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NEW MULTIPLIER SEQUENCES VIA DISCRIMINANT AMOEBAE

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To Vladimir Igorevich Arnold who left us too early

Abstract. In their classic 1914 paper, Polýa and Schur introduced and characterized two types of linear operators acting diagonally on the monomial basis of $\mathbb{R}[x]$, sending real-rooted polynomials (resp. polynomials with all nonzero roots of the same sign) to real-rooted polynomials. Motivated by fundamental properties of amoebae and discriminants discovered by Gelfand, Kapranov, and Zelevinsky, we introduce two new natural classes of polynomials and describe diagonal operators preserving these new classes. A pleasant circumstance in our description is that these classes have a simple explicit description, one of them coinciding with the class of log-concave sequences.

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1. Introduction

The theory of *linear preservers* (linear operators preserving certain families of matrices or polynomials) is a widely developed and active area of mathematics (see, e.g., [SU92] and the references therein). Linear preservers have found applications in many areas such as approximation theory, probability theory, and statistics (see, e.g., [Kar68]), and have even been used to give interesting reformulations of the Riemann Hypothesis [Cso03]. One of the most classical instances of the theory of linear preservers occurs in the setting of real-rooted polynomials, initiated in the late 19th century by Laguerre and Hermite.

Given a sequence of real numbers $\gamma = \{\gamma_j\}_{j=0}^{\infty}$ consider the linear operator $T_\gamma\colon \mathbb{R}[x] \to \mathbb{R}[x]$ acting on each x^j by multiplication by γ_j. We refer to such a T_γ as the *diagonal operator* corresponding to γ. Let $RR \subset \mathbb{R}[x]$ denote the collection of polynomials all of whose complex roots are real, i.e., *r̲eal-r̲ooted* polynomials. Following [PS14] we call γ a *multiplier sequence* (*"Faktorenfolge"*) *of the first kind*

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if $T_\gamma(RR) \subseteq RR$. Similarly, let SS denote the subset of RR consisting of polynomials p whose nonzero roots (all real, by assumption) are all of the same sign. A *multiplier sequence of the 2nd kind* is then a γ with $T_\gamma(SS) \subseteq RR$.

The following result of Polýa and Schur is fundamental.

Theorem A [PS14]**.** *Let $\gamma = \{\gamma_j\}_{j=0}^{\infty}$ be a sequence of real numbers and $T_\gamma : \mathbb{R}[x] \to \mathbb{R}[x]$ the corresponding diagonal operator. Then:*

(i) *γ is a multiplier sequence of the 1st kind (i.e., $T_\gamma(RR) \subseteq RR$) if and only if for all $n \in \mathbb{N}$ we have $T_\gamma((1+x)^n) \in SS$.*
(ii) *γ is a multiplier sequence of the 2nd kind (i.e., $T_\gamma(SS) \subseteq RR$) if and only if for all $n \in \mathbb{N}$ we have $T_\gamma((1+x)^n) \in RR$.* □

Remark 1. Polýa and Schur also obtained a transcendental characterization in terms of the generating function $\Phi_\gamma(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$.

There exist obvious versions of these notions for polynomials of bounded degree. In particular, a sequence $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_k)$ will be referred to as a *multiplier sequence of length* $k+1$ or simply a *finite multiplier sequence* if it has the above mentioned properties when acting on the linear space $\mathbb{R}_k[x]$ of real polynomials of degree at most k. In particular, we define $RR_k := RR \cap \mathbb{R}_k[x]$ and $SS_k := SS \cap RR_k$.

Craven and Csordas proved 60 years later that for a finite length multiplier sequence γ, checking whether γ is of first or second kind can be reduced to checking the image of just one polynomial under T_γ (see [CC77, Thm. 3.7] and [CC83, Thm. 3.1]).

Theorem B. *Let $\gamma = (\gamma_0, \ldots, \gamma_k)$ and T_γ the corresponding diagonal operator. Then for all $k \in \mathbb{N}$, we have:*

(i) $T_\gamma(RR_k) \subseteq RR_k$ *if and only if* $T_\gamma((1+x)^k) \in SS$.
(ii) $T_\gamma(SS_k) \subseteq RR_k$ *if and only if* $T_\gamma((1+x)^k) \in RR$.

Remark 2. While Assertion (i) is merely a rewording of [CC77, Thm. 3.7], Assertion (ii) appears to be new and follows upon a closer examination of Section 3 of [CC77].

Letting $q(x) := x^m(1+x)^2$, note that $q \in SS \subsetneq RR$ and q has -1 as a root of multiplicity 2. It then follows that if one decreases the coefficient of x^{m+1} in q (and leaves the coefficients of x^m and x^{m+2} fixed) then the resulting polynomial has non-real roots. With a little more work one then easily concludes that any multiplier sequence $\gamma = (\gamma_0, \gamma_1, \ldots)$ of first or second kind must satisfy *Turán's inequalities* (see, e.g., [CVV90] and [CC04, Problem 4.8]): $\gamma_j^2 \geqslant \gamma_{j-1}\gamma_{j+1}$ for all $j \geqslant 2$. Since we can naturally identify any finite multiplier sequence $(\gamma_0, \ldots, \gamma_k)$ with the infinite sequence $(\gamma_0, \ldots, \gamma_k, 0, 0, \ldots)$ the Turán Inequalities clearly hold for finite length multiplier sequences (of first or second kind) as well. The converse fails, however, as can be easily seen by perturbing the nonzero coefficients of $x^m(1+x)$ instead. The occurence of roots of multiplicity >1 here is one reason it is natural to start thinking of discriminants (see also Figures 1 and 2 below).

