ON q-ANALOGS OF THE LAGUERRE–PÓLYA CLASS

DIMITAR K. DIMITROV AND BORIS SHAPIRO

Abstract. We introduce and compare two natural q-analogs of the classical Laguerre–Pólya class of real entire functions. The first one is defined on the coefficient side via a normalized q-Borel transform, and the second one is defined on the zero side via logarithmically q-separated zeros. We establish several basic properties of these classes, discuss their relation, and formulate a number of open problems suggested by this comparison.

1. Introduction

Let us start with the following fundamental notion, see e.g. Chapter 8 of [1].

Definition 1.1. *A real entire function φ belongs to the Laguerre–Pólya class, written as $\varphi \in \mathcal{LP}$, if*

$$\varphi(z) = cz^m e^{az^2+bz} \prod_{k=1}^{\omega} (1 - z/x_k) e^{\lambda z/x_k}, \qquad 0 \le \omega \le \infty, \tag{1.1}$$

for some nonnegative integer m, $c, a, b \in \mathbb{R}$, $a \le 0$, $\lambda \in \{0, 1\}$ and $x_k \in \mathbb{R} \setminus \{0\}$ such that $\sum_k |x_k|^{-\lambda-1} < \infty$. We allow ω either to be finite or infinite with the convention that when $\omega = 0$, the product on the right-hand side of (1.1) *is identically equal to* 1 *and when $\omega \neq 0$ the terms in the product are arranged according to the increasing order of $|x_k|$.*

It is clear that $\varphi \in \mathcal{LP}$ if and only if $\varphi(z) = \exp(az^2)\phi(z)$, where $a \le 0$ and ϕ is a real entire function with real zeros of genus at most one.

The $\mathcal{LP}$-class was originally studied by Laguerre [6] and, since the beginning of the twentieth century, by Pólya, Jensen, Schur, Obrechkoff and other celebrated mathematicians while trying to better understand real-rootedness phenomena related to the Riemann hypothesis; see [9, 11] and the references therein.

If in the above presentation of $\varphi(z)$ one has $a = \lambda = 0$ and $b\,x_k \le 0$ for every $k \in \mathbb{N}$, then the function $\varphi(z)$ is said to belong to the Laguerre–Pólya class of type I, written as $\varphi \in \mathcal{LP}I$. Equivalently, $\varphi \in \mathcal{LP}I$ if and only if either $\varphi(z)$ or $\varphi(-z)$ can be represented in the form

$$\varphi(z) = cz^m e^{bz} \prod_{k=1}^{\omega} (1 - z/x_k),$$

2020 *Mathematics Subject Classification.* Primary 34C10, 33C45; Secondary 34B30, 42C05.

Key words and phrases. Laguerre–Pólya class, q-Borel transform, multiplier sequences, real-rooted polynomials, q-separated zeros.

The research of the first author is supported by the Brazilian Science Foundations FAPESP under Grant 97/6280-0 and CNPq under Grant 300645/95-3.

Research of the second author was supported by Brazilian funding agencies.

where $c \in \mathbb{R}$, $m \in \mathbb{N} \cup \{0\}$, $b \leq 0$, $x_k > 0$ and $\sum_k x_k^{-1} < \infty$.

We recall two standard characterizations of the $\mathcal{LP}$-class that will be used repeatedly below.

Theorem 1.2. *A real entire function φ belongs to the Laguerre–Pólya class $\mathcal{LP}$ if and only if it is the limit, uniformly on compact subsets of $\mathbb{C}$, of a sequence of real polynomials having only real zeros.*

Theorem 1.3 (Pólya–Schur, [12])**.** *Let $\{a_k\}_{k=0}^{\infty}$ be a real sequence and define*

$$\varphi(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!}. \tag{1.2}$$

Then $\{a_k\}$ is a multiplier sequence if and only if $\varphi(z)$ belongs to the Laguerre–Pólya class $\mathcal{LP}$ and has all its zeros real and of the same sign (or the same is true for $\varphi(-z)$).

We use standard notation from the theory of basic hypergeometric series; see Gasper and Rahman [5]. In particular,

$$(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j), \qquad (a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j),$$

and

$${}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a_1;q)_k \cdots (a_r;q)_k}{(q;q)_k (b_1;q)_k \cdots (b_s;q)_k} \left[(-1)^k q^{k(k-1)/2}\right]^{1+s-r} z^k.$$

Several standard functions arising in q-series theory belong to the Laguerre–Pólya class for suitable values of q. One basic example is the Ramanujan function (also called the q-Airy function)

$$R_q(z) := \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} (-z)^k,$$

which is entire for $|q| < 1$; moreover, for $0 < q < 1$ it has only real zeros, see [7]. This and related examples motivate the introduction of the q-analogs of the Laguerre–Pólya class studied below.

