DISCRIMINANTS, SYMMETRIZED GRAPH MONOMIALS, AND SUMS OF SQUARES

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Abstract. Motivated by the necessities of the invariant theory of binary forms J. J. Sylvester constructed in 1878 for each graph with possible multiple edges but without loops its symmetrized graph monomial which is a polynomial in the vertex labels of the original graph. We pose the question for which graphs this polynomial is a non-negative resp. a sum of squares. This problem is motivated by a recent conjecture of F. Sottile and E. Mukhin on discriminant of the derivative of a univariate polynomial, and an interesting example of P. and A. Lax of a graph with 4 edges whose symmetrized graph monomial is non-negative but not a sum of squares. We present detailed information about symmetrized graph monomials for graphs with four and six edges, obtained by computer calculations.

1. Introduction

In what follows by a *graph* we will always mean a (directed or undirected) graph with (possibly) multiple edges but no loops. The classical construction of J. J. Sylvester and J. Petersen [8, 9] associates to an arbitrary directed loopless graph a symmetric polynomial as follows.

Definition 1. *Let g be a directed graph, with vertices $x_1, \dots, x_n$ and adjacency matrix (a_{ij}), where a_{ij} is the number of directed edges connecting x_i and x_j. Define its graph monomial P_g as*

$$P_g(x_1, \dots, x_n) := \prod_{1 \le i,j \le n} (x_i - x_j)^{a_{ij}}.$$

The symmetrized graph monomial of g is defined as

$$\tilde{g}(\mathbf{x}) = \sum_{\sigma \in S_n} P_g(\sigma \mathbf{x}), \quad \mathbf{x} = x_1, \dots, x_n.$$

Notice that if the original g is undirected one can still define $\tilde{g}$ up to a sign by choosing an arbitrary orientation of its edges. Symmetrized graph monomials are closely related to SL_2-invariants and covariants and were introduced in the 1870's in attempt to find new tools in the invariant theory. Namely, to obtain an SL_2-coinvariant from a given $\tilde{g}(\mathbf{x})$ we have to perform two standard operations. First we express the symmetric polynomial $\tilde{g}(\mathbf{x})$ in n variables in terms of the elementary symmetric functions $e_1, ..., e_n$ and obtain the polynomial $\hat{g}(e_1, ..., e_n)$. Secondly, we perform the standard homogenization of a polynomial of a given degree d

$$Q_g(a_0, a_1, ..., a_n) := a_0^d \hat{g}\left(\frac{a_1}{a_0}, ..., \frac{a_n}{a_0}\right).$$

The following fundamental proposition apparently goes back to A. Cayley, see Theorem 2.4 of [11].

Theorem 1. (i) If g is a d-regular graph with n vertices then $Q_g(a_0, ..., a_n)$ is either an SL_2-invariant of degree d in n variables or it is identically zero.
(ii) Conversely, if $Q(a_0, ..., a_n)$ is an SL_2-invariant of degree d and order n then there exist d-regular graphs $g_1, ..., g_r$ with n vertices and integers $\lambda_1, ..., \lambda_r$ such that

$$Q = \lambda_1 Q_{g_1} + ... + \lambda_r Q_{g_r}.$$

Remark 1. *Recall that a graph is called d-regular if every its vertex has valency d. Notice that if g is an arbitrary graph then it is natural to interpret its polynomial $Q_g(a_0, ..., a_n)$ as the SL_2-coinvariant.*

The question about the kernel of the map sending g to $\tilde{g}(\mathbf{x})$ (or to Q_g) was already discussed by J. Petersen who claimed that he has found a necessary and sufficient condition when g belongs to the kernel, see [11]. This claim turned out to be false. (An interesting correspondence between J. J. Sylvester, D. Hilbert and F. Klein related to this topic can be found in [12].) The kernel of this map seems to be related to several open problems such as Alon-Tarsi [2] and the Rota basis conjecture [13]. (We want to thank Professor A. Abdesselam for this valuable information, see [1].)

In the present paper we are interested in examples of graphs whose symmetrized graph monomial are non-negative resp. sum of squares. Our interest in this matter has two sources.

