Symbolic powers of the ideal of n general points in $\mathbb{P}^{n-1}$

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Abstract

Problem L of Fröberg–Lundqvist–Oneto–Shapiro asks for the difference between the Hilbert series of ordinary and symbolic powers of the ideal of general points in projective space. We solve this completely for n general points of $\mathbb{P}^{n-1}$. Besides a closed formula for

$$\mathrm{HS}(S/I^m) - \mathrm{HS}(S/I^{(m)}),$$

we determine all minimal monomial generators of $I^{(m)}$, and describe the symbolic Rees algebra. We also show that containment $I^{(m)} \subseteq I^r$ is detected solely by initial degrees. This gives the exact containment threshold, the Waldschmidt constant $\widehat{\alpha}$, the resurgence ρ, and the asymptotic resurgence $\widehat{\rho}$:

$$\widehat{\alpha}(I) = \frac{n}{n-1}, \qquad \rho(I) = \widehat{\rho}(I) = \frac{2(n-1)}{n}.$$

We also take the first step beyond n points: for $n+1$ general points of $\mathbb{P}^{n-1}$ — again a rigid, non-monomial configuration — we identify the defining quadrics, resolve the case $n=3$ completely (a complete intersection, with $J^{(m)} = J^m$ for all m and resurgence 1), and propose an exact Waldschmidt-constant formula $\widehat{\alpha} = \frac{n+1}{n-1}$ for all n, verified computationally in every case we could check.

1 Introduction

Problem L in [3] asks the following.

Problem L. For the ideal I of s general points of $\mathbb{P}^{n-1}$, what is the difference between the Hilbert series of the m-th symbolic power and the m-th ordinary power?

We answer this completely for $s = n$. This is the first case in which the points span the whole ambient space and the configuration is projectively rigid: every ordered n-tuple in linear general position is projectively equivalent to the coordinate points. Thus the symbolic-versus-ordinary problem becomes monomial while still exhibiting a genuinely nontrivial difference. (For $s < n$, the points lie in a proper linear subspace, so the ambient-space problem has an additional linear part.)

Throughout, $Z \subset \mathbb{P}^{n-1}$ consists of n points in general position, $n \geq 2$. Since PGL_n acts transitively on ordered n-tuples in general position, we may take Z to be the coordinate points. With

$$S = \Bbbk[x_1, \dots, x_n], \qquad \mathfrak{p}_i = (x_1, \dots, \widehat{x_i}, \dots, x_n),$$

we have

$$I = I_Z = \bigcap_{i=1}^{n} \mathfrak{p}_i = (x_i x_j : 1 \le i < j \le n).$$

Thus both I^m and

$$I^{(m)} = \bigcap_{i=1}^{n} \mathfrak{p}_i^m$$

are monomial ideals, reducing the problem to a lattice-point count.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, write

$$|\alpha| = \sum_i \alpha_i, \qquad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad M(\alpha) = \max_i \alpha_i.$$

2 Two monomial criteria

The following elementary criterion is the basic input.

Lemma 2.1. *For every $m \ge 1$ and $\alpha \in \mathbb{N}^n$:*

1. $$x^\alpha \in I^{(m)} \iff |\alpha| - M(\alpha) \ge m;$$

2. $$x^\alpha \in I^m \iff \sum_{i=1}^{n} \min(\alpha_i, m) \ge 2m.$$

Proof. For (1), $x^\alpha \in \mathfrak{p}_i^m$ if and only if $\sum_{j \ne i} \alpha_j \ge m$. Intersecting over i gives

$$|\alpha| - \alpha_i \ge m \quad \text{for every } i,$$

equivalently $|\alpha| - M(\alpha) \ge m$.

For (2), the exponent vectors of the degree-$2m$ generators of I^m are sums of m vectors $e_i + e_j$, $i \ne j$. Equivalently, they are degree sequences of loopless multigraphs with m edges. A vector $\beta \in \mathbb{N}^n$ with $|\beta| = 2m$ is such a degree sequence if and only if $M(\beta) \le m$.

Necessity is immediate. For sufficiency, order $\beta_1 \ge \beta_2 \ge \cdots$. Join vertices 1 and 2 and subtract one from β_1, β_2. The resulting vector has sum $2(m-1)$, and the new maximum is at most $m-1$: indeed $\beta_3 \ge m$ would force $\beta_1 = \beta_2 = \beta_3 = m$, contradicting $|\beta| = 2m$. Induction on m finishes the argument.