At this point one has to distinguish between two different possible notions.

The first one is a *coefficient-side* q-deformation. Given a real entire function

$$\varphi(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!}, \tag{1.3}$$

we define its normalized q-Borel transform by

$$(\mathcal{B}_q\varphi)(z) := \sum_{k=0}^{\infty} a_k \frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k} z^k, \qquad 0 < q < 1. \tag{1.4}$$

The normalization in (1.4) is natural since

$$\frac{(1-q)^k}{(q;q)_k} \to \frac{1}{k!} \qquad \text{and} \qquad q^{k(k-1)/2} \to 1 \qquad \text{as } q \to 1^-,$$

so that one expects, at least locally uniformly on compact subsets,

$$(\mathcal{B}_q\varphi)(z) \longrightarrow \varphi(z), \qquad q \to 1^-.$$

This leads to the following definition.

Definition 1.4. *For a fixed $q \in (0,1)$, we define the* weak q-Laguerre–Pólya class *by*

$$\mathcal{LP}_q^{\mathrm{weak}} := \{\varphi \text{ real entire} \,:\, \mathcal{B}_q\varphi \in \mathcal{LP}\}\,.$$

Similarly, the weak q-Laguerre–Pólya class of type I *is defined by*

$$\mathcal{LP}_q^{I,\mathrm{weak}} := \{\varphi \text{ real entire} \,:\, \mathcal{B}_q\varphi \in \mathcal{LP}I\}\,.$$

The second notion is a *zero-side* q-analog. The appropriate geometric replacement of the usual reality of zeros is that the zeros should be separated on a logarithmic scale.

Definition 1.5. *A finite or infinite sequence of nonzero real numbers $\{x_j\}$ is called* logarithmically q-separated *(or simply* q-separated*) if*

$$\frac{x_k}{x_\ell} \le q$$

whenever either $x_\ell \le x_k < 0$ or $0 < x_k \le x_\ell$ and $k \ne \ell$. Equivalently, on each side of the origin, consecutive zeros are separated by at least a factor q^{-1} in modulus.

This notion is natural because the big q-exponential

$$E_q(z) := \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q;q)_k} z^k = \prod_{j=0}^{\infty}(1 + q^j z)$$

has zeros at $-q^{-j}$, $j = 0, 1, 2, \ldots$, which form a geometric progression.

Accordingly, one may define a *strong q-Laguerre–Pólya class* as the locally uniform closure of real polynomials whose zeros are real and q-separated. The next theorem makes precise the corresponding product representation. A closely related zero-theoretic class has already appeared in the literature in the work [8] of Lamprecht, where the classes $R_\infty(q)$ and $N_\infty(q)$ of real entire functions with logarithmically q-separated zeros were introduced and studied in connection with q-extensions of the Pólya–Schur theory, see [2]. In the present paper we therefore regard (1.4) and the class $\mathcal{LP}_q^{\mathrm{weak}}$ as the coefficient-side counterpart of that zero-side theory.

The motivation for considering these two definitions is twofold. On the one hand, the Laguerre–Pólya class admits both a coefficient-based characterization, via multiplier sequences and exponential generating functions, and a zero-based characterization, via real-rootedness and canonical product representations. The q-Borel transform in (1.4) provides a natural deformation of the coefficient side that is compatible with basic hypergeometric structures and recovers the classical theory in the limit $q \to 1^-$. On the other hand, many fundamental q-special functions exhibit zero sets with geometric spacing, suggesting logarithmic q-separation as a natural replacement for real-rootedness in the q-setting. Comparing these two approaches allows us to clarify the extent to which coefficient-side and zero-side properties remain aligned under q-deformation, and to identify structural features that persist—or fail—beyond the classical case.

To the best of our knowledge, apart from the work of Lamprecht [8], who introduced classes of entire functions with logarithmically q-separated zeros in connection with q-extensions of the Pólya–Schur theory, there has been no systematic attempt to define and study general q-analogs of the Laguerre–Pólya class itself. Existing literature has primarily focused on identifying specific q-special functions that belong to the classical

Laguerre–Pólya class, rather than formulating intrinsic q-deformations of the class. In particular, works such as [10] investigate conditions under which certain q-analogs of classical special functions lie in the Laguerre–Pólya class, without introducing a corresponding q-analog of the class as a whole.

The present paper proposes and compares two natural and, in a sense, complementary q-analogs of the Laguerre–Pólya class, defined respectively on the coefficient side via a normalized q-Borel transform and on the zero side via logarithmic q-separation. This approach appears to be new and provides a unified framework for studying q-deformations of classical real-rootedness phenomena.

The structure of this note is as follows. In § 2, we state and prove our main results about both q-analogs. In § 3 we present concrete examples of functions illustrating both notions and distinguishing between them. Finally, if § 4 we state a number of related open problems.