The first one is a recent conjecture of F. Sottile and E. Mukhin formulated on the AIM meeting "Algebraic systems with only real solutions" in October 2010.[1]

Conjecture 1. *The discriminant $\mathcal{D}_n$ of the derivative of a polynomial p of degree n is the sum of squares of polynomials in the differences of the roots of p.*

Based on our calculations and computer experiments we propose the following extension and strenghtening of the latter conjecture. We call an arbitrary graph with all edges of even multiplicity a *square graph.* Notice that the symmetrized graph monomial of a square graph is obviously a sum of squares.

Conjecture 2. *For any non-negative integer $0 \leq k \leq n-2$ the discriminant $\mathcal{D}_{n,k}$ of the kth derivative of a polynomial p of degree n is a finite positive linear combination of the symmetrized graph monomials where all underlying graphs are square graphs with n vertices. The vertices $x_1, \dots, x_n$ are the roots of p. In other words, $\mathcal{D}_{n,k}$ lies in the convex cone spanned by the symmetrized graph monomials of the square graphs with n vertices and $\binom{n-k}{2}$ edges.*

Observe that $\deg \mathcal{D}_{n,k} = (n-k)(n-k-1)$ and is, therefore, even. The following examples support the above conjectures. Below we use the following agreement. If a shown graph has fewer than n vertices, then we always assume that it is appended by the required number of isolated vertices to get totally n.

Example 1. *If $k = 0$ then $\mathcal{D}_{n,0}$ is proportional to $\tilde{g}$ where g is the complete graph on n vertices with all edges of multiplicity 2.*

Example 2. *For $k \geq 0$, the discriminant $\mathcal{D}_{k+2,k}$ equals*

$$\frac{(k+1)!}{2} \sum_{1 \leq i < j \leq k+2} (x_i - x_j)^2.$$

[1] After the initial version of this text was posted on arXiv, Conjecture 1 was settled in a preprint [7], see Corollary 14. We want to thank Professor B. Sturmfels for informing us about this result.

In other words, $\mathcal{D}_{k+2,k} = \frac{(k+2)!}{2}\tilde{g}$ where the graph g is given in Fig. 1 (appended with k isolated vertices).

FIGURE 1. The graph g for the case $\mathcal{D}_{k+2,k}$

Example 3. *For $k \geq 0$ we conjecture that the discriminant $\mathcal{D}_{k+3,k}$ equals*

$$(k!)^3\left[\frac{(k+1)^3(k+2)(k+6)}{72}\tilde{g}_1 + \frac{(k+1)^3k(k+2)}{12}\tilde{g}_2 + \frac{(k-1)k(k+1)^2(k+2)(k-2)}{96}\tilde{g}_3\right]$$

where the graphs g_1, g_2 and g_3 are given in Fig. 2. (This claim is verified for $k = 1, \ldots, 12$.)

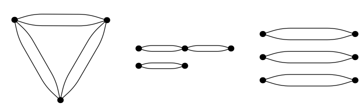

FIGURE 2. The graphs g_1, g_2 and g_3 for the case $\mathcal{D}_{k+3,k}$

Example 4. *The discriminant $\mathcal{D}_{5,1}$ is given by*

$$\mathcal{D}_{5,1} = \frac{19}{6}\tilde{g}_1 + 14\tilde{g}_2 + 2\tilde{g}_3$$

where g_1, g_2, g_3 are given in Fig. 3.

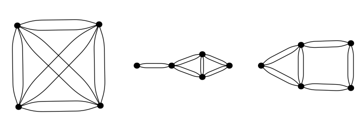

FIGURE 3. The graphs g_1, g_2 and g_3 for the case $\mathcal{D}_{5,1}$

Example 5. *Finally*

$$\mathcal{D}_{6,2} = 19200\tilde{g}_1 + 960\tilde{g}_2 + 3480\tilde{g}_3 + 3240\tilde{g}_4 + \frac{3440}{3}\tilde{g}_5 + 2440\tilde{g}_6$$

where $g_1, \ldots, g_6$ are given in Fig. 4. (Note that this representation as sum of graphs is not unique.)

It is classically known that for any given number n of vertices and d edges, the linear span of the symmetrized graph monomials coming from all graphs with n vertices and d edges coincides with the linear space $\mathbf{PST}_{n,d}$ of all symmetric translation-invariant polynomials of degree d in n variables.