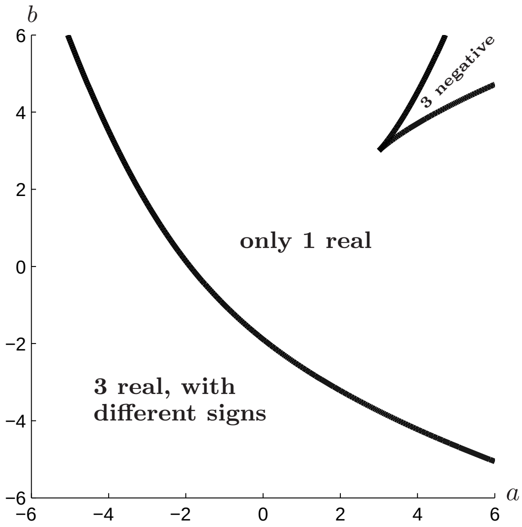

FIGURE 1. The discriminant variety of the family $1+ax+bx^2+x^3$ separates the coefficient space into regions according to the number of real roots.

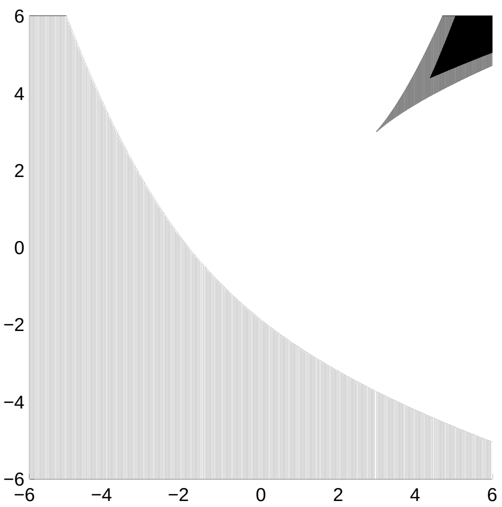

FIGURE 2. Corresponding slices of $SI_3^{\geqslant}$, SS_3, and RR_3: $SI_3^{\geqslant}$ is in black, $SI_3^{\geqslant} \subsetneqq SS_3 \subsetneqq RR_3$, and the complement of RR_3 is white.

Remark 3. Positive sequences satisfying Turán's inequalities are called *log-concave* and find frequent applications in combinatorics. An analoguous notion with the coefficients weighted by binomial coefficients is known as *ultra log-concavity* [Lig97], [KS06].

We will return to $x^m(1+x)$ momentarily but observe now that the polynomial $x^m(1+x)^2$ has the following special property: all polynomials obtained by arbitrary sign flips of its coefficients also belong to RR.

Definition 1. A real polynomial p is called *sign-independently real-rooted* if p is real-rooted and all polynomials obtained by arbitrary sign flips of the coefficients of p are real-rooted as well. We let SI denote the set of all sign-independently real-rooted polynomials and $SI^{\geqslant}$ denote the subset of SI consisting of polynomials with all coefficients nonnegative. Finally, we call γ a *multiplier sequence of the* 3rd *kind* if $T_\gamma(SI^{\geqslant}) \subseteq RR$.

Clearly, $SI^{\geqslant} \subsetneq SI$ and $SI^{\geqslant} \subsetneq SS \subsetneq RR$. Another simple example of a sign-independently real-rooted polynomial is $x^m(1+x)$ and less trivial examples can be found in Section 2.2. Similar to our earlier development we define $SI_k := SI \cap \mathbb{R}_k[x]$ and $SI_k^{\geqslant} := SI^{\geqslant} \cap \mathbb{R}_k[x]$. The sets $SI_3^{\geqslant}$, SS_3, and RR_3 are illustrated in Figure 2 below.

Our main results are summarized by the following two theorems and a corollary.

Theorem 1. *γ is a multiplier sequence of the third kind (finite or infinite) if and only if it is log-concave, i.e., $T_\gamma(x^n(1+x)^2) \in RR$ for all $n \in \mathbb{N}$. Moreover, any such γ satisfies $T_\gamma(SI^{\geqslant}) \subseteq SI^{\geqslant}$.*

Corollary 1. *If $p(x) = a_0 + a_1x + \cdots + a_kx^k \in SI_k^{\geqslant}$ then $a_\nu^2 \geqslant 4a_{\nu-1}a_{\nu+1}$ for all $\nu \in \{1, \ldots, k-1\}$, and any truncated polynomial $a_mx^m + a_{m+1}x^{m+1} + \cdots + a_nx^n$ obtained from p (for $0 \leqslant m < n \leqslant k$) has all its nonzero roots negative.*

Davenport and Polýa observed earlier [DP49] that log-concave positive sequences form a semigroup with respect to the *Hadamard product* $(\gamma_0, \gamma_1, \ldots)\cdot(\gamma_0', \gamma_1', \ldots) := (\gamma_0\gamma_0', \gamma_1\gamma_1', \ldots)$. In particular, it will be fruitful to observe later that the image of such sequences under coordinate-wise logarithm forms a cone.

More to the point, via *A-discriminant theory* [GKZ08], we can reinterpret the sets $SI_k^{\geqslant}$, SS_k, and RR_k in terms of the complement of an important hypersurface associated to k. This point of view yields yet another new family of multiplier sequences, in some sense dual to $SI^{\geqslant}$.

Definition 2. We define $II_k^{\geqslant}$ to be the set of those polynomials $p(x) = a_0 + a_1x + \cdots + a_kx^k$ such that (i) $a_j \geqslant 0$ for all j, (ii) $a_0, a_k > 0$, (iii) p has exactly 1 or 0 real roots according as k is odd or even, (iv) for any polynomial p^* obtained from p by multiplying any subset of the a_i with $i \in \{1, \ldots, k-1\}$ by -1, p^* also has maximally many imaginary roots in the sense of Condition (iii).

Note that for k even, any polynomial $p \in II_k^{\geqslant}$ is positive on all of $\mathbb{R}$, and any p^* obtained from p (as in Condition (iv) above) is also positive on all of $\mathbb{R}$.

