TREES, PARKING FUNCTIONS, AND DEFORMATIONS OF MONOMIAL IDEALS

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Abstract. For a graph G, we define G-parking functions and show that their number is equal to the number of spanning trees of G. We construct a certain monomial ideal and a certain ideal generated by powers of linear forms. The dimension of the quotient of the polynomial ring modulo either of these two ideals equals the number of spanning trees of G. The monomials corresponding to G-parking functions form linear bases in each of these two algebras. Then we investigate the general class of monotone monomial ideals and their deformations. We show that the Hilbert series of a monotone monomial ideal is always bounded by the Hilbert series of its deformation. We prove several formulas for the Hilbert series.

1. Introduction

The famous Cayley's result says that the number of trees on $n+1$ labelled vertices equals $(n+1)^{n-1}$. Remarkably, this number has several other interesting combinatorial interpretations. For example, it is equal to the number of parking functions of size n.

In this paper we present two algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ of dimension $(n+1)^{n-1}$. The algebra $\mathcal{A}_n$ is a quotient of the polynomial ring modulo a monomial ideal; and the algebra $\mathcal{B}_n$ is a quotient of the polynomial ring modulo powers of some linear forms. It is immediate that the set of monomials x^b, where b is a parking function, is the standard monomial basis of the algebra $\mathcal{A}_n$. On the other hand, the same set of monomials forms a basis of the algebra $\mathcal{B}_n$, which is a non-trivial result.

More generally, for any graph G, we define G-parking functions and describe two algebras $\mathcal{A}_G$ and $\mathcal{B}_G$. The number of G-parking functions equals the number of spanning trees of the graph G. The set of monomials corresponding to G-parking functions form bases in both these algebras. In particular, $\dim \mathcal{A}_G = \dim \mathcal{B}_G$ is the number of spanning trees of G.

All these pairs of algebras are instances of the general class of algebras given by monotone monomial ideals and their deformations. For such an algebra $\mathcal{A}$ and its deformation $\mathcal{B}$, we show that $\dim \mathcal{A} \geq \dim \mathcal{B}$ and the Hilbert series of $\mathcal{B}$ is termwise bounded by the Hilbert series of $\mathcal{A}$. There is a natural correspondence between polynomial generators of the ideal for $\mathcal{B}$ and monomial generators of the ideal for $\mathcal{A}$. However, these monomials are not the leading terms of the polynomial generators for any term order, because they are usually located at the center of the Newton polytope of the corresponding polynomial generators. The standard Gröbner bases technique cannot be applied for treatment of this class of algebras.

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The general outline of the paper follows. In Section 2 we define G-parking functions for a digraph G. We formulate Theorem 2.1 that says that the number of such functions equals the number of oriented spanning trees of G. Then we construct the algebra $\mathcal{A}_G$ as the quotient of the polynomial ring modulo certain monomial ideal. Elements of the standard monomial basis of $\mathcal{A}_G$ correspond to G-parking functions. In Section 3 we construct the algebra $\mathcal{B}_G$ as the quotient of the polynomial ring modulo the ideal generated by power of certain linear forms. Then we formulate Theorem 3.1 that implies that the algebras $\mathcal{A}_G$ and $\mathcal{B}_G$ have the same Hilbert series. In Section 4 we give two examples of these results. For the complete graph $G = K_{n+1}$ we recover the usual parking functions and the algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ of dimension $(n+1)^{n-1}$. For a little bit more general class of graphs we obtain two algebras of dimension $l\,(l+kn)^{n-1}$. Section 5 is devoted to description of monotone monomial ideals and their deformations. We formulate Theorem 5.2, which implies the inequality for the Hilbert series. In Section 6 we prove general formulas for Hilbert series and dimension of the algebra given by a monotone monomial ideal. Then in Section 7 we deduce Theorem 2.1. In Section 8 we construct the algebra $\mathcal{C}_G$ and prove Theorem 8.1 that claims that the dimension of this algebra equals the number of spanning trees. Actually, we will later see that $\mathcal{C}_G$ is isomorphic to the algebra $\mathcal{B}_G$. In Section 9 we prove Theorem 5.2. Then we finish the proof of Theorem 3.1, which goes as follows. By Theorem 5.2 and construction of $\mathcal{C}_G$ we know that $\operatorname{Hilb}\mathcal{A}_G \geq \operatorname{Hilb}\mathcal{B}_G \geq \operatorname{Hilb} C_G$. On the other hand, by Theorems 2.1 and 8.1, $\dim \mathcal{A}_G = \dim \mathcal{C}_G$ is the number of spanning trees of G. Thus the Hilbert series of these three algebras coincide. In Section 10 we discuss some results of our previous works and compare them with results of this paper. We mention a certain algebra, whose dimension equals the number of forests on $n+1$ vertices. This algebra originally appeared in the attempt to lift Schubert calculus of the flag manifold on the level of differential forms. In Section 11 we discuss a special class of monotone monomial ideals and their deformations. In this case we give a subtraction-free formula for the Hilbert series of the algebra $\mathcal{A}$ and list several cases when it is equal to the Hilbert series of $\mathcal{B}$.

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2. G-parking functions

A *parking function* of size n is a sequence $b = (b_1, \dots, b_n)$ of nonnegative integers such that its increasing rearrangement $c_1 \leq \dots \leq c_n$ satisfies $c_i < i$. Equivalently, we can formulate this condition as $\#\{i \mid b_i < r\} \geq r$, for $r = 1, \dots, n$. The parking functions of size n are known to be in a bijective correspondence with trees on $n+1$ labelled vertices, see [Krew]. Thus the total number of parking functions of size n equals $(n+1)^{n-1}$, which it Cayley's formula for the number of labelled trees. In this section we extend this statement to a more general class of functions.

A *graph* is given by specifying its set of vertices, set of edges, and a function that associates to each edge an unordered pair of vertices. A *directed graph*, or *digraph*, is given by specifying its set of vertices, set of edges, and a function that associates to each edge an ordered pair of vertices. Thus multiple edges and loops are allowed in graphs and digraphs. A *subgraph* H in a (directed) graph G is a (directed) graph on the same set of vertices whose set of edges is a subset of edges of G. We will write $H \subset G$ to denote that H is subgraph of G. For a subgraph $H \subset G$, the notation $G \setminus H$ denote the *complement subgraph* whose edge set is complementary to that of H. Also we will write $e \in G$ to show that e is an edge of the graph G.

Let G be a digraph on the set of vertices $0, 1, \ldots, n$. The vertex 0 will be the root of the graph G. The digraph G is determined by its *adjacency matrix* $A = (a_{ij})_{0 \le i,j \le n}$, where a_{ij} is the number of edges from the vertex i to the vertex j. We will regard graphs as a special case of digraphs with symmetric adjacency matrix A. Actually, almost everywhere in the paper, except for this section, we will assume that G is a graph.