We say that a pair (n, d) is stable if $n \geq 2d$. For stable (n, d), we suggest a natural basis in $\mathbf{PST}_{n,d}$ of symmetrized graph monomials which seems to be new, see Proposition 6 and Corollary 4.

In the case of even degree, there is a second basis in $\mathbf{PST}_{n,d}$ of symmetrized graph monomials consisting of only square graphs, see Proposition 8 and Corollary 5.

The second motivation of the present study is an interesting example of a graph whose symmetrized graph monomial is non-negative but not a sum of squares. Namely, the main result of [10] shows that $\tilde{g}$ for the graph given in Fig. 5 has this property.

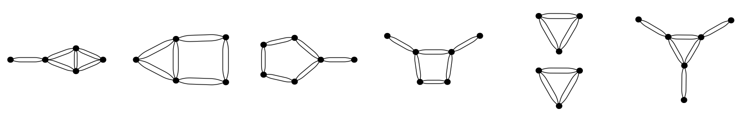

FIGURE 4. The graphs $g_1, \ldots, g_6$ for the case $\mathcal{D}_{6,2}$

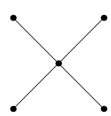

FIGURE 5. The Lax graph, i.e., the only 4-edged graph which yields a non-negative polynomial which is not SOS.

Finally, let us present our main computer-aided results regarding the case of graphs with 4 resp. 6 edges. Notice that there exist 23 graphs with 4 edges and 212 graphs with 6 edges. We say that two graphs are *equivalent* if their symmetrized graph monomials are non-vanishing identically and proportional. Note that two graphs do not need to be isomorphic to be equivalent, see e.g. the equivalence classes in Fig. 6.

Proposition 2. (i) 10 graphs with 4 edges have identically vanishing symmetrized graph monomial. (ii) The remaining 13 graphs are divided into 4 equivalence classes presented in Fig 6. (iii) The first two classes contain square graphs and, thus, their symmetrized monomials are non-negative. (iv) The third graph is non-negative (as a positive linear combination of the Lax graph and a polynomial obtained from a square graph). Since it effectively depends only on three variables, it is SOS, see [4]. (v) The last graph is the Lax graph which is thus the only non-negative graph with 4 edges not being a SOS.

Proposition 3. (i) 102 graphs with 6 edges have identically vanishing symmetrized graph monomial. (ii) The remaining 110 graphs are divided into 27 equivalence classes. (iii) 12 of these classes can be expressed as non-negative linear combinations

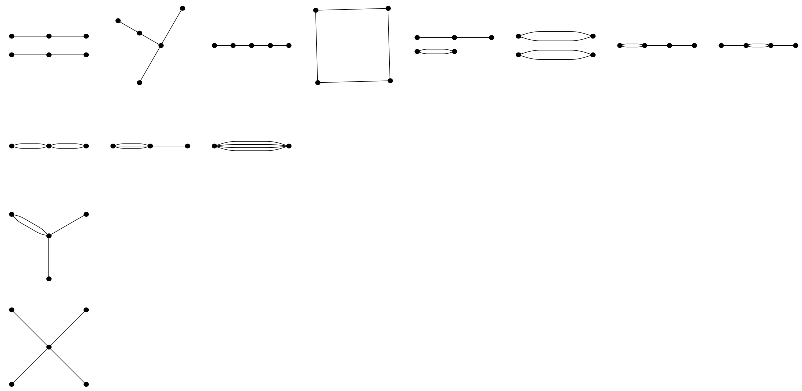

FIGURE 6. 4 equivalence classes of the 13 graphs with 4 edges, whose symmetrized graph monomials do not vanish identically.

of square graphs, i.e. lie in the convex cone spanned by the square graphs. (iv) Of the remaining 15 classes, the symmetrized graph monomial of 7 of them changes sign. (v) Of the remaining 8 classes (which are presented in Fig. 7) the first 5 are sums of squares. (Notice however that these symmetrized graph monomials do not lie in the convex cone spanned by the square graphs.) (vi) The last 3 classes contain all non-negative graphs with 6 edges, which are not SOS and, therefore, give new examples of graphs a'la Lax.

Proposition 2 is only a matter of straightforward computations. Case (i)-(iv) in Proposition 3 also follows from a longer calculation, by examining each of the 212 graphs. Proof of case (v) requires the notion of certificates.