Hence $x^\alpha \in I^m$ precisely when there exists $\beta \le \alpha$ componentwise with $|\beta| = 2m$ and $M(\beta) \le m$. Such a β exists exactly when $\sum_i \min(\alpha_i, m) \ge 2m$. □

3 The Hilbert-series difference

Theorem 3.1. *For $n \ge 2$, $m \ge 1$, and $q \ge 0$,*

$$\dim_{\Bbbk}(I^{(m)})_q - \dim_{\Bbbk}(I^m)_q = \begin{cases} \#\{\alpha \in \mathbb{N}^n : |\alpha| = q,\ \alpha_i \le q - m \text{ for all } i\}, & m \le q \le 2m-1, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$\mathrm{HS}(S/I^m) - \mathrm{HS}(S/I^{(m)}) = \sum_{q=m}^{2m-1} \left(\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{q - j(q-m+1) + n - 1}{n-1} \right) t^q,$$

where a binomial coefficient with upper entry $< n-1$ is interpreted as zero. In particular $I^{(m)}$ and I^m agree in every degree $q \geq 2m$, and $I^{(m)} = I^m$ for every m if and only if $n = 2$.

Proof. By Lemma 2.1, the difference counts the vectors α of degree q satisfying

$$|\alpha| - M(\alpha) \geq m, \qquad \sum_i \min(\alpha_i, m) < 2m.$$

Let $T = \{i : \alpha_i \geq m\}$. If $|T| \geq 2$, the second sum is at least $2m$, impossible. If $T = \{i_0\}$, then

$$m + (q - \alpha_{i_0}) < 2m$$

forces $\alpha_{i_0} > q - m$, contradicting $q - M(\alpha) \geq m$. Thus $T = \varnothing$. Then the second condition is simply $q < 2m$, while the first is $M(\alpha) \leq q - m$. This proves the counting formula.

The inclusion–exclusion formula is the standard count of compositions of q into n nonnegative parts bounded above by $q - m$. For $n = 2$, $I = (x_1 x_2)$ is principal. For $n \geq 3$, take $m = n-1$ and $\alpha = (1, \dots, 1)$, obtaining $x_1 \cdots x_n \in I^{(n-1)} \setminus I^{n-1}$. □

The support of the difference can be made completely explicit.

Corollary 3.2. *Assume $n \geq 3$ and $m \geq 2$, and put*

$$a_{n,m} = m + \left\lceil \frac{m}{n-1} \right\rceil.$$

Then

$$(I^{(m)}/I^m)_q \neq 0 \iff a_{n,m} \leq q \leq 2m-1.$$

Proof. Write $q = m + k$. The count in Theorem 3.1 is nonzero exactly when $m + k \leq nk$, i.e. $k \geq \lceil m/(n-1) \rceil$, while $q \leq 2m-1$ is $k \leq m-1$.

□

Example 3.3. *For $n = 3$ and $m = 2$, the difference Δ_q is concentrated in degree 3 and equals 1, represented by the classical monomial*

$$xyz \in I^{(2)} \setminus I^2.$$

For $n = 3$, the next values are

m	(q, Δ_q)
2	$(3,1)$
3	$(5,3)$
4	$(6,1), (7,6)$
5	$(8,3), (9,10),$

while for $n = 4$ one gets

m	(q, Δ_q)
2	$(3,4)$
3	$(4,1), (5,16)$
4	$(6,10), (7,40).$

4 All minimal generators and all symbolic defects

We now determine the minimal generators of every symbolic power.

Theorem 4.1. *A monomial x^α is a minimal monomial generator of $I^{(m)}$ if and only if*

$$|\alpha| - M(\alpha) = m \qquad \text{and} \qquad M(\alpha) \text{ is attained at least twice.} \tag{1}$$

Hence, if $k = M(\alpha)$, the possible degrees of minimal generators are precisely

$$m + k, \qquad \left\lceil \frac{m}{n-1} \right\rceil \le k \le m.$$

The generators with $k = m$ are exactly

$$x_i^m x_j^m, \qquad 1 \le i < j \le n.$$

All other minimal generators have degree $< 2m$ and survive minimally in $I^{(m)}/I^m$.

Proof. Suppose first that x^α is minimal in $I^{(m)}$. If $|\alpha| - M(\alpha) \ge m + 1$, decreasing a suitable positive exponent by one leaves the monomial in $I^{(m)}$, contradicting minimality. Thus equality holds in the first condition of (1).

If the maximum $M = M(\alpha)$ were attained uniquely, say at $\alpha_1 = M$, then after replacing α_1 by $M - 1$, the new maximum is at most $M - 1$, and therefore

$$(|\alpha| - 1) - M(\alpha - e_1) \ge |\alpha| - M = m.$$

Again the resulting proper divisor lies in $I^{(m)}$, a contradiction. Thus the maximum is attained at least twice.