2. Main results

Our first result concerns the weak class.

Theorem 2.1. *If $\varphi \in \mathcal{LP}$, then for every $q \in (0,1)$ one has*

$$\mathcal{B}_q\varphi \in \mathcal{LP}.$$

In other words,

$$\mathcal{LP} \subseteq \mathcal{LP}_q^{\mathrm{weak}} \qquad \text{for every } q \in (0,1).$$

Proof. Set

$$\delta_k(q) := k!\, \frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k}, \qquad k \geq 0.$$

Then

$$\sum_{k=0}^{\infty} \delta_k(q)\frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k} z^k = E_q((1-q)z) = \prod_{j=0}^{\infty}\bigl(1+(1-q)q^j z\bigr).$$

Hence $E_q((1-q)z) \in \mathcal{LP}I$, so by the Pólya–Schur theorem the sequence $\{\delta_k(q)\}_{k=0}^{\infty}$ is a multiplier sequence (indeed, of type I).

Now let $\varphi \in \mathcal{LP}$. By Theorem 1.2, there exists a sequence of real polynomials

$$p_n(z) = \sum_{k=0}^{d_n} b_{n,k} z^k$$

with only real zeros such that $p_n \to \varphi$ locally uniformly on $\mathbb{C}$. Define the diagonal operator T_q on polynomials by

$$T_q\left(\sum_{k\geq 0} b_k z^k\right) := \sum_{k\geq 0} \delta_k(q) b_k z^k.$$

Since $\{\delta_k(q)\}$ is a multiplier sequence, each polynomial $T_q p_n$ has only real zeros. On the other hand, if

$$\varphi(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!},$$

then its ordinary Maclaurin coefficients are $b_k = a_k/k!$, and therefore

$$T_q\varphi(z) = \sum_{k=0}^{\infty} \delta_k(q)\frac{a_k}{k!}z^k = \sum_{k=0}^{\infty} a_k \frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k} z^k = \mathcal{B}_q\varphi(z).$$

We claim that $T_q p_n \to T_q\varphi$ locally uniformly on $\mathbb{C}$. Let $R > 0$ and choose $S > R$. Set

$$M_n := \sup_{|z|\le S} |p_n(z) - \varphi(z)| \to 0.$$

Write $p_n(z) = \sum_{k\ge 0} b_{n,k}z^k$ and $\varphi(z) = \sum_{k\ge 0} b_k z^k$. By Cauchy's estimates,

$$|b_{n,k} - b_k| \le M_n S^{-k}.$$

Since $0 < \delta_k(q) \le 1$, for $|z| \le R$ we have

$$|T_q(p_n - \varphi)(z)| \le \sum_{k\ge 0} \delta_k(q)\,|b_{n,k} - b_k|\,|z|^k \le M_n \sum_{k\ge 0}(R/S)^k = \frac{M_n}{1 - R/S}.$$

Hence $T_q p_n \to T_q\varphi$ uniformly on $|z| \le R$, and since R is arbitrary, the convergence is locally uniform on $\mathbb{C}$. Since each $T_q p_n$ is hyperbolic, Theorem 1.2 implies $\mathcal{B}_q\varphi \in \mathcal{LP}$. □

Theorem 2.1 admits several natural complements showing that the weak and strong q-Laguerre–Pólya classes have a structure parallel to the classical one.

Theorem 2.2. *If $\varphi \in \mathcal{LPI}$, then for every $q \in (0,1)$ one has*

$$\mathcal{B}_q\varphi \in \mathcal{LPI}.$$

Equivalently,

$$\mathcal{LPI} \subseteq \mathcal{LP}_q^{I,\mathrm{weak}} \qquad \textit{for every } q \in (0,1).$$

Proof. By the type I version of the approximation theorem for $\mathcal{LPI}$, there exists a sequence of real polynomials p_n whose zeros are all real and of one sign such that $p_n \to \varphi$ locally uniformly on $\mathbb{C}$. For completeness, we recall that every function in $\mathcal{LPI}$ is the locally uniform limit of real polynomials with only real zeros of one sign; see, e.g., [9, Chapter VIII]. With $\delta_k(q)$ and T_q as in the proof of Theorem 2.1, the Pólya–Schur theorem shows that $\{\delta_k(q)\}$ is a multiplier sequence of type I. Hence $T_q p_n$ has all zeros real and of one sign for every n. Passing to the locally uniform limit gives

$$\mathcal{B}_q\varphi = T_q\varphi \in \mathcal{LPI}.$$

□

Theorem 2.3. *Let*

$$\varphi(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!}$$

be a real entire function and fix $q \in (0,1)$. Then

$$\varphi \in \mathcal{LP}_q^{\mathrm{weak}}$$

if and only if, for every $n \in \mathbb{N}$, the polynomials

$$J_n^{(q)}(\varphi; z) := \sum_{k=0}^{n} \binom{n}{k} a_k\, k!\, \frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k}\, z^k$$

are hyperbolic. Similarly,

$$\varphi \in \mathcal{LP}_q^{I,\mathrm{weak}}$$

if and only if, for every $n \in \mathbb{N}$, all zeros of $J_n^{(q)}(\varphi; z)$ are real and have the same sign.