Theorem 2. *A positive sequence $\gamma := (\gamma_0, \ldots, \gamma_k)$ satisfies $T_\gamma(II_k^{\geqslant}) \subseteq II_k^{\geqslant}$ if and only if*

$$\gamma_j^k \leqslant \left(\frac{\gamma_k}{\gamma_0}\right)^j \quad \textit{for all } j \in \{1, \ldots, k-1\}.$$

Within the next section, we will see how $SI_k^{\geqslant}$ and $II_k^{\geqslant}$ correspond naturally to opposite connected components of a particular *amoeba* complement.

2. Background on Discriminants and Amoebae

The first ingredient in the proof of our main results is the following construction: Consider the map $\mathrm{Log}\,|\cdot|\colon (\mathbb{C}^*)^{k+1} \to \mathbb{R}^{k+1}$ sending $\boldsymbol{a} = (a_0, a_1, \ldots, a_k) \in (\mathbb{C}^*)^{k+1}$ to $(\log|a_0|, \log|a_1|, \ldots, \log|a_k|)$. Notice that $\mathrm{Log}\,|\cdot|$ maps $\mathbb{R}_+^{k+1}$ diffeomorphically onto $\mathbb{R}^{k+1}$, where $\mathbb{R}_+$ is the set of all positive real numbers.

For any polynomial $q \in \mathbb{C}[a_0, \ldots, a_k]$ one defines its *amoeba* $\mathrm{Amoeba}(q)$ as the image of the complex algebraic hypersurface

$$H_q := \{\boldsymbol{a} = (a_0, \ldots, a_k) \in (\mathbb{C}^*)^{k+1} \mid q(\boldsymbol{a}) = 0\}$$

under $\mathrm{Log}\,|\cdot|$. Recall also that the *Newton polytope* of $q(x) := \sum_{\boldsymbol{\alpha}\in A} c_{\boldsymbol{\alpha}} x^\alpha$, written $\mathrm{Newt}(q)$, is the convex hull[2] of $\{\alpha \in \mathbb{Z}^{k+1} \mid c_\alpha \neq 0\}$, where the notation $x^\alpha := x_1^{\alpha_0} \ldots x_k^{\alpha_k}$ is understood. There is a natural 1-1 correspondence between unbounded connected components of the complement $\mathbb{R}^{k+1} \setminus \mathrm{Amoeba}(q)$ and the vertices of $\mathrm{Newt}(q)$.

Lemma 1 [GKZ08, Prop. 1.7 & Cor. 1.8, pp. 195–196]**.** *Suppose a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ has Newton polytope P and v is a vertex of P. Also let C denote the closure of the cone of inner normals to v. Then there is a unique unbounded connected component Γ of the complement to* $\mathrm{Amoeba}(f)$ *containing a translate of the cone C.* $\square$

The cone C above is also called the *recession cone* of Γ, since it consists of all translations $y \in \mathbb{R}^n$ with $y + \Gamma \subseteq \Gamma$.

Let Δ_k denote the discriminant of the family of polynomials $a_0 + \cdots + a_k x^k$, i.e., $\Delta_k \in \mathbb{Z}[a_0, \ldots, a_k]$ is the unique (up to sign) irreducible polynomial such that $a_0 + \cdots + a_k x^k$ has a root of multiplicity > 1 implies that $\Delta_k(a_0, \ldots, a_k) = 0$. For instance, $\Delta_3 := -27a_0^2a_3^2 + 18a_0a_1a_2a_3 + a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3$. More generally, Δ_k can be computed using a number of arithmetic operations polynomial in k (via a standard formula involving a $(2k-1) \times (2k-1)$ determinant), and is the special case $A = \{0, \ldots, k\}$ of an *A-discriminant* (see [GKZ08, Ch. 9 & 12] for further background).

Amoebae of A-discriminants have a more refined structure. For example, the boundary of $\mathrm{Amoeba}(\Delta_k)$ is contained in the image of the real part $H_{\Delta_k}^{\mathbb{R}}$ of the complex algebraic hypersurface H_{Δ_k} under $\mathrm{Log}\,|\cdot|$ (see Figure 3 below). The latter fact motivates the following definition.

Definition 3. For a complex algebraic hypersurface $H_q \subset \mathbb{C}^{k+1}$ defined by the equation $q(a_0, a_1, \ldots, a_k) = 0$ we define its *complete reflection* $H_q^\dagger$ as the union of the 2^{k+1} hypersurfaces given by $q(\pm a_0, \pm a_1, \ldots, \pm a_k) = 0$ for all 2^{k+1} possible choices of signs of coordinates (see, e.g., Figure 4 below).

Consider the restriction of the real part $(H_q^\dagger)^{\mathbb{R}}$ of $H_q^\dagger$ to $\mathbb{R}_+^{k+1}$. Notice that by the above remark each connected component of $\mathbb{R}_+^{k+1} \setminus (H_q^\dagger)^{\mathbb{R}}$ is mapped by $\mathrm{Log}\,|\cdot|$ diffeomorphically either onto a connected component of the complement $\mathbb{R}^{k+1} \setminus \mathrm{Amoeba}(q)$ or onto a connected subset of $\mathrm{Amoeba}(q)$ itself. One thus sees

[2] i.e., the smallest convex set containing...

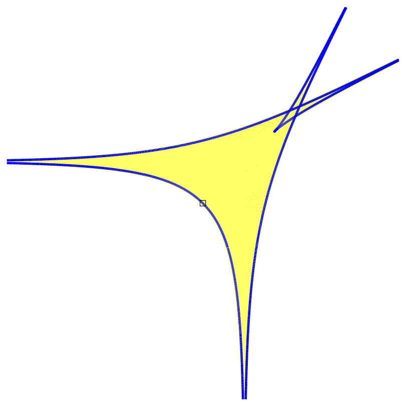

FIGURE 3. The amoeba of the specialized cubic discriminant $\Delta_3(1,\, a,\, b,\, 1)$ (the shaded area), and the image of $H^{\mathbb{R}}_{\Delta_3(1,a,b,1)}$ under $\mathrm{Log}\,|\cdot|$.