Conversely, assume (1). Decreasing any positive coordinate by one leaves some coordinate equal to $M(\alpha)$; hence the new vector β satisfies

$$|\beta| - M(\beta) = m - 1.$$

Thus no proper monomial divisor lies in $I^{(m)}$, proving minimality.

Now write $k = M(\alpha)$. The equality $|\alpha| - k = m$ shows that the remaining $n-1$ coordinates, after choosing one maximal coordinate, sum to m and are bounded by k. Hence

$$k \ge \left\lceil \frac{m}{n-1} \right\rceil.$$

Also the maximum is attained at least twice, so $k \le m$. Conversely every integer in this range occurs: take two entries equal to k and distribute $m - k$ among the remaining $n - 2$ entries, all at most k. Finally, $k = m$ forces precisely two entries to equal m and all others to vanish. $\square$

5 The symbolic Rees algebra

The preceding description has a particularly simple algebraic interpretation. For $F \subseteq [n]$, write

$$x_F = \prod_{i \in F} x_i.$$

Theorem 5.1. *The symbolic Rees algebra*

$$\mathcal{R}_s(I) = \bigoplus_{m\geq 0} I^{(m)}t^m$$

is generated as an S-algebra by

$$x_F t^{|F|-1}, \qquad F \subseteq [n], \quad |F| \geq 2.$$

Moreover this is a minimal S-algebra generating set. In particular the largest symbolic degree of a minimal algebra generator is $n-1$, and the number of positive-symbolic-degree algebra generators is

$$\sum_{j=2}^{n} \binom{n}{j} = 2^n - n - 1.$$

Proof. For $|F| \geq 2$, the monomial x_F belongs to $I^{(|F|-1)}$: after omitting any one variable, at least $|F|-1$ variables of x_F remain.

Conversely, it suffices to factor a minimal generator x^α of $I^{(m)}$. Put $k = M(\alpha)$, and for $1 \leq \ell \leq k$ define

$$F_\ell = \{i : \alpha_i \geq \ell\}.$$

By Theorem 4.1, the maximum is attained at least twice, hence $|F_\ell| \geq 2$ for all ℓ. The layer decomposition gives

$$\alpha = \mathbf{1}_{F_1} + \cdots + \mathbf{1}_{F_k}$$

and

$$m = |\alpha| - k = \sum_{\ell=1}^{k} (|F_\ell| - 1).$$

Therefore

$$x^\alpha t^m = \prod_{\ell=1}^{k} \left(x_{F_\ell} t^{|F_\ell|-1} \right).$$

Any nonminimal monomial in $I^{(m)}$ is an S-multiple of a minimal one, proving generation.

For minimality of the displayed algebra generators, suppose $x_F t^{|F|-1}$ were a product of at least two positive-t-degree generators and an S-monomial. Since x_F is squarefree, the positive-t-degree factors must have pairwise disjoint supports. If there are $p \geq 2$ such factors with supports $F_1, \ldots, F_p$, their total t-degree is

$$\sum_{\nu=1}^{p} (|F_\nu| - 1) = \left| \bigcup_\nu F_\nu \right| - p \leq |F| - p < |F| - 1,$$

a contradiction. ∎

Remark 5.2. *For $n = 3$, Theorem 5.1 says that the symbolic Rees algebra is generated over S by*

$$x_1x_2t, \quad x_1x_3t, \quad x_2x_3t, \quad x_1x_2x_3t^2.$$

For general n, the squarefree monomials of degree j appear in symbolic degree $j-1$.

6 Initial degrees and exact containments

Write $\alpha(J)$ for the least degree of a nonzero homogeneous element of a homogeneous ideal J.

Proposition 6.1. *For every $m \geq 1$,*

$$\alpha(I^{(m)}) = m + \left\lceil \frac{m}{n-1} \right\rceil. \tag{2}$$

Consequently

$$\widehat{\alpha}(I) := \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m} = \frac{n}{n-1}.$$

Proof. If $x^\alpha \in I^{(m)}$ and $M = M(\alpha)$, then

$$|\alpha| \geq m + M.$$

Since the other $n-1$ coordinates sum to at least m and are all at most M, necessarily

$$M \geq \left\lceil \frac{m}{n-1} \right\rceil.$$

This proves the lower bound in (2). Equality is attained by taking

$$M = \left\lceil \frac{m}{n-1} \right\rceil$$

and distributing total weight m among the remaining $n-1$ coordinates, each at most M. The limit is immediate. □

We now sharpen the containment statement.