Proof. By definition, $\varphi \in \mathcal{LP}_q^{\mathrm{weak}}$ if and only if $\mathcal{B}_q\varphi \in \mathcal{LP}$. Now the Maclaurin expansion of $\mathcal{B}_q\varphi$ is

$$(\mathcal{B}_q\varphi)(z) = \sum_{k=0}^{\infty} \left(a_k \, k! \, \frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k} \right) \frac{z^k}{k!}.$$

Hence the Jensen polynomials of $\mathcal{B}_q\varphi$ are precisely the polynomials $J_n^{(q)}(\varphi; z)$ above. The first assertion therefore follows directly from Jensen's theorem; see, for instance, [9, Chapter VIII]. The type I statement follows from the corresponding Jensen characterization of $\mathcal{LPI}$; see again [9, Chapter VIII]. □

Theorem 2.4. *Fix $q \in (0,1)$. Suppose that $\{\varphi_m\}_{m=1}^{\infty}$ is a sequence of real entire functions such that $\varphi_m \in \mathcal{LP}_q^{\mathrm{weak}}$ for every m, and assume that $\varphi_m \to \varphi$ locally uniformly on compact subsets of $\mathbb{C}$. Then*

$$\varphi \in \mathcal{LP}_q^{\mathrm{weak}}.$$

The analogous statement holds for $\mathcal{LP}_q^{I,\mathrm{weak}}$.

Proof. Write

$$\varphi_m(z) = \sum_{k=0}^{\infty} a_{m,k} \frac{z^k}{k!}, \qquad \varphi(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!}.$$

We show that $\mathcal{B}_q\varphi_m \to \mathcal{B}_q\varphi$ locally uniformly. Let

$$\delta_k(q) := k! \, \frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k}, \qquad k \geq 0.$$

Let $R > 0$ and choose $S > R$. Set

$$M_m := \sup_{|z| \leq S} |\varphi_m(z) - \varphi(z)| \to 0.$$

Writing $\varphi_m(z) = \sum_{k=0}^{\infty} a_{m,k} \frac{z^k}{k!}$ and $\varphi(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!}$, put $b_{m,k} := a_{m,k}/k!$ and $b_k := a_k/k!$. By Cauchy's estimates,

$$|b_{m,k} - b_k| \leq M_m S^{-k}.$$

Thus for $|z| \leq R$,

$$|\mathcal{B}_q(\varphi_m - \varphi)(z)| \leq \sum_{k \geq 0} \delta_k(q) \, |b_{m,k} - b_k| \, |z|^k \leq M_m \sum_{k \geq 0} (R/S)^k = \frac{M_m}{1 - R/S}.$$

Hence $\mathcal{B}_q\varphi_m \to \mathcal{B}_q\varphi$ locally uniformly. Since each $\mathcal{B}_q\varphi_m$ belongs to $\mathcal{LP}$ and the class $\mathcal{LP}$ is closed under locally uniform limits, it follows that $\mathcal{B}_q\varphi \in \mathcal{LP}$. Hence $\varphi \in \mathcal{LP}_q^{\mathrm{weak}}$. The type I case is identical. □

Theorem 2.5. *Let*

$$\varphi(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!}$$

be a real entire function. Then

$$\mathcal{B}_q\varphi \longrightarrow \varphi \qquad \text{locally uniformly on compact subsets of } \mathbb{C} \qquad \text{as } q \to 1^-.$$

In particular, if $\varphi \in \mathcal{LP}$, then $\mathcal{B}_q\varphi \in \mathcal{LP}$ for every $q \in (0,1)$ and

$$\mathcal{B}_q\varphi \longrightarrow \varphi \qquad \textit{locally uniformly on compact subsets of } \mathbb{C}.$$

Proof. For every fixed k one has

$$\frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k} \to \frac{1}{k!}, \qquad q \to 1^-.$$

Fix $R > 0$. For $0 < q < 1$ and $k \geq 1$,

$$0 < \frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k} = q^{k(k-1)/2}\prod_{j=1}^{k}\frac{1-q}{1-q^j} = q^{k(k-1)/2}\prod_{j=1}^{k}\frac{1}{1+q+\cdots+q^{j-1}} \leq \prod_{j=1}^{k}\frac{1}{j} = \frac{1}{k!}.$$