It is well-known that a polynomial is a sum of squares if and only if it may be represented as vQv^T where Q is positive semidefinite and v a monomial vector. Such a representation is called a certificate. Certificates for the 8 classes in Prop. 3, case (v) in the form of positive semi-definite matrices and corresponding monomial vectors are too large to be presented here and can be found in [14]. The simplest certificate, for the third class, is given by the vector v_3 below together with the positive semidefinite matrix Q_3.

Case (vi) is analysed with the Yalmip software which provides a second kind of certificates that shows that the last three classes are not SOS.

$$v_3 = \{x_3x_4^2, x_3^2x_4, x_2x_4^2, x_2x_3x_4, x_2x_3^2, x_2^2x_4, x_2^2x_3, x_1x_4^2, x_1x_3x_4, x_1x_3^2, x_1x_2x_4,$$
$$x_1x_2x_3, x_1x_2^2, x_1^2x_4, x_1^2x_3, x_1^2x_2\}$$

$$
Q_3 = \begin{pmatrix}
10 & -6 & -5 & -4 & 3 & 3 & -1 & -5 & -4 & 3 & 8 & 0 & -2 & 3 & -1 & -2 \\
-6 & 10 & 3 & -4 & -5 & -1 & 3 & 3 & -4 & -5 & 0 & 8 & -2 & -1 & 3 & -2 \\
-5 & 3 & 10 & -4 & -1 & -6 & 3 & -5 & 8 & -2 & -4 & 0 & 3 & 3 & -2 & -1 \\
-4 & -4 & -4 & 24 & -4 & -4 & -4 & 8 & -8 & 8 & -8 & -8 & 8 & 0 & 0 & 0 \\
3 & -5 & -1 & -4 & 10 & 3 & -6 & -2 & 8 & -5 & 0 & -4 & 3 & -2 & 3 & -1 \\
3 & -1 & -6 & -4 & 3 & 10 & -5 & 3 & 0 & -2 & -4 & 8 & -5 & -1 & -2 & 3 \\
-1 & 3 & 3 & -4 & -6 & -5 & 10 & -2 & 0 & 3 & 8 & -4 & -5 & -2 & -1 & 3 \\
-5 & 3 & -5 & 8 & -2 & 3 & -2 & 10 & -4 & -1 & -4 & 0 & -1 & -6 & 3 & 3 \\
-4 & -4 & 8 & -8 & 8 & 0 & 0 & -4 & 24 & -4 & -8 & -8 & 0 & -4 & -4 & 8 \\
3 & -5 & -2 & 8 & -5 & -2 & 3 & -1 & -4 & 10 & 0 & -4 & -1 & 3 & -6 & 3 \\
8 & 0 & -4 & -8 & 0 & -4 & 8 & -4 & -8 & 0 & 24 & -8 & -4 & -4 & 8 & -4 \\
0 & 8 & 0 & -8 & -4 & 8 & -4 & 0 & -8 & -4 & -8 & 24 & -4 & 8 & -4 & -4 \\
-2 & -2 & 3 & 8 & 3 & -5 & -5 & -1 & 0 & -1 & -4 & -4 & 10 & 3 & 3 & -6 \\
3 & -1 & 3 & 0 & -2 & -1 & -2 & -6 & -4 & 3 & -4 & 8 & 3 & 10 & -5 & -5 \\
-1 & 3 & -2 & 0 & 3 & -2 & -1 & 3 & -4 & -6 & 8 & -4 & 3 & -5 & 10 & -5 \\
-2 & -2 & -1 & 0 & -1 & 3 & 3 & 3 & 8 & 3 & -4 & -4 & -6 & -5 & -5 & 10
\end{pmatrix}
$$