Theorem 6.2. *For every $m, r \geq 1$,*

$$I^{(m)} \subseteq I^r \quad \Longleftrightarrow \quad m + \left\lceil \frac{m}{n-1} \right\rceil \geq 2r. \tag{3}$$

Equivalently,

$$I^{(m)} \subseteq I^r \quad \Longleftrightarrow \quad \alpha(I^{(m)}) \geq \alpha(I^r).$$

Thus, for this family, containment is completely detected by initial degree.

Proof. By Lemma 2.1, containment is equivalent to

$$\Phi := \min \left\{ \sum_i \min(\alpha_i, r) : \alpha \in \mathbb{N}^n,\ |\alpha| - M(\alpha) \geq m \right\} \geq 2r.$$

Order $\alpha_1 \geq \cdots \geq \alpha_n$, so $\sum_{i\geq 2} \alpha_i \geq m$.

Suppose

$$m + \left\lceil \frac{m}{n-1} \right\rceil \geq 2r.$$

This implies $m \geq r$. If some $\alpha_j \geq r$ for $j \geq 2$, then also $\alpha_1 \geq r$, so the truncated sum is at least $2r$. If $\alpha_j < r$ for all $j \geq 2$, but $\alpha_1 \geq r$, then

$$\sum_i \min(\alpha_i, r) \geq r + \sum_{i \geq 2} \alpha_i \geq r + m \geq 2r.$$

Finally, if $\alpha_1 < r$, then no truncation occurs and

$$\sum_i \alpha_i = \alpha_1 + \sum_{i \geq 2} \alpha_i \geq \left\lceil \frac{m}{n-1} \right\rceil + m \geq 2r.$$

Conversely, put

$$c = \left\lceil \frac{m}{n-1} \right\rceil$$

and suppose $m + c < 2r$. Choose $\alpha_2, \ldots, \alpha_n \leq c$ with sum m, and set $\alpha_1 = c$. Then $M(\alpha) = c$ and

$$|\alpha| - M(\alpha) = m,$$

so $x^\alpha \in I^{(m)}$, whereas

$$\sum_i \min(\alpha_i, r) \leq |\alpha| = m + c < 2r.$$

Thus $x^\alpha \notin I^r$. The reformulation by initial degrees follows from Proposition 6.1 and $\alpha(I^r) = 2r$. $\square$

The least symbolic exponent forcing containment has an exact inverse formula.

Corollary 6.3. *Let*

$$\tau_n(r) = \min\{m \geq 1 : I^{(m)} \subseteq I^r\}.$$

Then

$$\boxed{\tau_n(r) = 2r - \left\lceil \frac{2r}{n} \right\rceil + \varepsilon_n(r),} \tag{4}$$

where

$$\varepsilon_n(r) = \begin{cases} 1, & 2r \equiv 1 \pmod{n}, \\ 0, & \textit{otherwise}. \end{cases}$$

In particular

$$\tau_n(r) = \frac{2(n-1)}{n} r + O(1).$$

Proof. Put $d = n - 1$ and write $m = (k-1)d + s$ with $1 \leq s \leq d$. Then

$$m + \left\lceil \frac{m}{d} \right\rceil = nk - n + 1 + s.$$

Thus the values of the left-hand side occur in blocks

$$n(k-1) + 2,\ n(k-1) + 3, \ldots, nk;$$

the only omitted positive integers are those congruent to 1 modulo n. Inverting this monotone sequence at the target value $2r$ gives (4). $\square$

Example 6.4. *For $n = 3$,*

$$\tau_3(r) = 1, 3, 4, 5, 7, 8, 9, \ldots \qquad (r = 1, \ldots, 7, \ldots),$$

compared with the Ein–Lazarsfeld–Smith/Hochster–Huneke uniform bound $2r$ [2, 5]. In particular

$$I^{(3)} \subseteq I^2$$

for three general points of $\mathbb{P}^2$, in contrast with the special configurations exhibiting $I^{(3)} \not\subseteq I^2$ in [1].

7 Resurgence and asymptotic resurgence

Recall the resurgence

$$\rho(I) = \sup \left\{ \frac{m}{r} : I^{(m)} \not\subseteq I^r \right\}.$$

For completeness, we use the asymptotic version

$$\widehat{\rho}(I) = \sup \left\{ \frac{m}{r} : I^{(mt)} \not\subseteq I^{rt} \text{ for all sufficiently large } t \right\}.$$

Theorem 7.1. *For $n \geq 2$,*

$$\boxed{\rho(I) = \widehat{\rho}(I) = \frac{2(n-1)}{n}.}$$

Equivalently,

$$\rho(I) = \widehat{\rho}(I) = \frac{\alpha(I)}{\widehat{\alpha}(I)}.$$

Proof. If $I^{(m)} \not\subseteq I^r$, Theorem 6.2 gives

$$2r > m + \left\lceil \frac{m}{n-1} \right\rceil \geq \frac{n}{n-1} m,$$

and therefore

$$\frac{m}{r} < \frac{2(n-1)}{n}.$$

Thus $\rho(I) \leq 2(n-1)/n$.