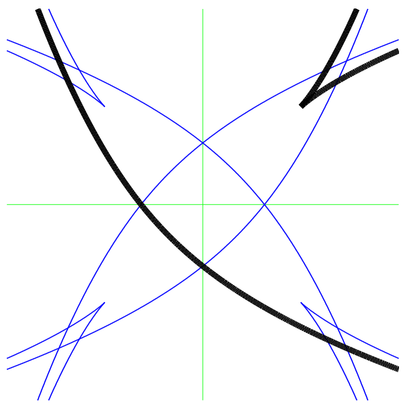

FIGURE 4. The real part of the discriminant variety of the family $1 + ax + bx^2 + x^3$ (bold) and its sign flips, i.e., $H^{\dagger}_{\Delta_3(1,a,b,1)}$.

that Amoeba(q) is the union of the images of some number of the latter connected components.

Returning to Δ_k, it is well known (see, e.g., [GKZ08, p. 271]) that Δ_k has the two homogeneities:

$$\Delta_k(\lambda a_0,\, \lambda a_1,\, \lambda a_2,\, \ldots,\, \lambda a_k) = \lambda^{2(k-1)}\Delta_k(\boldsymbol{a})$$

and

$$\Delta_k(a_0,\, \lambda a_1,\, \lambda^2 a_2,\, \ldots,\, \lambda^k a_k) = \lambda^{k(k-1)} \Delta_k(\boldsymbol{a}).$$

This immediately implies that $\mathrm{Newt}(\Delta_k)$ has codimension at least 2. In fact, the codimension is exactly 2, and it is then easy to see that $\mathrm{Amoeba}(\Delta_k)$ is an $\mathbb{R}^2$-bundle over a base that is an amoeba of smaller dimension. In particular, one can take the base to be the amoeba of $\Delta_k(1,\, a_1,\, \ldots,\, a_{k-1},\, 1)$, thus explaining why our illustrations for $k = 3$ are in the plane, as opposed to $\mathbb{R}^4$.

There is also a combinatorial formula for the monomials in Δ_k with exponents corresponding to vertices of $\mathrm{Newt}(\Delta_k)$ (see [GKZ08, pp. 300 & 302]). Namely, each such vertex monomial corresponds to a unique subdivision of the line segment $[0,\, k]$ into a collection of segments $\{[0,\, k_1],\, [k_1,\, k_2],\, \ldots,\, [k_m,\, k]\}$, with integers $0 < k_1 < k_2 < \ldots < k_m < k$. In particular, the finest subdivision $\{[0,\, 1],\, \ldots,\, [k-1,\, k]\}$ of $[0,\, k]$ into unit intervals is associated with the monomial

$$\pm a_1^2 a_2^2 \ldots a_{k-1}^2 = \pm (a_1 a_2 \ldots a_{k-1})^2, \tag{1}$$

whereas the second finest subdivisions, having one segment $[l-1,\, l+1]$ of length two and all other segments of unit length, correspond to the monomials

$$\pm 4\, a_{l-1} a_l^{-2} a_{l+1} (a_1 a_2 \ldots a_{k-1})^2, \quad l \in \{1,\, \ldots,\, k-1\}. \tag{2}$$

Moreover, thanks to Lemma 1, we obtain a trinity of associations (see also Figure 5 below):

$$\begin{tikzcd}[column sep=small] & \begin{array}{c}\text{certain unbounded connected components}\\ \text{of the complement of Amoeba}(\Delta_k)\end{array} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{vertex monomials of } \Delta_k \arrow[rr, leftrightarrow] & & \text{triangulations of } \{0,\, \ldots,\, k\} \end{tikzcd}$$

Combinatorially, $\mathrm{Newt}(\Delta_k)$ is a cube of dimension $k-1$ and the monomials (2) represent the vertices $v_0 + e_{l-1} - 2e_l + e_{l+1}$ neighboring the vertex $v_0 = (0,\, 2,\, 2,\, \ldots,\, 2,\, 0)$ corresponding to the monomial (1).

2.1. Archimedean Newton Polygons. An arguably more direct association between polynomials of degree k and subdivisions of the point set $\{0,\, \ldots,\, k\}$ can be obtained via the *Archimedean Newton polygon*, which dates back to work of Ostrowski in the 1940s [Ost40, pp. 106 & 132]. This particular kind of Newton polygon further elucidates the connection between Theorems 1 and 2, and we use the appelation "Archimedean" to complement the *non-Archimedean* Newton polygons coming from number theory and tropical geometry.

Definition 4. Given any polynomial $f(x) = a_0 + a_1 x + \cdots + a_k x^k$, its *Archimedean Newton polygon*, written $\mathrm{ArchNewt}(f)$, is the convex hull of the finite point set $\{(i,\, -\log|a_i|) \mid i \in \{0,\, \ldots,\, k\}\}$. We also call any edge of $\mathrm{ArchNewt}(f)$ a *lower* edge if it has an inner normal with positive last coordinate.

One can observe experimentally that there is a strong correlation between the slopes of the lower edges of $\mathrm{ArchNewt}(f)$ and the absolute values of the roots of f.