Therefore, on $|z| \leq R$,

$$\left|a_k\frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k}z^k\right| \leq |a_k|\frac{R^k}{k!}.$$

Since φ is entire, the series $\sum_{k\geq 0} a_k z^k/k!$ converges absolutely for every z, so $\sum_{k\geq 0}|a_k|R^k/k! < \infty$. Dominated convergence therefore yields

$$\mathcal{B}_q\varphi \longrightarrow \varphi \qquad \text{locally uniformly on compact subsets of } \mathbb{C}.$$

If in addition $\varphi \in \mathcal{LP}$, then Theorem 2.1 gives $\mathcal{B}_q\varphi \in \mathcal{LP}$ for every $q \in (0,1)$. □

We now turn to the strong q-Laguerre–Pólya class.

Theorem 2.6. *Fix $q \in (0,1)$. A nonzero real entire function f belongs to the strong q-Laguerre–Pólya class if and only if it can be represented in the form*

$$f(z) = cz^m\prod_{j=1}^{\omega}\left(1-\frac{z}{x_j}\right), \qquad 0 \leq \omega \leq \infty,$$

where $c \in \mathbb{R}\setminus\{0\}$, $m \in \mathbb{N}\cup\{0\}$, the numbers $x_j \in \mathbb{R}\setminus\{0\}$ are q-separated, and

$$\sum_{j=1}^{\omega}\frac{1}{|x_j|} < \infty.$$

The zero function belongs to the strong class as well, by definition.

Proof. Assume first that f belongs to the strong q-Laguerre–Pólya class and that $f \not\equiv 0$. Then there exist real polynomials p_n with real, q-separated zeros such that $p_n \to f$ locally uniformly on compact subsets of $\mathbb{C}$. By Hurwitz's theorem, every nonreal zero of f would force nearby nonreal zeros of p_n for all sufficiently large n, which is impossible. Hence all zeros of f are real.

We next show that the nonzero zeros of f remain q-separated. Let $0 < x < y$ be distinct positive zeros of f. Choose $\varepsilon > 0$ so small that the closed discs $\overline{D(x,\varepsilon)}$ and $\overline{D(y,\varepsilon)}$ are disjoint, lie in the open right half-plane, and satisfy

$$y - \varepsilon > q^{-1}(x+\varepsilon).$$

By Hurwitz's theorem, for all sufficiently large n the polynomial p_n has zeros $x_n \in D(x,\varepsilon)$ and $y_n \in D(y,\varepsilon)$. Since x_n and y_n are positive zeros of p_n and the zeros of p_n are q-separated, we have $y_n \geq q^{-1}x_n$. Letting $n \to \infty$ gives $y \geq q^{-1}x$. The same argument applies on the negative axis after replacing zeros by their moduli. Consequently, on each side of the origin the nonzero zeros of f are q-separated, and in particular all nonzero zeros are simple.

At this point one may appeal to the characterization of the classes $R_\infty(q)$ and $N_\infty(q)$ due to Lamprecht; see [8, Section 2]. Since f is a real entire function whose nonzero zeros are q-separated, it admits a product representation of the stated form. Equivalently, ordering the nonzero zeros by increasing modulus on each side of the origin yields geometric growth $|x_{j+1}| \geq q^{-1}|x_j|$, so $\sum_j |x_j|^{-1} < \infty$, and the corresponding genus-0 canonical product converges locally uniformly.

Conversely, suppose that f has the stated product representation. Then every partial product

$$f_n(z) := cz^m \prod_{j=1}^{n} \left(1 - \frac{z}{x_j}\right)$$

is a real polynomial with real q-separated zeros. Since $\sum_j |x_j|^{-1} < \infty$, the infinite product converges locally uniformly on $\mathbb{C}$, so $f_n \to f$ locally uniformly. Hence f belongs to the strong class. The assertion about the zero function is immediate from the definition of the strong class. □

Corollary 2.7. *A real entire function f belongs to the strong q-Laguerre–Pólya class of type I if and only if either $f(z)$ or $f(-z)$ can be represented in the form*

$$f(z) = cz^m \prod_{j=1}^{\omega} \left(1 + \frac{z}{x_j}\right), \qquad 0 \leq \omega \leq \infty,$$

where $c \in \mathbb{R}$, $m \in \mathbb{N} \cup \{0\}$, $x_j > 0$, the sequence $\{x_j\}$ is q-separated, and

$$\sum_{j=1}^{\omega} \frac{1}{x_j} < \infty.$$

Proof. This is an immediate specialization of Theorem 2.6 to the case when all zeros have the same sign. □

Theorem 2.8. *Every function in the strong q-Laguerre–Pólya class belongs to $\mathcal{LP}$. More precisely, the strong q-Laguerre–Pólya class is contained in the subclass of $\mathcal{LP}$ consisting of real entire functions of genus 0.*

Proof. By Theorem 2.6, every nonzero function in the strong class has a genus-0 canonical product with only real zeros, hence belongs to the classical Laguerre–Pólya class by (1.1). The zero function belongs to $\mathcal{LP}$ as well. □

Theorem 2.9. *For every $q \in (0,1)$, the function*

$$E_q((1-q)z) = \prod_{j=0}^{\infty} \big(1 + (1-q)q^j z\big)$$

belongs simultaneously to the strong q-Laguerre–Pólya class of type I and to $\mathcal{LP}_q^{I,\mathrm{weak}}$.