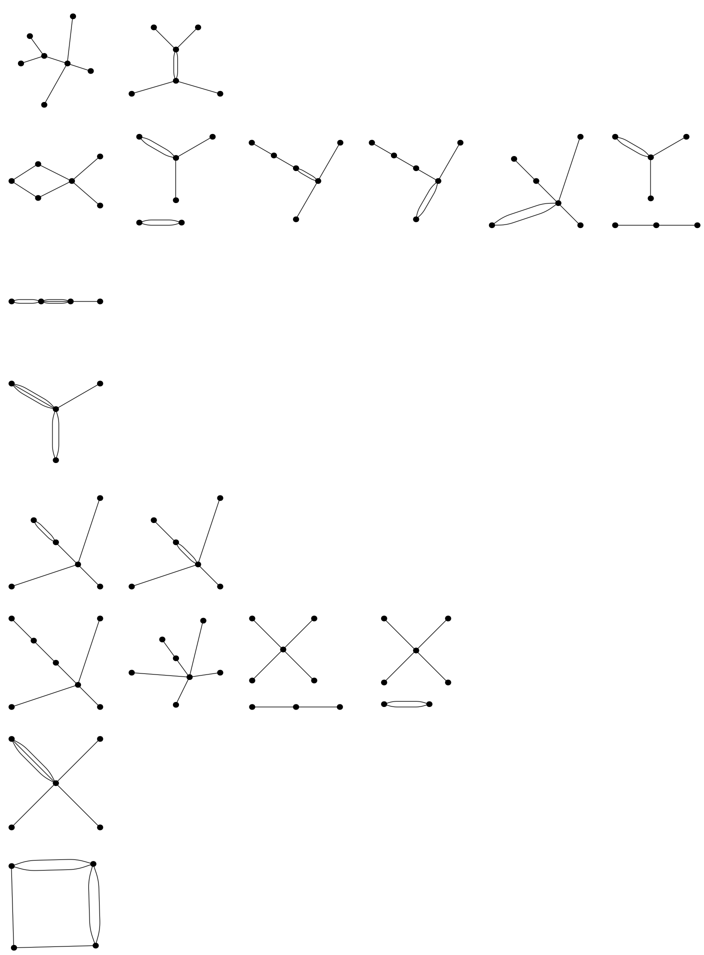

FIGURE 7. 8 equivalence classes of all non-negative graphs with 6 edges.

Finally, notice that translation invariant symmetric polynomials appeared also in the early 1970's in the study of integrable N-body problems in mathematical physics, particularly in the famous paper [3] of F. Calogero. A few much more recent papers related to the ring of such polynomials in connection with the investigation of multi-particle interactions and the quantum Hall effect were published since then, see e.g., [6], [5]. In particular, the ring structure and the dimensions of the homogeneous components of this ring were calculated. It was also shown in § IV of [6] and [5] that the ring of translation invariant symmetric polynomials (with integer coefficients) in $x_1, \dots, x_n$ is isomorphic as a graded ring to the polynomial ring $\mathbb{Z}[e_2, \dots, e_n]$ where e_i stands for the i-th elementary symmetric function in $x_1 - x_{avg}, \dots, x_n - x_{avg}$ with $x_{avg} = \frac{1}{n}(x_1 + \cdots + x_n)$.

From this fact one can easily show that the dimension of its d-th homogeneous component equals the number of distinct partitions of d where each part is strictly bigger than 1 and the number of parts is at most n. Several natural linear bases were also suggested for each such homogeneous component, see (29) in [6] and [5]. It seems that the authors of the latter papers were unaware of the mathematical developments in this field related to graphs.

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2. Some generalities on symmetrized graph monomials

Definition 2. *An* integer partition *of d is a d-tuple $(\alpha_1, \dots, \alpha_d)$ such that $\sum_i \alpha_i = d$ and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d \geq 0$.*

Definition 3. *Let g be a directed graph with d edges and n vertices $v_1, v_2, \dots, v_n$. Let $\alpha = (\alpha_1, \dots, \alpha_d)$ be an integer partition of d. A* partition-coloring *of g with α is an assignment of colors to the edges and vertices of g satisfying the following:*

- *For each color i, $1 \leq i \leq d$, we paint with the color i some vertex v_j and exactly α_i edges connected to v_j.*
- *Each edge of g is colored exactly once.*

An edge is called odd-colored *if it has color j and is directed towards a vertex with the same color. The coloring is said to be negative if there is an odd number of odd-colored edges in g, and positive otherwise.*

Definition 4. *Given a polynomial $P(\mathbf{x})$ and a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$, we use the notation $Coeff_\alpha(P(\mathbf{x}))$ to denote the coefficient in front of $\mathbf{x}^\alpha$ in $P(\mathbf{x})$.*

Note that we may view α as a partition of the sum of the exponents.