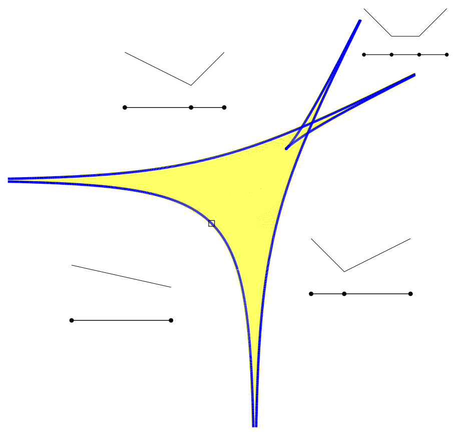

FIGURE 5. Lower hulls of ArchNewt(f), and associated subdivisions of $\{0, 1, 2, 3\}$, corresponding to the unbounded components of the complement of Amoeba$(\Delta_3(1, a, b, 1))$.

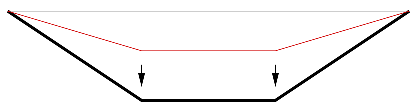

FIGURE 6. $1+2.9x+2.9x^2+x^3$ (the broken line above) does not lie in the upper right component of Figure 5, but $1+9x+9x^2+x^3$ (with a more "bowed" lower hull for its ArchNewt) does...

In particular, paraphrasing in more modern language, Ostrowski proved remarkable explicit bounds revealing how the slopes of the lower edges of ArchNewt(f) approximate the negatives of the logs of the norms of the roots of f [Ost40, pp. 106 & 132].

Even more directly, one notes that the lower hull of ArchNewt(f) naturally associates, via orthogonal projection onto the first coordinate, a triangulation of $\{0, \ldots, k\}$ to f. In particular, it is easy to derive that the strict log-concavity[3] of the sequence of coefficients of f is nothing more than the condition that ArchNewt(f) have exactly $k-1$ lower edges. However, unless ArchNewt(f) is sufficiently "bowed", a degree k polynomial f having ArchNewt(f) with $k-1$ edges need *not* correspond to a point in the corresponding component Γ of the complement of Amoeba(Δ_k). For instance, $1+2.9x+2.9x^2+x^3$ has only 1 real root, but $1+9x+9x^2+x^3$ has 3 real roots.

[3]Strict log-concavity for $(\gamma_0, \ldots, \gamma_k)$ simply means that $\gamma_j^2 > \gamma_{j-1}\gamma_{j+1}$ for all $j \in \{1, \ldots, k-1\}$.

As we will see in Lemma 2 of the next section, multiplier sequences can be used to make the lower hull of an ArchNewt(f) more bowed. Similarly, the sequences highlighted in Theorem 2 can clearly be identified with those f having ArchNewt(f) with exactly 1 lower edge. Thus, Theorem 1 (resp. Theorem 2) appears to relate maximal (resp. minimal) triangulations with polynomials having maximally (resp. minimally) many real roots.

2.2. Supporting Results on Real-Rooted Polynomials. Using the notation $x_l = \log|a_l|$ we see that $\mathrm{Amoeba}(\Delta_k)$ is the set of vectors $(x_0, \ldots, x_k) \in \mathbb{R}^{k+1}$ such that the torus $|a_0| = e^{x_0}, \ldots, |a_k| = e^{x_k}$ intersects the discriminant locus H_{Δ_k}.

Proposition 1. *The map* $\mathrm{Log}|\cdot|$ *is a diffeomorphism from* $SI_k^{\gtrless}$ *to the connected component of the complement of* $\mathrm{Amoeba}(\Delta_k)$ *corresponding to the monomial* (1).

The proof of this proposition is based on several additional statements. Along the way, we will also see some more examples of sign-independently real-rooted polynomials.

First consider the vector $s \in \mathbb{N}^{k-1}$ given by

$$s_j = \left(\left|\frac{k}{2} - j\right| + 1\right) + \left(\left|\frac{k}{2} - j\right| + 2\right) + \cdots + \frac{k}{2}, \quad j \in \{1, \ldots, k-1\},$$

for k even, and by

$$s_j = \left(j - \frac{k-1}{2}\right) + \left(j + 1 - \frac{k-1}{2}\right) + \cdots + \frac{k-1}{2}, \quad j \in \{1, \ldots, k-1\},$$

for k odd.

The first few instances of s are (1) for $k = 2$; (1, 1) for $k = 3$; (2, 3, 2) for $k = 4$; (2, 3, 3, 2) for $k = 5$; (3, 5, 6, 5, 3) for $k = 6$; (3, 5, 6, 6, 5, 3) for $k = 7$; (4, 7, 9, 10, 9, 7, 4) for $k = 8$; (4, 7, 9, 10, 10, 9, 7, 4) for $k = 9$; and (5, 9, 12, 14, 15, 14, 12, 9, 5) for $k = 10$.

Lemma 2. *The polynomial*

$$p_k(x) = 1 + \lambda^{s_1} x + \lambda^{s_2} x^2 + \cdots + \lambda^{s_{k-1}} x^{k-1} + x^k$$

of degree k *is sign-independently real-rooted for any sufficiently large value of the positive real parameter* λ.

Proof. This follows from the fact that for large λ the polynomial p_k has coefficients approaching the polynomial q_k given by:

$$q_k(x) = (x + \lambda^{-k/2})(x + \lambda^{1-k/2}) \ldots (x + \lambda^{k/2})$$

if k is even, and by

$$q_k(x) = (x + \lambda^{-(k-1)/2})(x + \lambda^{1-(k-1)/2}) \cdots (x + \lambda^{(k-1)/2})$$

if k is odd.