An *oriented spanning tree* T of the digraph G is a subgraph $T \subset G$ such that there exists a unique directed path in T from any vertex i to the root 0. The number N_G of such trees is given by the *matrix-tree theorem*, e.g., see [Sta2, Section 5.6]:

$$N_G = \det L_G, \tag{1}$$

where $L_G = (l_{ij})_{1 \le i,j \le n}$ the *Laplace matrix*, also known as the *Kirkhoff matrix*, given by

$$l_{ij} = \begin{cases} \displaystyle\sum_{r \in \{0,\ldots,n\} \setminus \{i\}} a_{ir} & \text{for } i = j, \\ -a_{ij} & \text{for } i \ne j. \end{cases}$$

If G is a graph, i.e., A is a symmetric matrix, then oriented spanning trees defined above are exactly the usual *spanning trees* of G, which are connected subgraphs of G without cycles.

For a subset I in $\{1, \ldots, n\}$ and $i \in I$, let $d_I(i) = \sum_{j \notin I} a_{ij}$, i.e., $d_I(i)$ is the number of edges from the vertex i to a vertex outside of the subset I. Let us say that a sequence $b = (b_1, \ldots, b_n)$ of nonnegative integers is a *G-parking function* if, for any nonempty subset $I \subseteq \{1, \ldots, n\}$, there exists $i \in I$ such that $b_i < d_I(i)$.

If $G = K_{n+1}$ is the complete graph on $n+1$ vertices then K_{n+1}-parking functions are the usual parking functions of size n defined in the beginning of this section.

Theorem 2.1. *The number G-parking functions equals the number $N_G = \det L_G$ of oriented spanning trees of the digraph G.*

We can reformulate the definition of G-parking functions in algebraic terms as follows. Throughout this paper we fix a field $\mathbb{K}$. Let $\mathcal{I}_G = \langle m_I \rangle$ be the monomial ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials

$$m_I = \prod_{i \in I} x_i^{d_I(i)}, \tag{2}$$

where I ranges over all nonempty subsets $I \subseteq \{1, \ldots, n\}$. Define the algebra $\mathcal{A}_G$ as the quotient $\mathcal{A}_G = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_G$.

A nonnegative integer sequence $b = (b_1, \ldots, b_n)$ is a G-parking function if and only if the monomial $x^b = x_1^{b_1} \cdots x_n^{b_n}$ is nonvanishing in the algebra $\mathcal{A}_G$.

For a monomial ideal $\mathcal{I}$, the set of all monomials that do not belong to $\mathcal{I}$ is a basis in the quotient of the polynomial ring modulo $\mathcal{I}$, called the *standard monomial*

basis. Thus the monomials x^b, where b ranges over G-parking functions, form the standard monomial basis of the algebra $\mathcal{A}_G$.

Corollary 2.2. *The algebra $\mathcal{A}_G$ is finite-dimensional as a linear space over $\mathbb{K}$. Its dimension is equal to the number of oriented spanning trees of the digraph G:*

$$\dim \mathcal{A}_G = N_G.$$

3. Power algebras

Let G be a graph on the set of vertices $0, 1, \dots, n$. In this case the dimension of the algebra $\mathcal{A}_G$ is equal to the number of usual spanning trees of G.

For a nonempty subset I in $\{1, \dots, n\}$, let $D_I = \sum_{i \in I,\, j \notin I} a_{ij} = \sum_{i \in I} d_I(i)$ be the total number of edges that join some vertex in I with a vertex outside of I. For any nonempty subset $I \subseteq \{1, \dots, n\}$, let

$$p_I = \left(\sum_{i \in I} x_i \right)^{D_I}. \tag{3}$$

Let $\mathcal{J}_G = \langle p_I \rangle$ be the ideal in the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ generated by the polynomials p_I for all nonempty subsets I. Define the algebra $\mathcal{B}_G$ as the quotient $\mathcal{B}_G = \mathbb{K}[x_1, \dots, x_n]/\mathcal{J}_G$.

The algebras $\mathcal{A}_G$ and $\mathcal{B}_G$, as well as all other algebras in this paper, are graded. For a graded algebra $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1 \oplus \mathcal{A}^2 \oplus \cdots$, the *Hilbert series* of $\mathcal{A}$ is the formal power series in q given by

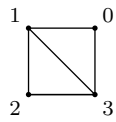
$$\operatorname{Hilb} \mathcal{A} = \sum_{k \geq 0} q^k \dim \mathcal{A}^k.$$

Theorem 3.1. *The monomials x^a, where a ranges over G-parking functions, form a linear basis of the algebra $\mathcal{B}_G$. Thus the Hilbert series of the algebras $\mathcal{A}_G$ and $\mathcal{B}_G$ coincide termwise. In particular, both these algebras are finite-dimensional as linear spaces over $\mathbb{K}$ and*

$$\dim \mathcal{A}_G = \dim \mathcal{B}_G = N_G$$

is the number of spanning trees of the graph G.

For example, for $n = 3$ and the graph

$G =$ 

,

we have

$$\mathcal{I}_G = \left\langle x_1^3, x_2^2, x_3^3, x_1^2 x_2, x_1^2 x_3^2, x_2 x_3^2, x_1 x_2^0 x_3 \right\rangle,$$

$$\mathcal{J}_G = \left\langle x_1^3, x_2^2, x_3^3, (x_1 + x_2)^3, (x_1 + x_3)^4, (x_2 + x_3)^3, (x_1 + x_2 + x_3)^2 \right\rangle,$$

the standard monomial basis of the algebra $\mathcal{A}_G$ is $\{1, x_1, x_2, x_3, x_1^2, x_1 x_2, x_2 x_3, x_3^2\}$, $\dim \mathcal{A}_G = \dim \mathcal{B}_G = 8$ is equal to the number of spanning trees of G, and $\operatorname{Hilb} \mathcal{A}_G = \operatorname{Hilb} \mathcal{B}_G = 1 + 3q + 4q^2$.

We will refine Theorem 3.1 and interpret dimensions of graded components of the algebras $\mathcal{A}_G$ and $\mathcal{B}_G$ in terms of certain statistics on spanning trees. Let us fix a linear ordering of all edges of the graph G. For a spanning tree T of G, an edge $e \in G \setminus T$ is called *externally active* if there exists a cycle C in the graph G such that e is the minimal edge of C and $(C \setminus \{e\}) \subset T$. The *external activity* of a

spanning tree is the number of externally active edges. Let N_G^k denote the number of spanning trees $T \subset G$ of external activity k. Even though the notion of external activity depends on a particular choice of ordering of edges, the numbers N_G^k are known to be invariant on the choice of ordering.