Lemma 4. *Let g be a directed graph with d edges and vertices $v_1, v_2, \dots, v_n$. Then $Coeff_\alpha(\tilde{g})$ is given by the difference of the numbers of positive and negative partition-colorings of g with α.*

Proof. See [11, Lemma 2.3]. □

2.1. Bases for $\mathbf{PST}_{n,d}$.

It is known that the dimension of $\mathbf{PST}_{n,d}$ with $n \geq 2d$ is given by the number of integer partitions of d where each nonzero part is at least of size 2, see [5]. Such integer partition will be called a 2-partition.

To each 2-partition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $\alpha_i \neq 1$, we associate the following graph b_α. For each $\alpha_i \geq 2$, we have a connected component of b_α consisting of a root vertex, connected to α_i other vertices, with the edges directed away from the root vertex. Since α is an integer partition of d, it follows that b_α has exacly d edges. This type of graph will be called a *partition graph*.

The dimension of $\mathbf{PST}_{n,d}$ is independent of n (as long as $n \geq 2d$), and we only deal with homogenous symmetric polynomials of degree d. Thus, each monomial is essentially determined only by the way the powers of the variables are partitioned. The variables themselves become unimportant since every permutation of the variables is present. For example, the monomials x^3zw and xy^3w are always present simultaneously with the same coefficient, while x^3z^2 is different from the previous two.

2.2. **Partition graphs.**

Definition 5. *Let $P(\mathbf{x})$ be a polynomial in $|\mathbf{x}|$ variables. We use the notation*

$$Sym_{(\mathbf{x}\cup\mathbf{y})}P = \sum_{(\tau_1,\tau_2,\dots,\tau_n)\subseteq \mathbf{x}\cup\mathbf{y}} P(\tau_1,\tau_2,\dots,\tau_n)$$

where we sum over all possible permutations *and choices of n variables among the $|\mathbf{x}|+|\mathbf{y}|$ variables.*

Lemma 5. *Let $P(\mathbf{x})$ be a polynomial. Then*

$$Sym_{(\mathbf{x}\cup\mathbf{y})}P = \sum_{i=0}^{|\mathbf{y}|} \sum_{\substack{\sigma\subseteq\mathbf{y}\\ |\sigma|=i}} \sum_{\substack{\tau\subseteq\mathbf{x}\\ |\tau|=|\mathbf{x}|-i}} Sym_{(\tau\cup\sigma)}P.$$

Here, the two inner sums denote choices of all subsets *of certain size.*

Proof. Obvious. ☐

Corollary 1. *If $Sym_{\mathbf{x}}P$ is non-negative, then $Sym_{(\mathbf{x}\cup\mathbf{y})}P$ is non-negative.*

Corollary 2. *If $Sym_{\mathbf{x}}P$ is a sum of squares, then $Sym_{(\mathbf{x}\cup\mathbf{y})}P$ is a sum of squares.*

Corollary 3. *If $\sum_i \lambda_i Sym_{\mathbf{x}}P_i = 0$ then $\sum_i \lambda_i Sym_{(\mathbf{x}\cup\mathbf{y})}P_i = 0$.*

We will use the notation that every symmetric polynomial $\tilde{g}$ associated with a graph on d edges is symmetrized over $2d$ variables. Corollary 3 says that if a relation holds for the symmetrizations in $2d$ variables, it will also hold for $2d+k$ variables ($k \geq 0$). Therefore, each relation derived in this section also holds for $2d+k$ variables.

Proposition 6. *Let b_α be a partition graph with d edges, $\alpha = (\alpha_1,\dots,\alpha_d)$, and let $\beta = (\beta_1,\beta_2,\dots,\beta_d)$ be a 2-partition.*

Then $Coeff_\beta(\tilde{b_\alpha})$ equals

$$\begin{cases} 0 \text{ if } \beta \neq \alpha \\ \prod_{j=0}^{d} (\#\{i|\alpha_i = j\})! \text{ if } \beta = \alpha. \end{cases}$$

Proof. We will try to color the graph b_α with β :

Since $\beta_i \neq 1$, we may only color the roots of b_α. Hence, all edges in each component of b_α must have the same color as the corresponding root. It is clear that such a coloring is impossible if $\alpha \neq \beta$. If $\alpha = \beta$, we see that each coloring has positive sign, since only roots are colored and all connected edges are directed outwards.