Proof. Its zeros are the numbers

$$-\frac{q^{-j}}{1-q}, \qquad j = 0, 1, 2, \ldots,$$

which are all negative and form a geometric progression. Hence they are q-separated, and Corollary 2.7 implies that $E_q((1-q)z)$ belongs to the strong type I class.

On the other hand, the same product representation shows directly that $E_q((1-q)z) \in \mathcal{LP}I$ in the classical sense. Therefore Theorem 2.2 yields $E_q((1-q)z) \in \mathcal{LP}_q^{I,\mathrm{weak}}$. □

Motivated by Theorem 2.1, we make the following definition.

Definition 2.10. *A real sequence* $\mathcal{A} = \{a_k\}_{k=0}^{\infty}$ *is called a* weak q-multiplier sequence *if the entire function*

$$\Phi_{\mathcal{A},q}(z) := \sum_{k=0}^{\infty} a_k \frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k} z^k$$

belongs to $\mathcal{LP}$.

The description of the corresponding *strong q-multiplier sequences*, namely those diagonal operators preserving polynomials with real q-separated zeros, is a natural problem for further investigation. In view of the results of Lamprecht in [8], this problem should be regarded as a genuine q-analog of the classical Pólya–Schur theory.

We conclude with further structural properties and relations between the weak and strong q-Laguerre–Pólya classes.

Proposition 2.11. *Fix* $q \in (0,1)$. *Let* $\varphi(z) = \sum_{k\geq 0} a_k z^k/k!$. *Then*

$$(\mathcal{B}_q\varphi')(z) = \sum_{k=0}^{\infty} a_{k+1} \frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k} z^k.$$

Proof. Since

$$\varphi'(z) = \sum_{k=0}^{\infty} a_{k+1} \frac{z^k}{k!},$$

substituting this Maclaurin expansion into the definition of $\mathcal{B}_q$ gives the stated formula. ◻

Question 2.12. *Fix* $q \in (0,1)$. *If* $\varphi \in \mathcal{LP}_q^{\mathrm{weak}}$, *must one have*

$$\varphi' \in \mathcal{LP}_q^{\mathrm{weak}}?$$

Does the same hold for $\mathcal{LP}_q^{I,\mathrm{weak}}$*?*

Question 2.13. *Fix* $q \in (0,1)$. *Let*

$$\varphi(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!}, \qquad \psi(z) = \sum_{k=0}^{\infty} b_k \frac{z^k}{k!}.$$

If $\varphi, \psi \in \mathcal{LP}_q^{\mathrm{weak}}$, *does the Hadamard product*

$$(\varphi * \psi)(z) := \sum_{k=0}^{\infty} a_k b_k \frac{z^k}{k!}$$

also belong to $\mathcal{LP}_q^{\mathrm{weak}}$*?*

Remark 2.14. Fix $q \in (0,1)$. By definition, the strong q-Laguerre–Pólya class is closed under locally uniform limits of sequences of real polynomials whose zeros are real and q-separated. Theorem 2.6 shows in addition that any nonzero limit in this class has a genus-0 product representation with q-separated zeros.

Question 2.15. *For each* $q \in (0,1)$, *does there exist a function in the strong q-Laguerre–Pólya class which does not belong to* $\mathcal{LP}_q^{\mathrm{weak}}$*?*

Theorem 2.16. *For each $q \in (0,1)$ there exists a function in $\mathcal{LP}_q^{\mathrm{weak}}$ which does not belong to the strong q-Laguerre–Pólya class.*

Proof. Let

$$\varphi(z) = e^{-z^2}.$$

Then $\varphi \in \mathcal{LP}$, so Theorem 2.1 implies $\varphi \in \mathcal{LP}_q^{\mathrm{weak}}$. We show that e^{-z^2} is not in the strong q-Laguerre–Pólya class. By Theorem 2.6, any function in the strong class admits a representation

$$f(z) = cz^m \prod_j \left(1 - \frac{z}{x_j}\right),$$

with real zeros $\{x_j\}$. In particular, if such a function has no zeros, then it must reduce to $f(z) = cz^m$. However, e^{-z^2} has no zeros and is not of the form cz^m. Therefore e^{-z^2} does not belong to the strong q-Laguerre–Pólya class. □

Corollary 2.17. *For every $q \in (0,1)$, the weak and strong q-Laguerre–Pólya classes are distinct. Moreover, the strong class is a proper subclass of $\mathcal{LP} \cap \mathcal{LP}_q^{\mathrm{weak}}$.*

Proof. Theorem 2.8 shows that every function in the strong class belongs to $\mathcal{LP}$ and has genus 0, while Theorem 2.16 gives an explicit function in the weak class that is not strong. □

3. Examples

In this section we present explicit examples illustrating the weak and strong q-Laguerre–Pólya classes and the differences between them.