Let $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ be the k-th graded components of the algebras $\mathcal{A}_G$ and $\mathcal{B}_G$, correspondingly.

Theorem 3.2. *The dimensions of the k-th graded components $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ are equal to*

$$\dim \mathcal{A}_G^k = \dim \mathcal{B}_G^k = N_G^{|G|-n-k},$$

the number of spanning trees of G of external activity $|G|-n-k$, where $|G|$ denotes the number of edges of G.

4. Examples

4.1. **Two algebras of dimension $(n+1)^{n-1}$.** Suppose that $G = K_{n+1}$ is the complete graph on $n+1$ vertices. As we have already mentioned, the K_{n+1}-parking functions are the usual parking functions of size n defined in the beginning of Section 2.

Let $\mathcal{I}_n = \langle m_I \rangle$ and $\mathcal{J}_n = \langle p_I \rangle$ be the ideals in the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ generated by the monomials m_I and the polynomials p_I, correspondingly, given by

$$m_I = (x_{i_1} \cdots x_{i_r})^{n-r+1},$$

$$p_I = (x_{i_1} + \cdots + x_{i_r})^{r(n-r+1)},$$

where in both cases $I = \{i_1, \dots, i_r\}$ runs over all nonempty subsets of $\{1, \dots, n\}$. Let $\mathcal{A}_n = \mathbb{K}[x_1, \dots, x_n]/\mathcal{I}_n$ and $\mathcal{B}_n = \mathbb{K}[x_1, \dots, x_n]/\mathcal{J}_n$.

Corollary 4.1. *The graded algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ have the same Hilbert series. They are finite-dimensional, as linear spaces over $\mathbb{K}$. Their dimensions are equal to*

$$\dim \mathcal{A}_n = \dim \mathcal{B}_n = (n+1)^{n-1}.$$

The images of the monomials x^b, where b ranges over parking functions of size n, form linear bases in both algebras $\mathcal{A}_n$ and $\mathcal{B}_n$.

An *inversion* in a tree T on the $n+1$ vertices labelled $0, \dots, n$ is a pair of vertices labelled i and j such that $i > j$ and the vertex i belongs to the shortest path in T that joins the vertex j with the root 0.

Corollary 4.2. *The dimension $\dim \mathcal{A}_n^k = \dim \mathcal{B}_n^k$ of the k-th graded components of the algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ is equal to*

(A) *the number of parking functions b of size n such that $b_1 + \cdots + b_n = k$;*
(B) *the number of trees on $n+1$ vertices with external activity $\binom{n}{2} - k$;*
(C) *the number of trees on $n+1$ vertices with $\binom{n}{2} - k$ inversions.*

It is well known that the numbers (A), (B), and (C) are equal, see [Krew]. The *inversion polynomial* is defined as the sum $I_n(q) = \sum_T q^{\#\text{ of inversions in } T}$ over all trees T on $n+1$ labelled vertices. Thus the Hilbert series of the algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ are equal to

$$\operatorname{Hilb} \mathcal{A}_n = \operatorname{Hilb} \mathcal{B}_n = q^{\binom{n}{2}} I_n(q^{-1}).$$

4.2. **Two algebras of dimension $l\,(l+kn)^{n-1}$.** It is possible to extend the previous example as follows. Fix two nonnegative integers k and l. Let $G = K_{n+1}^{k,l}$ be the complete graph on the vertices $0, 1, \ldots, n$ with the edges (i,j), $i,j \neq 0$, of multiplicity k and the edges $(0,i)$ of multiplicity l. The $K_{n+1}^{k,l}$-parking functions are the nonnegative integer sequences $b = (b_1, \ldots, b_n)$ such that, for $r = 1, \ldots, n$,

$$\#\{i \mid b_i < l + k(r-1)\} \geq r.$$

The definition of these functions can be also formulated as $c_i < l + (i-1)\,k$, where $c_1 \leq \cdots \leq c_n$ is the increasing rearrangement of elements of b. Such functions were studied in [PiSt] and then in [Yan]. These authors demonstrated that their number equals $l\,(l+kn)^{n-1}$. One can show, using for example the matrix-tree theorem (1), that the number of spanning trees in the graph $K_{n+1}^{k,l}$ equals $l\,(l+kn)^{n-1}$. Thus Theorem 2.1 recovers the above formula of the number of $K_{n+1}^{k,l}$-parking functions.

Let $\mathcal{I}_{n,k,l} = \langle m_I \rangle$ and $\mathcal{J}_{n,k,l} = \langle p_I \rangle$ be the ideals in the ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials m_I and the polynomials p_I, correspondingly, given by

$$m_I = (x_{i_1} \cdots x_{i_r})^{l+k(n-r)},$$

$$p_I = (x_{i_1} + \cdots + x_{i_r})^{r(l+k(n-r))},$$

where in both cases $I = \{i_1, \ldots, i_r\}$ runs over all nonempty subsets of $\{1, \ldots, n\}$. Let $\mathcal{A}_{n,k,l} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_{n,k,l}$ and $\mathcal{B}_{n,k,l} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_{n,k,l}$.

Corollary 4.3. *The graded algebras $\mathcal{A}_{n,k,l}$ and $\mathcal{B}_{n,k,l}$ have the same Hilbert series. They are finite-dimensional, as linear spaces over $\mathbb{K}$. Their dimensions are*

$$\dim \mathcal{A}_{n,k,l} = \dim \mathcal{B}_{n,k,l} = l\,(l+kn)^{n-1}.$$

The images of the monomials x^b, where b ranges over $K_{n+1}^{k,l}$-parking functions, form linear bases in both algebras $\mathcal{A}_{n,k,l}$ and $\mathcal{B}_{n,k,l}$.

5. Monotone monomial ideals and their deformations

Let us say that a collection of monomials $\{m_I \mid I \in \Sigma\}$ in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ labelled by a set Σ of nonempty subsets in $\{1, \ldots, n\}$ is a *monotone collection* if the following three conditions hold:

(M1) If $I, J \in \Sigma$ then $I \cup J \in \Sigma$.
(M2) For $I \in \Sigma$, we have $m_I = \prod_{i \in I} x_i^{\nu_I(i)}$.
(M3) For $I, J \in \Sigma$ such that $I \subset J$ and $i \in I$, we have $\deg_{x_i}(m_I) \geq \deg_{x_i}(m_J)$.

For a monotone collection of monomials, the corresponding *monotone monomial ideal* $\mathcal{I}$ is the ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials m_I, $I \in \Sigma$. For example, the monomial ideals $\mathcal{I}_G$ constructed in Section 2 are monotone.