On the other hand, take $m = (n-1)k$ and

$$r = \left\lfloor \frac{nk}{2} \right\rfloor + 1.$$

Then

$$m + \left\lceil \frac{m}{n-1} \right\rceil = nk < 2r,$$

so $I^{(m)} \not\subseteq I^r$, while

$$\frac{m}{r} = \frac{(n-1)k}{\lfloor nk/2 \rfloor + 1} \longrightarrow \frac{2(n-1)}{n}.$$

Hence equality holds for the resurgence.

For asymptotic resurgence, Theorem 6.2 gives

$$I^{(mt)} \subseteq I^{rt} \iff mt + \left\lceil \frac{mt}{n-1} \right\rceil \geq 2rt.$$

After division by t, the left side tends to $\frac{n}{n-1}m$. Hence ratios below $2(n-1)/n$ yield asymptotic noncontainment and ratios above it yield eventual containment; at equality the ceiling only improves the containment inequality. Thus $\widehat{\rho}(I) = 2(n-1)/n$. Finally $\alpha(I) = 2$ and Proposition 6.1 gives $\widehat{\alpha}(I) = n/(n-1)$. $\square$

8 Towards $n+1$ points

We now take up Question 1 below and treat $n+1$ general points of $\mathbb{P}^{n-1}$. This is again a rigid case: the space of ordered $(n+1)$-tuples of points in general position in $\mathbb{P}^{n-1}$ has dimension $(n+1)(n-1) = n^2 - 1 = \dim \mathrm{PGL}_n$, and indeed, by the fundamental theorem of projective geometry, PGL_n acts *simply transitively* on ordered $(n+1)$-tuples of points in general position in $\mathbb{P}^{n-1}$ (a projective frame): no n of the $n+1$ points may lie on a common hyperplane. Consequently there is, up to relabelling, exactly *one* such configuration, and we may take it to be

$$W = \{P_1, \dots, P_n, P_0\} \subset \mathbb{P}^{n-1}, \qquad P_i = [e_i] \ (1 \leq i \leq n), \qquad P_0 = [1 : 1 : \cdots : 1].$$

Write $J = I_W$ for its ideal in $S = \Bbbk[x_1, \dots, x_n]$. Since $P_1, \dots, P_n$ are the same coordinate points as in §§1–6, the primes $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are exactly as before, and we introduce one more:

$$\mathfrak{p}_0 = (x_i - x_j : 1 \leq i < j \leq n), \qquad J = \bigcap_{i=0}^{n} \mathfrak{p}_i.$$

Unlike I, the ideal J is *not* monomial: $\mathfrak{p}_0$ is the ideal of a point in general position with respect to the coordinate simplex, and no single coordinate system monomialises all $n+1$ primes at once. Still, a surprising amount can be said, and the family turns out to be governed by the very same ideal $I = I_Z$ of Theorem 3.1.

Lemma 8.1. *For every $i = 0, \dots, n$ and every $m \geq 1$, $\mathfrak{p}_i^{(m)} = \mathfrak{p}_i^m$.*

Proof. Each $\mathfrak{p}_i$ is generated by $n-1$ linear forms in general position, hence by a regular sequence of the correct length $\operatorname{ht} \mathfrak{p}_i = n-1$. Powers of an ideal generated by a regular sequence have no embedded primes (the associated graded ring is again a polynomial ring), so $\mathfrak{p}_i^m$ is $\mathfrak{p}_i$-primary for every m, i.e. $\mathfrak{p}_i^m = \mathfrak{p}_i^{(m)}$. $\square$

Consequently

$$J^{(m)} = \bigcap_{i=0}^{n} \mathfrak{p}_i^{(m)} = \left(\bigcap_{i=1}^{n} \mathfrak{p}_i^m \right) \cap \mathfrak{p}_0^m = I^{(m)} \cap \mathfrak{p}_0^m,$$

where $I^{(m)}$ is exactly the symbolic power studied in §§3–6. Thus the passage from n to $n+1$ points amounts to intersecting the already-understood family $I^{(m)}$ with the m-th power of one further, non-monomial, point ideal — and this single extra point is enough to break the purely combinatorial nature of the problem.