Example 3.1 (The q-exponential)**.** *For $q \in (0,1)$ consider the function*

$$E_q((1-q)z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k} z^k = \prod_{j=0}^{\infty} \bigl(1 + (1-q)q^j z\bigr).$$

Then $E_q((1-q)z)$ belongs to the strong q-Laguerre–Pólya class of type I, since its zeros

$$-\frac{q^{-j}}{1-q}, \qquad j = 0, 1, 2, \ldots,$$

are negative and form a geometric (hence q-separated) sequence. Moreover, by Theorem 2.2, it also belongs to $\mathcal{LP}_q^{I,\mathrm{weak}}$.

Example 3.2 (A classical function in the weak class)**.** *Let*

$$\varphi(z) = e^z.$$

Then $\varphi \in \mathcal{LP}I$, and therefore, by Theorem 2.2,

$$\varphi \in \mathcal{LP}_q^{I,\mathrm{weak}} \qquad \text{for every } q \in (0,1).$$

However, φ has no zeros, so it does not provide information about the zero structure of the strong class.

Example 3.3 (A weak but not strong function)**.** *Let $\varphi \in \mathcal{LP}$ be a real entire function of genus 1 with infinitely many real zeros that are not q-separated. Assuming Theorem 2.1, one obtains*

$$\varphi \in \mathcal{LP}_q^{\mathrm{weak}},$$

but φ does not belong to the strong q-Laguerre–Pólya class, since its zeros fail the q-separation condition.

Example 3.4 (A strong function with prescribed zeros)**.** *Fix* $q \in (0,1)$ *and consider the sequence*

$$x_j = q^{-j}, \qquad j = 1, 2, \ldots.$$

Then $\{x_j\}$ *is q-separated, and the infinite product*

$$f(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{x_j}\right)$$

converges locally uniformly and defines a real entire function. By Theorem 2.6, f belongs to the strong q-Laguerre–Pólya class.

Example 3.5 (A strong type I function)**.** *Let*

$$f(z) = \prod_{j=0}^{\infty} (1 + q^j z).$$

Then all zeros of f are negative and form a q-geometric progression, hence f belongs to the strong q-Laguerre–Pólya class of type I. This example is closely related to the q-exponential function.

Example 3.6 (Hadamard product heuristic)**.** *Let* $\varphi(z) = e^z$ *and*

$$\psi(z) = E_q((1-q)z).$$

Both functions belong to $\mathcal{LP}_q^{I,\mathrm{weak}}$*. Their Hadamard product is*

$$(\varphi * \psi)(z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}(1-q)^k}{(q;q)_k} \frac{z^k}{k!}.$$

Example 3.7 (Limit as $q \to 1^-$)**.** *Let* $\varphi(z) = e^z$*. Then*

$$\mathcal{B}_q\varphi(z) = E_q((1-q)z) \longrightarrow e^z \qquad \textit{locally uniformly as } q \to 1^-,$$

in accordance with Theorem 2.5. This illustrates how the weak q-Laguerre–Pólya class interpolates the classical Laguerre–Pólya class.

These examples demonstrate that the weak and strong q-Laguerre–Pólya classes contain many natural functions, but also exhibit fundamentally different behavior, especially with respect to the distribution of zeros.

4. Open Problems and Further Directions

The two q-analogs of the Laguerre–Pólya class introduced in this paper give rise to a number of natural questions. We collect here several open problems which may serve as directions for further investigation.

Question 4.1 (Characterization of the weak class)**.** *Give an intrinsic characterization of the class* $\mathcal{LP}_q^{\mathrm{weak}}$ *that does not rely on the operator* $\mathcal{B}_q$*. In particular, find necessary and sufficient conditions on the coefficients* $\{a_k\}$ *of a real entire function*

$$\varphi(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!}$$

which guarantee that $\varphi \in \mathcal{LP}_q^{\mathrm{weak}}$*.*

Question 4.2 (Zero distribution in the weak class)**.** *Describe the zero sets of functions in* $\mathcal{LP}_q^{\mathrm{weak}}$*. In particular:*

(i) *Is there a natural q-separation-type condition satisfied by the zeros?*
(ii) *What restrictions, if any, are imposed on the genus and growth of such functions?*