Let $I = \{i_1, \ldots, i_r\}$. For a monomial $m \in \mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$, an *$I$-deformation* of m is a homogeneous polynomial $p \in \mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$ of degree $\deg(p) = \deg(m)$ satisfying the *generosity condition*

$$\mathbb{K}[x_{i_1}, \ldots, x_{i_r}] = \langle R_m \rangle \oplus (p), \tag{4}$$

where $\langle R_m \rangle$ is the linear span of the set R_m of monomials in $\mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$ which are not divisible by m, (p) is the ideal in $\mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$ generated by p, and "$\oplus$" stands for a direct sum of subspaces. Notice that the generosity condition is satisfied for a Zarisky open set of polynomials in $\mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$ of degree $\deg(m)$. For example,

the polynomial $p = a\,x_1 + b\,x_2$ is a $\{1,2\}$-deformation of the monomial $m = x_1$ if and only if $a \neq 0$.

The following lemma describes a class of I-deformations of monomials.

Lemma 5.1. *Let $I = \{i_1, \dots, i_r\}$, let m be a monomial in $\mathbb{K}[x_{i_1}, \dots, x_{i_r}]$, and let $\alpha_1, \dots, \alpha_r \in \mathbb{K} \setminus \{0\}$. Then the polynomial*

$$p = (\alpha_1 x_{i_1} + \cdots + \alpha_r x_{i_r})^{\deg m}$$

is an I-deformation of the monomial m.

Proof. Let $m = x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}$. The generosity condition (4) is equivalent to the condition that the operator

$$\begin{aligned} A &: \mathbb{K}[x_{i_1}, \dots, x_{i_r}] \to \mathbb{K}[x_{i_1}, \dots, x_{i_r}] \\ A &: f \mapsto (\partial/\partial x_{i_1})^{a_1} \cdots (\partial/\partial x_{i_r})^{a_r}(p \cdot f) \end{aligned}$$

has zero kernel. Let us change the coordinates to $y_1 = x_{i_1}, \dots, y_{r-1} = x_{i_{r-1}}$, $y_r = \alpha_1 x_{i_1} + \cdots + \alpha_r x_{i_r}$. The operator A can be written in these coordinates as

$$A(f) = (\tilde{\partial}_1 + \alpha_1 \tilde{\partial}_r)^{a_1} \cdots (\tilde{\partial}_{r-1} + \alpha_{r-1} \tilde{\partial}_r)^{a_{r-1}} (\alpha_r \tilde{\partial}_r)^{a_r} (y_r^{a_1 + \cdots + a_r} \cdot f)$$

where $\tilde{\partial}_j = \partial/\partial y_j$. Then $A(f) = c \cdot f + g$, where c is a nonzero constant and $\deg_{y_r}(g) < \deg_{y_r}(f)$. Thus, in an appropriate basis, the operator A is given by a triangular matrix with nonzero diagonal elements. This implies that $\operatorname{Ker} A = 0$. $\square$

A *deformation* of a monotone monomial ideal $\mathcal{I}$ generated by monomials $m_I, I \in \Sigma$, is an ideal $\mathcal{J}$ generated by polynomials p_I, $I \in \Sigma$, such that p_I is an I-deformation of m_I for each $I \in \Sigma$. For example, according to Lemma 5.1, the ideal $\mathcal{J}_G$ given in Section 3 is a deformation of the monotone monomial ideal $\mathcal{I}_G$.

Theorem 5.2. *Let $\mathcal{I}$ be a monotone monomial ideal, and R be the standard monomial basis of the algebra $\mathcal{A} = \mathbb{K}[x_1, \dots, x_n]/\mathcal{I}$, i.e., R is the set of monomials that do not belong to $\mathcal{I}$. Let $\mathcal{J}$ be a deformation of the ideal $\mathcal{I}$, and $\mathcal{B} = \mathbb{K}[x_1, \dots, x_n]/\mathcal{J}$.*

Then the monomials in R linearly span the algebra $\mathcal{B}$.

Remark that the set of monomials R may or may not be a basis for $\mathcal{B}$.

Corollary 5.3. *Let $\mathcal{I}$ be a monotone monomial ideal, $\mathcal{J}$ be a deformation of the ideal $\mathcal{I}$, $\mathcal{A} = \mathbb{K}[x_1, \dots, x_n]/\mathcal{I}$, and $\mathcal{B} = \mathbb{K}[x_1, \dots, x_n]/\mathcal{J}$. Then we have the following termwise inequalities for the Hilbert series:*

$$\operatorname{Hilb} \mathcal{I} \leq \operatorname{Hilb} \mathcal{J} \qquad \textit{or, equivalently,} \qquad \operatorname{Hilb} \mathcal{A} \geq \operatorname{Hilb} \mathcal{B}.$$

In same cases the Hilbert series are actually equal to each other. According to Theorem 3.1, $\operatorname{Hilb} \mathcal{A}_G = \operatorname{Hilb} \mathcal{B}_G$, for any graph G. However in general the Hilbert series may not be equal to each other. It would be interesting to describe a general class of monotone monomial ideals and their deformations with equal Hilbert series.

There is an obvious correspondence between the generators m_I of a monotone monomial ideal $\mathcal{I}$ and the generators p_I of its deformation $\mathcal{J}$. Notice however that (except for very special cases) the monomial generator m_I does not belong to the boundary of the Newton polytopes of it polynomial deformation p_I. Thus the monomial m_I can not be the leading term of the polynomial p_I for any term order. This shows that our results can not be tackled by the standard Gröbner bases technique.

6. Hilbert series of monotone monomial ideals

In this section we present a formula for the Hilbert series of an arbitrary monotone monomial ideal.

We first give a general formula from [Naru] related to semilattices. Let $\mathcal{P}$ be a finite *join-semilattice*, i.e, $\mathcal{P}$ is a poset on a finite set of elements such that for any $X, Y \in \mathcal{P}$ there exists a unique minimal element in the set $\{Z \in \mathcal{P} \mid Z \geq X,\ Z \geq Y\}$, called *join* of X and Y and denoted $X \vee Y$. Let $\mathcal{M}$ be a set together with a grading $\gamma : \mathcal{M} \to \mathbb{Z}_{\geq 0}$. For a subset $M \subset \mathcal{M}$, the generating function of M is the formal power series in q given by

$$\mathrm{GF}(M) = \sum_{a \in M} q^{\gamma(a)}.$$

Proposition 6.1. [Naru] *Let M_X, $X \in \mathcal{P}$, be a collection of subsets of $\mathcal{M}$ such that, for any $X, Y \in \mathcal{P}$,*

$$M_X \cap M_Y \subseteq M_{X \vee Y}. \tag{5}$$

Then we have

$$\mathrm{GF}(\mathcal{M} \setminus \bigcup_{x \in P} M_X) = \mathrm{GF}(\mathcal{M}) + \sum_{k \geq 1} (-1)^k \sum_{X_1 < \cdots < X_k} \mathrm{GF}(M_{X_1} \cap \cdots \cap M_{X_k}),$$

where the second sum is over all strictly increasing chains $X_1 < \cdots < X_k$ in the semilattice $\mathcal{P}$.

