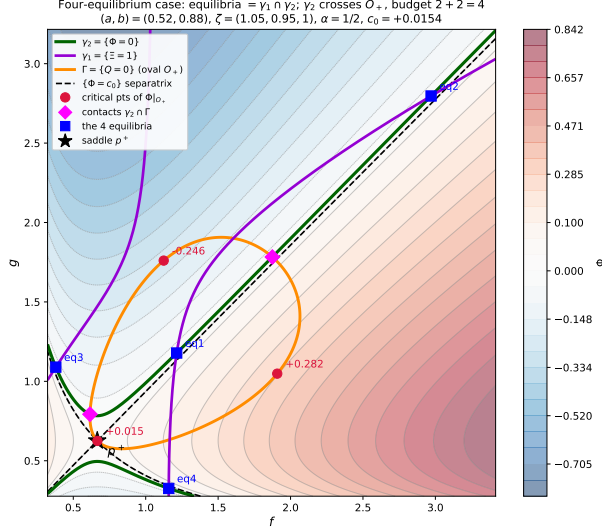


FIGURE 1. A positive-quadrant configuration with four equilibria. The orange curve is the oval $O^+ \subset \Gamma$; the green and purple curves are γ_2 and γ_1 , respectively; and the dashed curve is the separatrix level $\Phi = c_0$. Red points mark critical points of $\Phi|_{O^+}$, magenta diamonds mark intersections $\gamma_2 \cap \Gamma$, blue squares mark the four equilibria $\gamma_1 \cap \gamma_2$, and the black star marks the saddle p^+ .

The exact calculations at (5.2) are short and use only the definitions (2.4)–(2.15). A reproducible exact-arithmetic script, `verify_genericity.py`, accompanies the source of this article.

Proof of Theorem 1.1. For parameters in the dense open set of Proposition 5.1, Proposition 2.1 applies in every open quadrant. The argument in (2.18)–(2.19) therefore gives the bound 6 for generic noncollinear configurations.

Now let an arbitrary admissible potential have k nondegenerate equilibrium points. By the implicit function theorem, all k points persist under sufficiently small perturbations of the positions, the charges, and, if necessary, α . The perturbed parameters may be chosen noncollinear and in the dense open set of Proposition 5.1, while preserving positivity of the charges and the inequality $\alpha > 0$. The perturbed configuration has at least k equilibrium points and at most 6, so $k \leq 6$. Finally, write a point as $x = y + z$, where y belongs to the affine span of the charges and z is orthogonal to it. The normal component of $\nabla V_\alpha(x)$ is a nonzero scalar multiple of $z \sum_i \zeta_i \rho_i^{-\alpha-1}$; hence it vanishes only when $z = 0$. Thus every equilibrium point lies in the affine span of the charges, and the problem reduces to dimension at most 2. $\square$

The proof is independent of numerical experimentation. Figure 1 illustrates the positive-quadrant geometry but is not used in the argument.

6. SHARPNESS OF THE METHOD AND COMPUTATIONAL CHECKS

We record the following computational checks because they clarify where the two-step Rolle–Khovanskii method loses information. Exact rational constructions of Q and R , exact resultants, high-precision root isolation, and independent tracing of the real ovals were performed for ten parameter sets. These examples cover $\alpha \in [1/20, 20]$, flat, tall, and obtuse charge triangles, and charge ratios up to $100 : 1$. In every tested example, the unique zero of η_2 in each quadrant is a nondegenerate saddle, and $Q = R = 0$ has exactly four real solutions in each open quadrant. Thus there are 16 real torus solutions. In the same examples the mixed-volume count is saturated:

$$16 + 2 \cdot 6 = 28. \quad (6.1)$$

The relevant Newton polygon areas are 12 and 15, while the area of their Minkowski sum is 55, giving the mixed volume 28.

The remaining slack lies in the Rolle–Khovanskii inequalities. The positive-quadrant oval O^+ carries four critical points of $\Phi|_{O^+}$. Depending on the level defining γ_2 , the curve γ_2 meets O^+ in 0, 2, or 4 points.

All steps in the count are generically sharp except for the combined Rolle estimates

$$N \leq 2 + \#(\gamma_2 \cap \Gamma \cap \mathbb{R}_{>0}^2). \quad (6.2)$$

When the level of γ_2 lies between the two middle critical values of $\Phi|_{O^+}$, the curve has four crossings with the oval. To reach Maxwell’s conjectural bound 4 by this route, one would have to show that in precisely this situation the first Rolle step overcounts by at least 2.

Combining Theorem 1.1 with (1.2) gives $m_0 \leq 2$. Hence the number of nondegenerate equilibria is 2, 4, or 6. An improvement to $m_0 \leq 1$ would be equivalent to Maxwell’s conjectured bound for three charges.

7. FINAL REMARKS

1. Edelsbrunner, Fillmore, and Oliveira [3] found a counterexample to Conjecture 1.8 of [4], which asserted that, for every $\alpha > 0$, the number of nondegenerate equilibrium points of a finite configuration of positive charges does not exceed the number of its effective Voronoi cells. One of the main results of [4] proves this assertion for every fixed configuration and all sufficiently large α . Thus it remains possible that some configuration of three charges has exactly six equilibrium points for special positive values of α .

2. In view of a recent counterexample [1] the question of finding new conjectures/results for the maximal number of points of equilibrium for configurations of point charges has resurfaced anew.

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