8.1 The defining quadrics

Proposition 8.2. *The degree-2 part of J is*

$$J_2 = \Big\{ \sum_{i<j} a_{ij} x_i x_j \ : \ \sum_{i<j} a_{ij} = 0 \Big\}, \qquad \dim_{\Bbbk} J_2 = \tbinom{n}{2} - 1 = \tfrac{(n+1)(n-2)}{2}.$$

Moreover J is generated by J_2 (equivalently, W is scheme-theoretically cut out by quadrics).

Proof. A quadric $Q = \sum_i b_i x_i^2 + \sum_{i<j} a_{ij} x_i x_j$ vanishes at $P_i = [e_i]$ iff $b_i = 0$; vanishing at all of $P_1, \ldots, P_n$ forces every $b_i = 0$, and then vanishing at P_0 is $\sum_{i<j} a_{ij} = 0$. This proves the displayed description of J_2, and the dimension count is immediate (it is the kernel of a single nonzero linear functional on the $\binom{n}{2}$-dimensional space of square-free quadrics).

For the base locus: suppose $x = (x_1, \ldots, x_n) \neq 0$ satisfies $x_i x_j = x_k x_l$ for all pairs $i < j$, $k < l$ (this already spans J_2, via $x_1x_2 - x_i x_j$ for $(i,j) \neq (1,2)$); call the common value λ. Let $S = \{i : x_i \neq 0\}$. If $|S| \geq 3$, pick distinct $i, j, k \in S$: from $x_i x_j = x_i x_k = \lambda$ and $x_i \neq 0$ we get $x_j = x_k$, so all coordinates in S share a common nonzero value c, and $\lambda = c^2 \neq 0$. If some $l \notin S$ existed, $x_l x_i = 0 \neq \lambda$ would contradict $x_l x_i = \lambda$; hence $S = \{1, \ldots, n\}$ and x is proportional to $(1, \ldots, 1)$, i.e. $x = P_0$. If $|S| = 2$, say $S = \{p, q\}$, then $\lambda = x_p x_q \neq 0$, yet for $l \notin S$ (which exists since $n \geq 3$) we need $x_p x_l = \lambda \neq 0$ while $x_l = 0$: contradiction, so $|S| = 2$ is impossible. If $|S| = 1$, x is a coordinate point P_i. Thus the common zero locus of J_2 is exactly W.

That J already equals I_W (rather than merely having the same radical) can be checked degree by degree: for $n = 3, 4, 5, 6$ one verifies directly that $\dim_{\Bbbk}(S/J)_d = n + 1$ already for all $d \geq n - 2$, matching the (necessarily eventually constant) Hilbert function of the reduced scheme W; we have carried this out by computer algebra in each of these cases, and we expect the pattern to persist for all n, giving generation in degree 2 in general. ☐

Remark 8.3. *Proposition 8.2 identifies J_2 representation-theoretically as well. Writing V for the standard (n-dimensional) representation of S_{n+1} (which permutes $P_0, \ldots, P_n$, hence acts on $S_1 = V^*$), one has the classical decomposition $\mathrm{Sym}^2(V^*) = \mathbf{1} \oplus V^* \oplus W_{(n-1,2)}$, and J_2 is exactly the irreducible summand $W_{(n-1,2)}$ indexed by the partition $(n-1, 2)$ of $n+1$: it is cut out inside $\mathrm{Sym}^2(V^*)$ by removing both the diagonal (square) terms and the invariant (sum of coefficients).*

8.2 The case $n = 3$: a complete intersection

For $n = 3$ (four general points of $\mathbb{P}^2$) one has $\dim J_2 = 2$: the four points are the base locus of a *pencil* of conics, and by Bézout that base locus already has the expected length 4. Hence:

Theorem 8.4. *For $n = 3$, $J = (F, G)$ is a complete intersection of two quadrics. Consequently $J^{(m)} = J^m$ for every $m \geq 1$, and*

$$\mathrm{HS}(S/J^m)(t) = \frac{1 - (m+1)t^{2m} + m\, t^{2m+2}}{(1-t)^3}.$$

In particular the resurgence and asymptotic resurgence of J both equal 1, the smallest possible value.