Proof. According to the *inclusion-exclusion principle,* see [Sta1, Section 2.1], we have

$$\mathrm{GF}(\mathcal{M} \setminus \bigcup_{X \in \mathcal{P}} M_X) = \mathrm{GF}(\mathcal{M}) - \sum_X \mathrm{GF}(M_X) + \sum_{X,Y} \mathrm{GF}(M_X \cap M_Y) - \cdots .$$

The general summand in this expression is $s_{\mathcal{X}} = (-1)^k \,\mathrm{GF}(M_{X_1} \cap \cdots \cap M_{X_k})$, where $\mathcal{X} = \{X_1, \ldots, X_k\}$ is an unordered k-tuple of distinct elements in $\mathcal{P}$. We argue that if we take the summation only over *increasing chains* $X_1 < \cdots < X_k$ in $\mathcal{P}$ we get exactly the same answer. Indeed, let us show that the contribution of all other tuples $\mathcal{X}$ is zero. We will use the *involution principle,* see [Sta1, Section 2.6]. Let us construct an involution ι on the set of all unordered k-tuples $\mathcal{X}$ of all possible sizes $k \geq 0$ such that the elements of $\mathcal{X}$ cannot be arranged in an increasing chain. Let us fix a linear order on elements of $\mathcal{P}$. Find the lexicographically minimal pair of elements X and Y in $\mathcal{X}$ such that $X \not\leq Y$ and $Y \not\leq X$. Let $Z = X \vee Y$. If $Z \notin \mathcal{X}$, then set $\iota : \mathcal{X} \mapsto \mathcal{X} \cup \{Z\}$, otherwise set $\iota : \mathcal{X} \mapsto \mathcal{X} \setminus \{Z\}$. Then ι is an involution. According to condition (5), we have $s_{\mathcal{X}} = -s_{\iota(\mathcal{X})}$. Thus all summands $s_{\mathcal{X}}$ corresponding to non-chains cancel each other. □

Let $\mathcal{I}$ be a monotone monomial ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by monomials $m_I = \prod_{i \in I} x_i^{\nu_I(i)}$, $I \in \Sigma$, and let $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$. The set Σ is partially ordered by containment.

Proposition 6.2. *The Hilbert series of the algebra $\mathcal{A}$ equals*

$$\operatorname{Hilb} \mathcal{A} = \frac{1 + \sum_{k \geq 1} (-1)^k \sum_{I_1 \subsetneq \cdots \subsetneq I_k} q^{S(I_1, \ldots, I_k)}}{(1-q)^n},$$

where the sum is over all strictly increasing chains $I_1 \subsetneq \cdots \subsetneq I_k$ *in* Σ *and*

$$S(I_1,\dots,I_k) = \sum_{i_1\in I_1} \nu_{I_1}(i_1) + \sum_{i_2\in I_2\setminus I_1} \nu_{I_2}(i_2) + \cdots + \sum_{i_k\in I_k\setminus I_{k-1}} \nu_{I_k}(i_k).$$

According to condition (M1), Σ is a semilattice with join $I \vee J = I \cup J$. For $I \in \Sigma$, let M_I denote the set of monomials in $\mathbb{K}[x_1,\dots,x_n]$ divisible by m_I.

Lemma 6.3. *For any* $I, J \in \Sigma$*, we have* $M_I \cap M_J \subseteq M_{I\cup J}$.

Proof. By condition (M3), for $i \in I$, we have $\deg_{x_i} m_I \geq \deg_{x_i} m_{I\cup J}$. Analogously, for any $j \in J$, we have $\deg_{x_j} m_J \geq \deg_{x_j} m_{I\cup J}$. Thus any common divisor of m_I and m_J is divisible by $m_{I\cup J}$. □

Proof of Proposition 6.2. Let $\mathcal{M}$ be the set of monomials in $\mathbb{K}[x_1,\dots,x_n]$, and let $\gamma(m) = \deg(m)$ for $m \in \mathcal{M}$. Then $\operatorname{Hilb}\mathcal{A} = \operatorname{GF}(\mathcal{M} \setminus \bigcup M_I)$. Lemma 6.3 shows that all conditions of Proposition 6.1 are satisfied, where $\mathcal{P} = \Sigma$. Notice that $\operatorname{GF}(\mathcal{M}) = (1-q)^{-n}$. For an increasing chain $I_1 \subsetneq \cdots \subsetneq I_k$, the least common multiple of the monomials $m_{I_1},\dots,m_{I_k}$ equals

$$\prod_{i_1\in I_1} x_{i_1}^{\nu_{I_1}(i_1)} \times \prod_{i_2\in I_2\setminus I_1} x_{i_2}^{\nu_{I_2}(i_2)} \times \cdots \times \prod_{i_k\in I_k\setminus I_{k-1}} x_{i_k}^{\nu_{I_k}(i_k)}. \tag{6}$$

Its degree is $S(I_1,\dots,I_k)$. Thus $\operatorname{GF}(M_{I_1}\cap\cdots\cap M_{I_k}) = q^{S(I_1,\dots,I_k)}(1-q)^{-n}$.

The claim of Proposition 6.2 now follows from Proposition 6.1. □

Lemma 6.4. *The algebra* $\mathcal{A}$ *is finite-dimensional as a linear space over* $\mathbb{K}$ *if and only if* Σ *is the set of all nonempty subsets in* $\{1,\dots,n\}$.

Proof. If there is $i \in \{1,\dots,n\}$ such that $\{i\} \notin \Sigma$ then the powers x_i^a form an infinite linearly independent subset in $\mathcal{A}$. Thus $\mathcal{A}$ is infinite-dimensional. Otherwise, if Σ contains all one-element subsets, then, by condition (M1), Σ should contain all nonempty subsets. In this case, the algebra $\mathcal{A}$ is finite-dimensional. Indeed, a monomial $x_1^{a_1}\dots x_n^{a_n}$ vanishes in $\mathcal{A}$ unless $a_1 < \nu_{\{1\}}(1),\dots,a_n < \nu_{\{n\}}(n)$. □

Let us now assume that the set Σ consists of all nonempty subsets in $\{1,\dots,n\}$, and let $\nu(i) = \nu_{\{i\}}(i)$.