Indeed, in order to see that p_k is real-rooted for large positive λ, one observes that the roots of q_k are all real, and since they are given by distinct powers of λ, there are k of different magnitude. Hence, under the small change of real coefficients that is needed to deform q_k to the original polynomial p_k, the roots remain well apart, and hence cannot form any conjugate pair of complex roots. Now, one can easily check that for sufficiently large λ changing arbitrarily signs of roots of q_k one obtains 2^k polynomials close to 2^k polynomials obtained from q_k by arbitrary sign changes of its coefficients. Thus, any change of signs of some of the coefficients of p_k just corresponds to an appropriate sign change in some of the roots of q_k, and the preceding argument again shows that the polynomials are still real-rooted. □

Lemma 3. *The set $SI_k^{\geqslant}$ is fibered over $SI_{k-1}^{\geqslant}$ with contractible 1-dimensional fibers.*

Proof. Notice that the restriction of $SI_k^{\geqslant}$ to the hyperplane $a_0 = 0$ is in obvious 1-1 correspondence with $SI_{k-1}^{\geqslant}$ obtained by dividing a polynomial $p(x) = a_1x + \cdots + a_kx^k$ from the former set by the variable x. To finish the proof we show that for any $p(x) = a_0+a_1x+\cdots+a_kx^k$ belonging to $SI_k^{\geqslant}$ the family of polynomials $p_\tau = p-a_0\tau$, $\tau \in [0, 1]$ belong to $SI_k^{\geqslant}$ thus forming the required fiber of the projection in question. Indeed, consider for any real rooted polynomial $p(x) = a_0 + a_1x + \cdots + a_kx^k$ the family $p_\varepsilon(x) = p(x) + \varepsilon$, where $\varepsilon \in \mathbb{R}$. It is obvious that $p_\varepsilon(x)$ is real-rooted if and only if $\varepsilon \in [v_{\min}, V_{\max}]$, where $v_{\min}$ is the maximal local minimum of $p(x)$ and $V_{\max}$ is its minimal local maximum. Now take $p \in SI_k^{\geqslant}$ and consider its family $p_\varepsilon(x)$. Since all the a_i are now nonnegative, consider $p_-(x) = -a_0 + a_1x + \cdots + a_kx^k$, which must also be real-rooted. Thus at least for ε in the interval $[-2a_0, 0]$ one has that $p_\varepsilon(x)$ is real-rooted. Exactly the same argument works for all $p_\pm$ obtained from p by arbitrary sign changes of its coefficients proving that the family $p - a_0\tau$, $\tau \in [0, 1]$ sits inside $SI_k^{\geqslant}$. □

2.3. Finding Recession Cones. Denote by Γ_k the connected component of $\mathbb{R}^{k+1} \setminus \mathrm{Amoeba}(\Delta_k)$ corresponding to the monomial (1), and let C_k denote the recession cone of Γ_k. We now prove the following crucial result.

Lemma 4. *The cone C_k is given by the inequalities $2x_l \geqslant x_{l-1} + x_{l+1}$, for $l \in \{1, \ldots, k-1\}$.*

Proof. Recall that for a polynomial $p(\boldsymbol{z})$ in n complex variables $\boldsymbol{z} = (z_1, \ldots, z_n)$, one defines its *Ronkin function* $N_p(x)$, in n real variables $\bar{x} = (x_1, \ldots, x_n)$, by the formula

$$\frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(\mathbf{x})} \log|p(\boldsymbol{z})| \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n},$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$. It is known that the Ronkin function is convex, and it is affine on each connected component of the complement of the amoeba $\mathrm{Amoeba}(p)$. Equivalently, N_p is given by the integral

$$N_p(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \log|p(\boldsymbol{z})|\, d\theta_1 \ldots d\theta_n,$$

where

$$\boldsymbol{z} = \left(e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n}\right)$$

(see [PT04]). As an example, the Ronkin function of a monomial $p(\boldsymbol{z}) = az_1^{l_1} \ldots z_n^{l_n}$, $a \neq 0$, is given by

$$N_p(\boldsymbol{x}) = \log|a| + l_1x_1 + \cdots + l_nx_n.$$

From general results proved in [PR04] one knows that the Ronkin function of Δ_k is equal to $\log|c_v| + \langle v,\, x\rangle$ in the component corresponding to a vertex monomial $c_v x^v$.

In particular, in the components of the special vertex monomials (1) and (2), the Ronkin function coincides with the affine linear functions

$$2x_1 + \cdots + 2x_{k-1} = 2\,(x_1 + \cdots + x_{l-1})$$

and

$$x_{l-1} - 2x_l + x_{l+1} + 2\,(x_1 + \cdots + x_{k-1})$$

respectively. Now, by [PST05] one knows that the amoeba of $\Delta_k(1,\, a_1,\, \ldots,\, a_{k-1},\, 1)$ does not have any unbounded connected components for its complement other than those corresponding to the vertices of $\mathrm{Newt}(\Delta_k(1,\, a_1,\, \ldots,\, a_{k-1},\, 1))$. Now let S_{Δ_k} denote the spine (see [PR04] for its definition) and let S_∞ denote a sufficiently small neighborhood of S_{Δ_k} about infinity. It then follows that S_∞ is exactly a neighborhood about infinity of the corner locus of the pieceswise linear convex function (or tropical polynomial)

$$\max_v \bigl(\log|c_v| + \langle v,\, x\rangle\bigr),$$

where v ranges over the vertices of the Newton polytope of Δ_k. The unbounded connected components of the complement of the spine S_{Δ_k} are convex polyhedral cones where one of the affine linear functions dominates all the others, and the closure of such a cone is the recession cone of the unbounded connected component of the complement to $\mathrm{Amoeba}(\Delta_k)$. For the special vertex monomial (1) we obtain in this way that the recession cone C_k of Γ_k is given by the inequalities:

$$2(x_1 + \cdots + x_{k-1}) \geqslant x_{l-1} - 2x_l + x_{l+1} + 2(x_1 + \cdots + x_{k-1}), \qquad l \in \{1,\, \ldots,\, k-1\},$$

or, equivalently, $2x_l \geqslant x_{l-1} + x_{l+1}$ for $l \in \{1,\, \ldots,\, k-1\}$. □

We will later need the following refinement of Lemma 4 that characterizes the unique translate C_k^s of C_k *supporting* Γ_k.