The only difference between two colorings must be the assignment of the colors to the roots. Hence, components with the same size can permute colors, which yields

$$\prod_{j=0}^{d}(\#\{i|\alpha_i = j\})!$$

ways to color g with the partition $(\alpha_1, \ldots, \alpha_d)$. □

Corollary 4. *All partition graphs yield linearly independent polynomials, since each partition graph b_α contributes with the unique monomial $\mathbf{x}^\alpha$. The number of partition graphs on d edges equals the dimension of $\mathbf{PST}_{d,n}$, and, therefore, when $n \geq 2d$ must span the entire vector space.*

2.3. **Square graphs.** We will use the notation $\alpha = (\alpha_1, \ldots, \alpha_k|\alpha_{k+1}, \ldots, \alpha_d)$ to denote a partition where $\alpha_1, \ldots, \alpha_q$ are the odd parts in non-increasing order, and $\alpha_{q+1}, \ldots, \alpha_p$ are the even parts in non-increasing order. (Notice that this convention differs from the standard one for partitions.) As before, parts are allowed to be equal to 0, so that α can be used as a multi-index over d variables.

Now we define a second type of graphs which we associate with 2-paritions of even integers:

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k|\alpha_{k+1}, \ldots, \alpha_d)$, $\alpha_i \neq 1$, be a 2-partition of d. Since this is a partition of an even integer, k must be even.

For each even $\alpha_i \geq 2$, we have a connected component of h_α consisting of a root, connected to $\alpha_i/2$ other vertices, with the edges directed away from the root, and with multiplicity 2.

For each pair $\alpha_{2j-1}, \alpha_{2j}$ of odd parts, $j = 1, 2, \ldots, \frac{k}{2}$ we have a connected component consisting of two roots v_{2j-1} and v_{2j}, such that v_i is connected to $\lfloor \alpha_i/2 \rfloor$ other vertices for $i = 2j-1, 2j$ with edges of multiplicity 2, and the roots are connected with a double edge. This type of component will be called a *glued component.*

Thus, each edge in h_α has multiplicity 2, and the number of edges, counting multiplicity, is d. This type of multigraph will be called a *partition square graph.* Note that all edges have even multiplicity, so $\tilde{h}_\alpha(\mathbf{x})$ is a sum of squares.

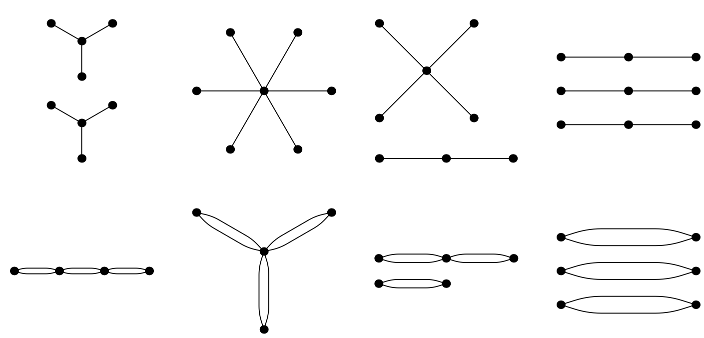

FIGURE 8. A base of partition graphs and a base of partition square graphs in the stable case with 6 edges.

Lemma 7. *Let h_α be a partition square graph where $\alpha = (\alpha_1, \dots, \alpha_d)$. Then*

$$Coeff_\alpha(\tilde{h}_\alpha) = (-1)^{\frac{1}{2}\#\{i|\alpha_i \equiv_2 1\}} 2^{\#\{i|\alpha_i = 2\}} \prod_{j=0}^{n} (\#\{i|\alpha_i = j\})!$$

Proof. Similarly to Proposition 6, it is clear that a coloring of h with p colors requires that each root is colored.

The root of a component with only two vertices is not uniquely determined, so we have $2^{\#\{i|\alpha_i=2\}}$ choices of the root.

It is clear that each glued component contributes with exacly one odd edge for every coloring, and, therefore the sign is the same for each coloring. The number of glued components is precisely $\frac{1}{2}\#\{i|\alpha_i \equiv_2 1\}$.