Proposition 6.5. *The dimension of the algebra* $\mathcal{A}$ *is given by the following polynomial in the variables* $\{\nu_I(i) \mid I \in \Sigma, i \in I\}$*:*

$$\begin{aligned}\dim\mathcal{A} = \sum_{I_1\subsetneq\cdots\subsetneq I_k} (-1)^k \prod_{i_1\in I_1}(\nu(i_1)-\nu_{I_1}(i_2)) \times \prod_{i_2\in I_2\setminus I_1}(\nu(i_2)-\nu_{I_2}(i_2))\times \\ \times\cdots\times \prod_{i_k\in I_k\setminus I_{k-1}}(\nu(i_k)-\nu_{I_k}(i_k)) \times \prod_{i_{k+1}\notin I_k}\nu(i_{k+1}),\end{aligned} \tag{7}$$

where the sum is over all strictly increasing chains $I_1 \subsetneq \cdots \subsetneq I_k$ *of nonempty subsets in* $\{1,\dots,n\}$*, including the empty chain of size* $k = 0$.

Proof. Let $\tilde{\mathcal{M}}$ be the set of monomials $x_1^{a_1}\cdots x_n^{a_n}$ such that $a_1 < \nu(1),\dots,a_n < \nu(n)$. A monomial x^a vanishes in the algebra $\mathcal{A}$ unless $x^a \in \tilde{\mathcal{M}}$. Let $\tilde{M}_I = M_I \cap \tilde{\mathcal{M}}$. Proposition (6.1) for $\tilde{\mathcal{M}}$ and $\gamma \equiv 0$ implies that

$$\dim\mathcal{A} = |\tilde{\mathcal{M}}| + \sum_{k\geq 1}(-1)^k \sum_{I_1\subsetneq\cdots\subsetneq I_k} |\tilde{M}_{I_1}\cap\cdots\cap\tilde{M}_{I_k}|.$$

According to (6), $(-1)^k\,|\tilde{M}_{I_1}\cap\cdots\cap\tilde{M}_{I_k}|$ is equal to the summand in (7). $\square$

Remark that if $I_1,\dots,I_k$ is not a chain of subsets then $|\tilde{M}_{I_1}\cap\cdots\cap\tilde{M}_{I_k}|$ may not be a polynomial in the $\nu_I(i)$. It can include expressions like $\min(\nu_I(i),\nu_J(i))$. Thus the inclusion-exclusion principle does not immediately produce a polynomial expression for $\dim\mathcal{A}$. Miraculously, all non-polynomial terms cancel each other.

7. Proof of Theorem 2.1

Let us specialize Proposition 6.5 to the algebra $\mathcal{A}_G$, where G is a digraph on the vertices $0,\dots,n$. Recall that a_{ij} are entries of the adjacency matrix of G.

Corollary 7.1. *The dimension of the algebra $\mathcal{A}_G$ is given by the following polynomial in the a_{ij}:*

$$\begin{aligned}\dim\mathcal{A}_G = \sum_{I_1\subsetneq\cdots\subsetneq I_k}(-1)^k\prod_{i_1\in I_1}\left(\sum_{j_1\in I_1}a_{i_1j_1}\right)\times\prod_{i_2\in I_2\setminus I_1}\left(\sum_{j_2\in I_2}a_{i_2j_2}\right)\times\\ \times\cdots\times\prod_{i_k\in I_k\setminus I_{k-1}}\left(\sum_{j_k\in I_k}a_{i_kj_k}\right)\times\prod_{i_{k+1}\in\{1,\dots,n\}\setminus I_k}\left(\sum_{j_{k+1}\in\{0,\dots,n\}}a_{i_{k+1}j_{k+1}}\right),\end{aligned}\tag{8}$$

where the sum is over all strictly increasing chains $I_1\subsetneq\cdots\subsetneq I_k$ of nonempty subsets in $\{1,\dots,n\}$, including the empty chain of size $k=0$. In this formula, we assume that $a_{ii}=0$.

Proof of Theorem 2.1. Let us show that the expression (8) for $\dim\mathcal{A}_G$ is equal to the number of oriented spanning trees of G. We will use the involution principle again.

Let us first give a combinatorial interpretation of the right-hand side of (8). The summand s_{I_*} that corresponds to an increasing chain $I_*=I_1\subsetneq\cdots\subsetneq I_k$ equal to $(-1)^k$ times the number of subgraphs H of G such that

(i) H contains exactly n directed edges $(i,f(i))$ for $i=1,\dots,n$.
(ii) If $i\in I_r\setminus I_{r-1}$ then $f(i)\in I_r$. (Here we assume that $I_0=\emptyset$.)

For such a subgraph H, let $J_H\subseteq\{1,\dots,n\}$ be the set of vertices i such $f^p(i)=0$ for some power p, i.e., J_H is the set of vertices i such that there is a directed path in H from i to the root 0. Notice that if $i\in\bigcup I_r$ then $f^p(i)\in\bigcup I_r$ for any p. Thus $I_1,\dots,I_k\subseteq\overline{J_H}=\{1,\dots,n\}\setminus J_H$. Also notice that H is an oriented spanning tree of G if and only if $J_H=\{1,\dots,n\}$.

Let us now construct an involution κ on the set of pairs (I_*,H) such that H is not an oriented spanning tree. In other words, the involution κ acts on the set of pairs (I_*,H) with nonempty $\overline{J_H}$. If $\overline{J_H}\notin I_*$ then define $\kappa:(I_*,H)\mapsto(I_*\cup\{\overline{J_H}\},H)$; otherwise define $\kappa:(I_*,H)\mapsto(I_*\setminus\{\overline{J_H}\},H)$. The contribution of the pair (I_*,H) to the right-hand side of (8) is opposite to the contribution of $\kappa(I_*,H)$. Thus the contributions of all subgraphs H which are not oriented spanning trees cancel each other. This implies that $\dim\mathcal{A}_G$ is the number of oriented spanning trees. $\square$

It would be interesting to find a combinatorial proof of Theorem 2.1. In other words, one would like to present a bijection between G-parking functions and oriented spanning trees of G. There are several known bijections between the

usual parking functions and trees. One such bijection is relatively easy to construct. There is a more elaborate bijection that maps parking functions b with $b_1 + \cdots + b_n = k$ into trees with $\binom{n}{2} - k$ inversions, see [Krew].

8. Algebra $\mathcal{C}_G$

Let G be a graph on the set of vertices $0, \ldots, n$. We will say that a subgraph $H \subset G$ of the graph G is *slim* if the complement subgraph $G \setminus H$ is connected. Let us associate commutative variables ϕ_e, $e \in G$, with edges of the graph G, and let Φ_G be the algebra over $\mathbb{K}$ generated by the ϕ_e with the defining relations:

$$(\phi_e)^2 = 0, \quad \text{for any edge } e;$$

$$\prod_{e \in H} \phi_e = 0, \quad \text{for any non-slim subgraph } H \subset G.$$

Clearly, the square-free monomials $\phi_H = \prod_{e \in H} \phi_e$, where H ranges over all slim subgraphs in G, form a linear basis of the algebra Φ_G. Thus the dimension of Φ_G is equal to the number of connected subgraphs in G.