Lemma 5. *The cone C_k^s defined by the inequalities $2x_l \geqslant x_{l-1} + x_{l+1} + \log 4$ for all $l \in \{1,\, \ldots,\, k-1\}$ contains Γ_k, but $y + C_k^s$ does not contain Γ_k for any y in the interior of C_k.*

Proof. First note that each polynomials $x^m(1+x)^2$, for $m \in \{0,\, \ldots,\, k-2\}$, lies on a unique facet of the cone C_k^s, and that this cone has exactly $k-1$ facets. So to conclude, we need only show that each such polynomial lies on the boundary of Γ_k. However, the last statement was already observed in the introduction, during our discussion of perturbing middle coefficients. □

Proof of Proposition 1. From our earlier discussion, we know that the set $SI_k^{\geqslant}$ (if non-empty) consists of some number of connected components of the complement $\mathbb{R}^{k+1} \setminus \Delta_k^{\dagger}$, where $\Delta_k^{\dagger}$ is the reflected discriminant of Δ_k (see, e.g., Figure 4). Indeed, $SI_k^{\geqslant}$ is the intersection of the set of all degree k real-rooted polynomials having only simple zeros with all similar sets obtained by all possible sign changes of the coefficients. By Lemmas 2 and 3 the set $SI_k^{\geqslant}$ is non-empty and connected, so $SI_k^{\geqslant}$ coincides with a unique connected component of $\mathbb{R}^{k+1} \setminus \Delta_k^{\dagger}$.

To conclude, we have to show that the image of $SI_k^{\geqslant}$ under $\mathrm{Log}\,|\cdot|$ coincides with the component of the complement to Amoeba(Δ_k) corresponding to the monomial (1). We show that the vector $s \in \mathbb{N}^{k-1}$ from Lemma 2 is an interior point in the recession cone of the unbounded connected component Γ_k of the complement of the discriminant amoeba corresponding to the finest subdivision of $\{0, \ldots, k\}$. Indeed, this recession cone is defined by the inequalities $2x_j \geqslant x_{j-1} + x_{j+1}$, $j \in \{1, \ldots, k-1\}$ with the dehomogenizing convention $x_0 = x_k = 0$, thanks to Lemma 4. This means that the coefficients λ^{s_j} of the polynomial p_k from Lemma 2, for large enough λ, represent a point in Γ_k. But the polynomial p_k was seen to be sign-independently real-rooted for large λ, and this concludes the proof. □

3. Proofs of Main Results

Proof of Theorem 1. The proof of the "only if" direction is easy, as outlined in the introduction: If $T_\gamma(SI^{\geqslant}) \subseteq RR$ then we must certainly have $T_\gamma(x^m(1+x)^2) \in RR$ for all m, since $x^m(1+x)^2 \in SI^{\geqslant}$ for all m. Thus, γ must be log-concave.

The proof of the "if" direction is more intricate but now follows easily from our preceding development: By Proposition 1 and Lemma 4, $\mathrm{Log}\,|\cdot|$ of the set of log-concave $\gamma = (\gamma_0, \ldots, \gamma_k)$ is precisely the recession cone C_k of Γ_k, and $\mathrm{Log}\,|\cdot|\colon SI_k^{\geqslant} \to \Gamma_k$ is a diffeomorphism. So any such γ satisfies $T_\gamma(SI_k^{\geqslant}) \subseteq SI_k^{\geqslant}$, and we are done. □

Proof of Corollary 1. The first part follows immediately from Lemma 5. The second part follows easily by applying Lemma 3 inductively. □

Our proof of Theorem 2 will be completely parallel to that of Theorem 1, so let us start with some analogues of Γ_k and C_k: First, let us denote by Γ'_k the connected component of $\mathbb{R}^{k+1} \setminus \mathrm{Amoeba}(\Delta_k)$ corresponding to the trivial (single-celled) subdivision of $\{0, \ldots, k\}$. Also let C'_k denote the recession cone of Γ'_k.

Lemma 6. *The cone C'_k is given by the inequalities $kx_j \leqslant j(x_k - x_0)$, for $j \in \{1, \ldots, k-1\}$.* □

Lemma 6 follows easily from the development of [GKZ08], [PT04] just like Lemma 4, so we proceed to an analogue of Proposition 1:

Proposition 2. *The map $\mathrm{Log}\,|\cdot|\colon II_k^{\geqslant} \to \Gamma'_k$ is a diffeomorphism.* □

Proposition 2 is proved in exactly the same way as Proposition 1, save that one uses a different deformation argument along the way: Lemma 2 is replaced by the observation that (a) $q_k(x) := 1 + \lambda^{-1}x + \cdots + \lambda^{-1}x^{k-1} + x^k \in II_k^{\geqslant}$ for all sufficiently large λ, and (b) the roots of q_k approach those of $x^k + 1$ as $|\lambda| \to \infty$.

We are now ready to prove Theorem 2:

Proof of Theorem 2. The "only if" direction can be proved as follows: For any $j \in \{1, \ldots, k-1\}$, consider the polynomial $p_j(x) := (k-j) - kx^j + jx^k$. It is then easily checked that (a) p_j has a unique degenerate real root, (b) p_j has exactly 1 or 2 real roots according as k is even or odd, (c) $p^-_{j,\varepsilon}(x) := (k-j) - k(1-\varepsilon)x^j + jx^k \in II_k^{\geqslant}$ for all $\varepsilon \in (0, 1]$, and (d) $p^+_{j,\varepsilon}(x) := (k-j) - k(1+\varepsilon)x^j + jx^k \notin II_k^{\geqslant}$ for all $\varepsilon > 0$ (To prove (a)–(d) one can simply apply Descartes' Rule of Signs and a clever formula for the discriminant of a trinomial from [GKZ08, Prop. 1.2, p. 217].) Thus, should the stated inequalities involving $(\gamma_0, \gamma_j, \gamma_k)$ fail to hold, we can easily find an $\varepsilon > 0$ such that $T_\gamma(p^-_{j,\varepsilon}) \notin II^{\geqslant}_k$ (with $\gamma := (\gamma_0, \underbrace{1, \ldots, 1}_{j-1}, \gamma_j, \underbrace{1, \ldots, 1}_{k-j-1}, \gamma_k))$ and obtain a contradiction.