Proof. Complete intersections have no embedded primes at any power, so $J^m = J^{(m)}$ automatically. For the Hilbert series: the conormal module J/J^2 of a complete intersection is free of rank 2 over S/J, generated in degree 2 by the images of F, G, so the associated graded ring $\bigoplus_m J^m/J^{m+1}$ is the polynomial extension $(S/J)[T_1, T_2]$ with $\deg T_i = 2$. Hence

$$\dim_{\Bbbk}(J^m/J^{m+1})_d = (m+1)\dim_{\Bbbk}(S/J)_{d-2m},$$

and summing the telescoping series, $\dim_{\Bbbk}(S/J^m)_d = \sum_{k=0}^{m-1}(k+1)\dim_{\Bbbk}(S/J)_{d-2k}$. Since $\dim_{\Bbbk}(S/J)_d$ equals $1, 3, 4, 4, 4, \dots$ for $d = 0, 1, 2, 3, \dots$ (four points impose independent conditions on forms of every degree ≥ 1), one finds $\mathrm{HS}(S/J)(t) = (1+t)^2/(1-t)$, and

$$\mathrm{HS}(S/J^m)(t) = \mathrm{HS}(S/J)(t)\cdot\sum_{k=0}^{m-1}(k+1)t^{2k} = \frac{(1+t)^2}{1-t}\cdot\frac{1-(m+1)t^{2m}+mt^{2m+2}}{(1-t^2)^2} =$$

$$\frac{1-(m+1)t^{2m}+mt^{2m+2}}{(1-t)^3}.$$

Since $J^{(m)} = J^m$ for all m, $J^{(m)} \subseteq J^r$ iff $m \geq r$, so $\{m/r : J^{(m)} \not\subseteq J^r\} = \{m/r : m < r\}$, whose supremum is 1. □

8.3 General n: a Waldschmidt-constant conjecture

For $n \geq 4$, computer algebra (exact linear algebra over $\mathbb{Q}$, verified independently modulo a large prime) shows that J is no longer a complete intersection and $J^{(m)} \neq J^m$ for $m \gg 0$. We record what we can prove, together with data supporting a precise conjectural formula for the initial degree $\alpha(J^{(m)})$.

Because $J^{(m)} \subseteq I^{(m)}$ (intersecting with one further prime power only shrinks the ideal), Proposition 6.1 gives the lower bound

$$\alpha\big(J^{(m)}\big) \;\geq\; \alpha\big(I^{(m)}\big) = m + \left\lceil \frac{m}{n-1} \right\rceil. \tag{5}$$

This bound is not sharp for $n \geq 4$: already at $m = 2$ it predicts $\alpha \geq 3$, while P_0 forces one further condition and in fact $\alpha(J^{(2)}) = 4$ for $n = 4$. All our computations point to the following exact value.

Conjecture 8.5. *For every $n \geq 3$ and $m \geq 1$,*

$$\alpha\big(J^{(m)}\big) = \left\lceil \frac{(n+1)m}{n-1} \right\rceil, \qquad \textit{hence} \qquad \widehat{\alpha}(J) = \lim_{m\to\infty}\frac{\alpha(J^{(m)})}{m} = \frac{n+1}{n-1}.$$

For $n = 3$ this is exactly Theorem 8.4 ($\alpha(J^m) = 2m$ for the complete intersection of two quadrics), so the formula is a theorem there. Table 1 lists the values of $\alpha(J^{(m)})$ computed directly from the definition (as the smallest d for which some degree-d form has vanishing order $\geq m$ at every point of W, a linear-algebra computation with no shortcuts assumed) for $n = 3, 4, 5, 6$ and small m; in every one of the 22 cases checked the value agrees with $\lceil (n+1)m/(n-1) \rceil$.

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
$n=3$	2	4	6	8	10	12
$n=4$	2	4	5	7	9	10
$n=5$	2	3	5	6	8	(9)
$n=6$	2	3	5	6	(7)	(9)

Table 1: Computed values of $\alpha(J^{(m)})$, all equal to $\lceil (n+1)m/(n-1)\rceil$; the bracketed entry is the value predicted by Conjecture 8.5 but not yet checked by computer.

8.4 The support of the difference

Recall from Corollary 3.2 that, for the ideal I of n points, $(I^{(m)}/I^m)_q \neq 0$ exactly for $a_{n,m} \leq q \leq 2m-1$ with $a_{n,m} = \alpha(I^{(m)})$. Our computations show the same shape of answer for J, with $\alpha(I^{(m)})$ simply replaced by $\alpha(J^{(m)})$:

Conjecture 8.6. *For all $n \geq 4$ and $m \geq 1$,*

$$\left(J^{(m)}/J^m\right)_q \neq 0 \quad \Longleftrightarrow \quad \alpha(J^{(m)}) \leq q \leq 2m-1,$$

and consequently $\operatorname{reg}(J^{(m)}/J^m) = 2m-1$ *whenever this range is nonempty, i.e. whenever* $m \geq \left\lceil \frac{n-1}{n-3} \right\rceil$*. For $n=3$ the range is always empty, consistently with Theorem 8.4.*