For $i = 1, \ldots, n$, let

$$X_i = \sum_{e \in G} c_{i,e}\, \phi_e,$$

where

$$c_{i,e} = \begin{cases} 1 & \text{if } e = (i,j),\ i < j; \\ -1 & \text{if } e = (i,j),\ i > j; \\ 0 & \text{otherwise.} \end{cases}$$

Define $\mathcal{C}_G$ as the subalgebra in Φ_G generated by the elements $X_1, \ldots, X_n$.

Fix a linear ordering of edges of the graph G. Recall that N_G^k denotes the number of spanning trees of G with external activity k, see Section 3.

Theorem 8.1. (1) *The dimension of the algebra $\mathcal{C}_G$ as a linear space over $\mathbb{K}$ equals the number of spanning trees in the graph G.*

(2) *The dimension of the k-th graded component $\mathcal{C}_G^k$ of the algebra $\mathcal{C}_G$ equals the number $N_G^{|G|-n-k}$ of spanning trees of G with external activity $|G| - n - k$.*

Recall that, for a nonempty subset $I \subset \{1, \ldots, n\}$, $D_I = \sum_{i \in I,\, j \notin I} a_{ij}$ is the number of edges in G that connect a vertex inside I with a vertex outside of I, see Section 3.

Lemma 8.2. *For any nonempty subset $I \subset \{1, \ldots, n\}$, the following relation holds in the algebra $\mathcal{C}_G$:*

$$\left(\sum_{i \in I} X_i \right)^{D_I} = 0.$$

This lemma shows that the algebra $\mathcal{C}_G$ is a quotient of the algebra $\mathcal{B}_G$. We will eventually see that $\mathcal{B}_G = \mathcal{C}_G$, but let us pretend that we do not know this yet.

Proof. Let $H_I \subset G$ be the subgraph that consists of all edges that connect a vertex in I with a vertex outside of I. We have $\sum_{i \in I} X_i = \sum_{e \in H_I} \pm \phi_e$. Thus $\left(\sum_{i \in I} X_i\right)^{D_I} = \pm \prod_{e \in H_I} \phi_e = 0$, because H_I is not a slim subgraph of G. $\square$

Let $\mathcal{S}_G$ be the subspace in $\mathbb{K}[y_1, \dots, y_n]$ spanned by the elements

$$\alpha_H = \prod_{e \in H} \alpha_e,$$

for all slim subgraphs $H \subset G$, where $\alpha_e = y_i - y_j$, for an edge $e = (i, j)$, $i < j$. Let $\mathcal{S}_G^k$ denote the k-th graded component of the space $\mathcal{S}_G$.

Lemma 8.3. *For any graph G and any k, we have* $\dim \mathcal{C}_G^k = \dim \mathcal{S}_G^k$.

Proof. Let $b_{H,a}$ be the coefficient $\prod_{e \in H} \phi_e$ in the expansion of $X_1^{a_1} \cdots X_n^{a_n}$, where $a = (a_1, \dots, a_n)$. Then $\dim \mathcal{C}_G^k$ is equal to the rank of the matrix $B = (b_{H,a})$, where H ranges over all slim subgraphs in G with k edges and a ranges over nonnegative integer n-element sequences with $a_1 + \cdots + a_n = k$. On the other hand, $b_{H,a}$ is also equal to the coefficients of $y_1^{a_1} \cdots y_n^{a_n}$ in the expansion of α_H. Thus $\dim \mathcal{S}_G^k$ equals the rank of the same matrix $B = (b_{H,a})$. □

For a spanning tree T in G, let T^+ denote the graph obtained from T by adding all externally active edges. In virtue of Lemma 8.3, the following claim implies Theorem 8.1.

Proposition 8.4. *The collection of elements $\alpha_{G \setminus T^+}$, where T ranges over all spanning trees of G, forms a linear basis of the space $\mathcal{S}_G$.*

Let us first prove a weaker version of this claim.

Lemma 8.5. *The elements $\alpha_{G \setminus T^+}$, where T ranges over all spanning trees of G, spans the space $\mathcal{S}_G$.*

Proof. Suppose not. Let H be the lexicographically maximal slim subgraph of G such that α_H cannot be expressed as a linear combination of the $\alpha_{G \setminus T^+}$. Then there exists a cycle $C = \{e_1, \dots, e_l\} \subset G$ with the minimal element e_1 such that $H \cap C = \{e_1\}$. Then α_{e_1} is a linear combination of $\alpha_{e_2}, \dots, \alpha_{e_l}$. Let H_i be the graph obtained from H by replacing the edge e_1 with e_i. For $i = 2, \dots, l$, the graph H_i is a slim subgraph of G, which is lexicographically greater than H_1. Then α_H can be expressed as a linear combination of $\alpha_{H_2}, \dots, \alpha_{H_l}$. Contradiction. □

Proof of Proposition 8.4. Recall that N_G denote the number of spanning trees in the graph G. In view of Lemma 8.5 it is enough to show that $\dim \mathcal{S}_G = N_G$. We will prove this statement by induction on the number of edges in G.

If G is a disconnected graph then it has no slim subgraphs and $\dim \mathcal{S}_G = N_G = 0$. If G is a tree then $\dim \mathcal{S}_G = N_G = 1$. This establishes the base of induction.

Suppose that G is a graph with at least one edge. Pick an edge e of G. Let $G \setminus e$ be the graph obtained from G by removing the edge e, also let G/e be the graph obtained from G by contracting the edge e. Then $N_G = N_{G \setminus e} + N_{G/e}$. Indeed, for a spanning tree T in G, we have either $e \notin T$ or $e \in T$. The former trees are exactly the spanning trees of $G \setminus e$. The later trees are in a bijective correspondence with spanning trees of G/e. This correspondence is given by contracting the edge e. Assume by induction that the statement is true for both graphs $G \setminus e$ and G/e.