The proof of the "if" direction is more intricate, but follows easily from our development: By Proposition 2 and Lemma 6, $\mathrm{Log}\,|\cdot|$ of the set of $\gamma = (\gamma_0, \ldots, \gamma_k)$ satisfying the stated inequalities is precisely the recession cone C'_k of Γ'_k, and $\mathrm{Log}\,|\cdot|\colon II^{\geqslant}_k \to \Gamma'_k$ is a diffeomorphism. So any such γ satisfies $T_\gamma(II^{\geqslant}_k) \subseteq II^{\geqslant}_k$, and we are done. □

4. Some Open Questions

Problem 1. How does one count connected components of the complement to the reflected discriminant of a given discriminant? In particular, is it true that the number of connected components of the complement to the reflected discriminant of univariate polynomials of degree k restricted to $\mathbb{R}^k_+$ equals 2^k?

Problem 2. Find an elementary proof of Theorem 1 avoiding the use of discriminant amoebae.

Regarding the last problem, we observe that we first derived our characterization of the recession cone relevant to $SI^{\geqslant}_k$ via some quick, informal calculations using the so-called Horn–Kapranov uniformization [Kap91], [PT04]. (The Horn–Kapranov uniformization is a remarkably useful rational parametrization of the A-discriminant variety. An intriguing fact is that the resulting parametric formula for H_{Δ_k} has size polynomial in k, while Δ_k has a number of monomials (and coefficient bit-sizes) exceeding 2^{k-1} [BHPR10].) It is likely that the Horn–Kapranov uniformization can yield an alternative proof of Theorem 1 without Ronkin functions. This could be seen as a step toward solving Problem 2.

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References

[BHPR10] O. Bastani, C. Hillar, D. Popov, and J. M. Rojas, *Randomization, sums of squares, and faster real root counting for tetranomials and beyond*, Preprint, Texas A & M University, 2010.

[CC77] T. Craven and G. Csordas, *Multiplier sequences for fields*, Illinois J. Math. **21** (1977), no. 4, 801–817. MR 0568321

[CC83] T. Craven and G. Csordas, *Location of zeros. I. Real polynomials and entire functions*, Illinois J. Math. **27** (1983), no. 2, 244–278. MR 694643

[CC04] T. Craven and G. Csordas, *Composition theorems, multiplier sequences and complex zero decreasing sequences*, Value distribution theory and related topics, Adv. Complex Anal. Appl., vol. 3, Kluwer Acad. Publ., Boston, MA, 2004, pp. 131–166. MR 2173299

[Cso03] G. Csordas, *Complex zero decreasing sequences and the Riemann hypothesis. II*, Analysis and applications—ISAAC 2001 (Berlin), Int. Soc. Anal. Appl. Comput., vol. 10, Kluwer Acad. Publ., Dordrecht, 2003, pp. 121–134. MR 2022744

[CVV90] G. Csordas, R. S. Varga, and I. Vincze, *Jensen polynomials with applications to the Riemann ζ-function*, J. Math. Anal. Appl. **153** (1990), no. 1, 112–135. MR 1080122

[DP49] H. Davenport and G. Pólya, *On the product of two power series*, Canadian J. Math. **1** (1949), 1–5. MR 0027306

[GKZ08] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2008. MR 2394437. Reprint of the 1994 edition.

[Hut23] J. I. Hutchinson, *On a remarkable class of entire functions*, Trans. Amer. Math. Soc. **25** (1923), no. 3, 325–332. MR 1501248 Not cited?

[Kap91] M. M. Kapranov, *A characterization of A-discriminantal hypersurfaces in terms of the logarithmic Gauss map*, Math. Ann. **290** (1991), no. 2, 277–285. MR 1109634

[Kar68] S. Karlin, *Total positivity. Vol. I*, Stanford University Press, Stanford, Calif, 1968. MR 0230102

[KS06] V. Kostov and B. Shapiro, *On the Schur–Szegö composition of polynomials*, C. R. Math. Acad. Sci. Paris **343** (2006), no. 2, 81–86. MR 2242036

[Lig97] T. M. Liggett, *Ultra logconcave sequences and negative dependence*, J. Combin. Theory Ser. A **79** (1997), no. 2, 315–325. MR 1462561

[Ost40] A. Ostrowski, *Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent. Chapitres III et IV*, Acta Math. **72** (1940), 157–257. MR 0004222

[PR04] M. Passare and H. Rullgård, *Amoebas, Monge–Ampère measures, and triangulations of the Newton polytope*, Duke Math. J. **121** (2004), no. 3, 481–507. MR 2040284

[PST05] M. Passare, T. Sadykov, and A. Tsikh, *Singularities of hypergeometric functions in several variables*, Compos. Math. **141** (2005), no. 3, 787–810. MR 2135529

[PT04] M. Passare and A. Tsikh, *Algebraic equations and hypergeometric series*, The legacy of Niels Henrik Abel, Springer, Berlin, 2004, pp. 653–672. MR 2077589

[PS14] G. Pólya and I. Schur, *Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen*, J. Reine Angew. Math. **144** (1914), 89–113.

[SU92] *A survey of linear preserver problems*, Linear and Multilinear Algebra **33** (1992), no. 1–2. MR 1346777.

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