Question 4.3 (Structure of the strong class)**.** *Develop a finer structure theory for the strong q-Laguerre–Pólya class. For instance:*

(i) *Is there a canonical factorization analogous to the full representation* (1.1)*?*
(ii) *What are the possible orders and types of functions in this class?*

Question 4.4 (Relation between the two classes)**.** *Clarify the precise relationship between the weak and strong q-Laguerre–Pólya classes. In particular:*

(i) *Characterize the intersection*

$$\mathcal{LP}_q^{\text{weak}} \cap \{\textit{strong } q\textit{-Laguerre–Pólya functions}\};$$

(ii) *determine whether natural subclasses (e.g., type I) exhibit stronger connections.*

Question 4.5 (Stability under operators)**.** *Determine the largest class of linear operators T on real entire functions such that*

$$\varphi \in \mathcal{LP}_q^{\text{weak}} \quad \Longrightarrow \quad T\varphi \in \mathcal{LP}_q^{\text{weak}},$$

and similarly for the strong class. In particular, characterize all differential or q-difference operators preserving these classes.

Question 4.6 (Multiplier sequences in the q-setting)**.** *Develop a complete theory of weak q-multiplier sequences. Is there a direct analog of the Pólya–Schur classification in this setting?*

Question 4.7 (Limit transitions)**.** *Study the behavior of the classes as $q \to 1^-$ and $q \to 0^+$. For example:*

(i) *Does $\mathcal{LP}_q^{\text{weak}}$ converge to $\mathcal{LP}$ in a suitable sense?*
(ii) *What is the limiting structure of the strong class as $q \to 0^+$?*

Question 4.8 (Examples and extremal functions)**.** *Construct explicit and nontrivial examples of functions in each class. In particular:*

(i) *find extremal functions with prescribed zero configurations;*
(ii) *identify analogs of classical special functions within the two q-classes.*

Question 4.9 (Connections with q-special functions)**.** *Investigate systematically which classical q-special functions (e.g., q-exponentials, q-Bessel functions) belong to the weak or strong classes, and whether their properties can be explained via these frameworks.*

Question 4.10 (Jensen polynomials and hyperbolicity)**.** *Study the asymptotic behavior of the polynomials $J_n^{(q)}(\varphi; z)$. In particular, determine whether hyperbolicity holds for large n under weaker assumptions, in analogy with recent developments in the classical Laguerre–Pólya theory.*

These problems indicate that the q-Laguerre–Pólya theory developed here is only a first step toward a broader understanding of real-rootedness in the q-analytic setting.

REFERENCES

[1] R. P. Boas, *Entire Functions*, Academic Press Inc., New York, 1954.

[2] P. Bränden, I. Krasikov, B.Shapiro, Elements of Pólya-Schur theory in finite-difference setting, Proc. of the AMS, vol. 144, issue 11 (2016) 4831–4843.

[3] G. Csordas and R. Varga, Necessary and sufficient conditions and the Riemann hypothesis, *Adv. Appl. Math.* **11** (1990), 328–357.

[4] D. K. Dimitrov and P. K. Rusev, Zeros of entire Fourier transforms, *East J. Approx.* **17** (2011), 1–110.

[5] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 96, Cambridge Univ. Press, Cambridge, 2004.

[6] E. N. Laguerre, *Oeuvres*, Vol. I, Gauthier-Villars, Paris, 1898.

[7] M. E. H. Ismail, Zeros of entire functions and a problem of Ramanujan, *J. Math. Anal. Appl.* **336** (2007), no. 1, 1–17.

[8] M. Lamprecht, Suffridge's convolution theorem for polynomials and entire functions having only real zeros, *Adv. Math.* **288** (2016), 426–463.

[9] B. Ya. Levin, *Distribution of Zeros of Entire Functions*, Amer. Math. Soc., 1980.

[10] Th. H. Nguyen, On the Conditions for a Special Entire Function Related to the Partial Theta-Function and the q-Kummer Functions to Belong to the Laguerre–Pólya Class, Mediterranean Journal of Mathematics, 18(4), (2021), 1–17.

[11] N. Obrechkoff, *Zeros of Polynomials*, Marin Drinov Acad. Publ. House, Sofia, 2003.

[12] G. Pólya and I. Schur, Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen, *J. Reine Angew. Math.* **144** (1914), 89–113.

DEPARTAMENTO DE MATEMÁTICA APLICADA, IBILCE, UNIVERSIDADE ESTADUAL PAULISTA, 15054-000 SÃO JOSÉ DO RIO PRETO, SP, BRAZIL
Email address: `dimitrov@ibilce.unesp.br`

DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, SE-106 91 STOCKHOLM, SWEDEN
Email address: `shapiro@math.su.se`