Lastly, we may permute the colors corresponding to the roots of the same degree. These observations together yield the formula

$$(-1)^{\frac{1}{2}\#\{i|\alpha_i \equiv_2 1\}} 2^{\#\{i|\alpha_i = 2\}} \prod_{j=0}^{n} \#\{i|\alpha_i = j\}!. \qquad \square$$

Define a total order on 2-partitions as follows:

Definition 6. *Let $\alpha = (\alpha_1, \dots, \alpha_k|\alpha_{k+1}, \dots, \alpha_d)$ and $\alpha' = (\alpha'_1, \dots, \alpha'_{k'}|\alpha'_{k'+1}, \dots, \alpha_d)$ be 2-partitions. We say that $\alpha \prec \alpha'$ if $\alpha_i = \alpha'_i$ for $i = 1, \dots, j-1$, $j \geq 1$ and one of the following holds:*

- *$\alpha_j > \alpha'_j$ and $\alpha_j = \alpha'_j \mod 2$*
- *α_j is odd and α'_j is even.*

This generalizes to $\alpha \preceq \alpha' \Leftrightarrow \alpha \prec \alpha'$ or $\alpha = \alpha'$.

Proposition 8. *Let h_α be a square graph. Then we may write*

$$\tilde{h}_\alpha = \sum_\beta \lambda_\beta \tilde{b}_\beta, \quad b_\beta \text{ is a partition graph,} \tag{1}$$

where $\lambda_\beta = 0$ if $\beta \prec \alpha$.

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_q|\alpha_{q+1} \dots, \alpha_d)$ and let $\beta = (\beta_1, \dots, \beta_r|\beta_{r+1}, \dots, \beta_d)$, with $\beta \prec \alpha$. Consider equation (1) and apply $Coeff_\beta$ on both sides. Lemma 6 implies

$$Coeff_\beta(\tilde{h}_\alpha) = \lambda_\beta \cdot C_\beta, \text{ where } C_\beta > 0.$$

It suffices to show that there is no partition-coloring of h_α with β if $\beta \prec \alpha$, since this implies $\lambda_\beta = 0$.

We now have three cases to consider:

Case 1 & 2: $\alpha_i = \beta_i$ for $i = 1, \dots, j-1$ and $\beta_j > \alpha_j$ where α_j and β_j are either both odd or both even. We must color a root and β_j connected edges, since $\beta_j > \alpha_j \geq 2$. There is no vacant root in h_α with degree at least β_j, all such roots have already been colored with the colors $1, \dots, j-1$. Hence a coloring is impossible in this case.

Case 3: $\alpha_i = \beta_i$ for $i = 1, \dots, j-1$, β_j is odd and α_j is even. This condition impies that $r < q$.

Every component of h_α has an even number of edges, and only vertices of degree at least three can be colored with an odd color. Therefore, glued components must be colored with exactly zero or two odd colors, and non-glued component must have an even number of edges of each present color. This implies that a coloring is only possible if $r \leq q$, contradiction.

Hence, there is no coloring of h_α with the colors given by β, and therefore, $Coeff_\beta(\tilde{h}_\alpha) = 0$ implying $\lambda_\beta = 0$. □

Corollary 5. *The polynomials obtained from the partition square graphs with d edges is a basis for $\mathbf{PST}_{d,n}$, for even d.*

Proof. Let $\alpha_1 \prec \cdots \prec \alpha_k$ be the 2-partitions of d. Since $\tilde{b}_{\alpha_1}, \dots, \tilde{b}_{\alpha_k}$ is a basis, there is a uniquly determined matrix M such that

$$(\tilde{h}_{\alpha_1}, \dots, \tilde{h}_{\alpha_k})^T = M(\tilde{b}_{\alpha_1}, \dots, \tilde{b}_{\alpha_k})^T.$$

Lemma 8 implies that M is lower-triangular. Proposition 6 and Lemma 7 implies that the entry at (α_i, α_i) in M is given by

$$(-1)^{\frac{1}{2}\#\{j|\alpha_{ij} \equiv_2 1\}} 2^{\#\{j|\alpha_{ij}=2\}}$$

which is non-zero. Hence M has an inverse and the square graphs is a basis. □

3. Final remarks

Some obvious challenges related to this project are as follows.

1. Prove Conjecture 2.

2. Describe the boundary of the convex cone spanned by all square graphs with a given number of (double) edges and vertices.

3. Find more examples of graphs a'la Lax.

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