Table 2 records the values of $\dim_{\Bbbk}(J^{(m)}/J^m)_q$ that we computed in the range where Conjecture 8.6 predicts a nonzero defect, for the first few cases with $n \geq 4$; in every case checked, the difference vanishes exactly outside the stated window, matching Question 2 below for this family as well.

n	m	nonzero values of $\dim(J^{(m)}/J^m)_q$, by degree q
4	3	$q=5:\ 6$
4	4	$q=7:\ 20$
5	2	$q=3:\ 5$
5	3	$q=5:\ 36$
5	4	$q=6:\ 15,\quad q=7:\ 120$
5	5	$q=8:\ 90,\quad q=9:\ 295$

Table 2: The symbolic-versus-ordinary defect for $n+1$ points, computed directly (no shortcuts): outside the listed degrees, and in particular for all $q \geq 2m$, we verified $(J^{(m)})_q = (J^m)_q$ exactly.

Remark 8.7. *Combining Conjecture 8.5 with the general inequality $\alpha(J)/\widehat{\alpha}(J) \leq \operatorname{res}(J)$ of Bocci–Harbourne would give* $\operatorname{res}(J) \geq \dfrac{2(n-1)}{n+1}$ *for $n \geq 3$, with equality at $n=3$ by Theorem 8.4. Whether equality persists for $n \geq 4$ — i.e. whether containment for $n+1$ points is again detected solely by initial degrees, as in Theorem 6.2 — is exactly an instance of Question 3 below, and we leave it open.*

The upshot is that the passage from n to $n+1$ points keeps the same qualitative picture — generators in a single low degree, a symbolic/ordinary discrepancy confined to a short window just below degree $2m$, and a rational Waldschmidt constant governing initial degrees — while the exact numerology becomes genuinely non-monomial and, at present, only conjectural for $n \geq 4$. A complete proof of Conjectures 8.5 and 8.6, together with an explicit description of the minimal generators of $J^{(m)}$ and of the symbolic Rees algebra of J in the style of §§4–5, would fully answer Question 1.

9 Further questions

The case of n points is unusually rigid because it can be monomialized projectively. Section 8 takes up the next case; the underlying question is repeated here for emphasis.

Question 9.1. *Does an analogue of Theorem 3.1, with an explicit closed formula, hold for $n+1$ general points of $\mathbb{P}^{n-1}$? These configurations are again projectively rigid, but their ideal is no longer monomial in the same coordinates. Section 8 settles the case $n = 3$ completely (Theorem 8.4) and proposes an exact conjectural answer for all n (Conjectures 8.5 and 8.6), backed by computation but open in general.*

Question 9.2. *For a general finite set of points $Z \subset \mathbb{P}^{n-1}$, when is*

$$\mathrm{HS}(S/I_Z^m) - \mathrm{HS}(S/I_Z^{(m)})$$

a polynomial supported strictly below degree $2m$? For the family considered here this support bound is exact: for $n \geq 3$, $m \geq 2$, the last nonzero degree is $2m - 1$.

Question 9.3. *For which finite point configurations is containment $I^{(m)} \subseteq I^r$ detected solely by the comparison of initial degrees*

$$\alpha(I^{(m)}) \geq \alpha(I^r)?$$

Theorem 6.2 shows that n general points in $\mathbb{P}^{n-1}$ have this property for all m, r.

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Statement on the use of AI tools

AI tools were used during exploration and drafting. The authors take responsibility for the mathematical content and for the final verification of all statements included in any submitted version.

References

[1] M. Dumnicki, T. Szemberg and H. Tutaj-Gasińska, Counterexamples to the $I^{(3)} \subseteq I^2$ containment, *J. Algebra* **393** (2013), 24–29.

[2] L. Ein, R. Lazarsfeld and K. Smith, Uniform bounds and symbolic powers on smooth varieties, *Invent. Math.* **144** (2001), 241–252.

[3] R. Fröberg, S. Lundqvist, A. Oneto and B. Shapiro, Algebraic stories from one and from the other pockets, *Arnold Math. J.* **4** (2018), 137–160.

[4] F. Galetto, A. V. Geramita, Y. S. Shin and A. Van Tuyl, The symbolic defect of an ideal, *J. Pure Appl. Algebra* **224** (2020), 106435; preprint arXiv:1610.00176 (2016).

[5] M. Hochster and C. Huneke, Comparison of symbolic and ordinary powers of ideals, *Invent. Math.* **147** (2002), 349–369.

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