Let $\mathcal{S}'_G \subset \mathcal{S}_G$ be the span of the $\alpha_{H'}$ with slim subgraphs $H' \subset G$ such that $e \in H'$ and let $\mathcal{S}''_G \subset S_G$ be the span of the $\alpha_{H''}$ with slim subgraphs $H'' \subset G$ such that $e \notin H''$. Then the space $\mathcal{S}_G$ is spanned by $\mathcal{S}'_G$ and $\mathcal{S}''_G$. Thus

$$\dim \mathcal{S}_G = \dim \mathcal{S}'_G + \dim \mathcal{S}''_G - \dim(\mathcal{S}'_G \cap \mathcal{S}''_G). \tag{9}$$

We have $\mathcal{S}'_G = (y_i - y_j)\,\mathcal{S}_{G\setminus e}$, where $e = (i,j)$. Thus $\dim \mathcal{S}'_G = \dim \mathcal{S}_{G\setminus e}$. Let $p : f(y_1,\dots,y_n) \mapsto f(y_1,\dots,y_n) \bmod (y_i - y_j)$ be the natural projection. Then $p(\mathcal{S}''_G) = \mathcal{S}_{G/e}$ and $\mathcal{S}'_G \cap \mathcal{S}''_G \subset \mathrm{Ker}(p)$. Thus

$$\dim \mathcal{S}''_G = \dim \mathcal{S}_{G/e} + \dim \mathrm{Ker}(p) \geq \dim \mathcal{S}_{G/e} + \dim(\mathcal{S}'_G \cap \mathcal{S}''_G). \tag{10}$$

Combining (9) and (10), we get

$$\dim \mathcal{S}_G \geq \mathcal{S}_{G\setminus e} + \dim \mathcal{S}_{G/e}.$$

By the induction hypothesis, the right-hand side of this expression equals $N_{G\setminus e} + N_{G/e} = N_G$. Thus $\dim \mathcal{S}_G \geq N_G$. On the other hand, Lemma 8.5 implies that $\dim \mathcal{S}_G \leq N_G$. Thus $\dim \mathcal{S}_G = N_G$, as needed. □

9. Proof of Theorems 3.1, 3.2, and 5.2

Let m_I, $I \in \Sigma$, be a monotone collection of monomials, and let $\mathcal{I} \subset \mathbb{K}[x_1,\dots,x_n]$ be the monomial ideal generated by the m_I, $I \in \Sigma$.

For a subset $I = \{i_1,\dots,i_r\} \in \{1,\dots,n\}$, let $\mathbb{K}[x_I] = \mathbb{K}[x_{i_1},\dots,x_{i_k}]$, and let Mon_I denote the set of all monomials in the variables x_i, $i \in I$. For $I \in \Sigma$, let $M_I = m_I \cdot \mathrm{Mon}_{\{1,\dots,n\}}$ be the set of all monomials in $\mathbb{K}[x_1,\dots,x_n]$ divisible by m_I. The standard monomial basis R of the algebra $\mathcal{A} = \mathbb{K}[x_1,\dots,x_n]/\mathcal{I}$ is the set of monomials

$$R = \mathrm{Mon}_{\{1,\dots,n\}} \setminus \bigcup_{I\in\Sigma} M_I$$

that survive in the algebra $\mathcal{A}$.

For $I, J \in \Sigma$, denote by $m_{J/I}$ the monomial obtained from m_J by removing all x_i's with $i \in I$, and let $M_{J/I} = m_{J/I} \cdot \mathrm{Mon}_{\overline{I}}$, where $\overline{I} = \{1,\dots,n\} \setminus I$. Let $\mathcal{I}_I$ be the monomial ideal in the polynomial ring $\mathbb{K}[x_{\overline{I}}]$ generated by the monomials $\{m_{J/I} \mid J \in \Sigma, J \not\subset I\}$. It follows from the monotonicity condition (M3) that the ideal $\mathcal{I}_I$ is also generated by the set of monomials $\{m_{J/I} \mid J \in \Sigma, J \supsetneq I\}$. Let R_I be the standard monomial basis of the algebra $\mathcal{A}_I = \mathbb{K}[x_{\overline{I}}]/\mathcal{I}_I$:

$$R_I = \mathrm{Mon}_{\overline{I}} \setminus \bigcup_{J\supsetneq I} M_{J/I}.$$

Proposition 9.1. *The polynomial ring $\mathbb{K}[x_1,\dots,x_n]$ decomposes into the direct sum of subspaces:*

$$\mathbb{K}[x_1,\dots,x_n] = \langle R\rangle \oplus \bigoplus_{I\in\Sigma} m_I\,\mathbb{K}[x_I]\,\langle R_I\rangle\,,$$

where $\langle R\rangle$ and $\langle R_I\rangle$ denote the linear spans of monomials in R and R_I, respectively.

Lemma 9.2. *For any monomial $x^a = x_1^{a_1}\cdots x_n^{a_n}$ in the ideal $\mathcal{I}$ there is a unique maximal by inclusion subset $J \in \Sigma$ such that $x^a \in M_I$.*

Proof. Let $\Sigma' = \{I \in \Sigma \mid x^a \in M_I\}$, and let $J = \bigcup_{I\in\Sigma'} I$. Then, by condition (M1), $J \in \Sigma$. According to Lemma 6.3, we have $x^a \in \bigcap_{I\in\Sigma'} M_I \subseteq M_J$. Thus J is the maximal by inclusion element of Σ such that $x^a \in M_J$. □

Proof of Proposition 9.1. For $I \in \Sigma$, let $M_I^{\max}$ be the following set of monomials:

$$M_I^{\max} = M_I \setminus \bigcup_{J\supsetneq I} M_J,$$

i.e., $M_I^{\max}$ is the set of monomials $x^a \in \mathrm{Mon}_{\{1,\dots,n\}}$ such that I is the maximal by inclusion subset $I \in \Sigma$ with $x^a \in M_I$, see Lemma 9.2. Thus the set of all monomials in $\mathbb{K}[x_1,\dots,x_n]$ decomposes into the disjoint union

$$\mathrm{Mon}_{\{1,\dots,n\}} = R \,\dot\cup\, \dot{\bigcup_{I\in\Sigma}} M_I^{\max}. \tag{11}$$

Using monotonicity condition (M3), we obtain, for $I \subsetneq J$,

$$M_I \setminus M_J = m_I \times \mathrm{Mon}_I \times (\mathrm{Mon}_{\overline{I}} \setminus M_{J/I}),$$

where the notation "$\times$" means that every monomial in the left-hand side decomposes uniquely into the product of monomials. Thus we have

$$M_I^{\max} = \bigcap_{J\supsetneq I} (M_I \setminus M_J) = m_I \times \mathrm{Mon}_I \times \bigcap_{J\supsetneq I} (\mathrm{Mon}_{\overline{I}} \setminus M_{J/I}) = m_I \times \mathrm{Mon}_I \times R_I. \tag{12}$$

Formulas (11) and (12) imply the required statement. □

Let p_I, $I \in \Sigma$, be a collection of polynomials such that p_I is an I-deformation of the monomial m_I. Remarkably, a similar statement is valid for the polynomials p_I.

Proposition 9.3. *The polynomial ring $\mathbb{K}[x_1,\dots,x_n]$ decomposes into the direct sum of subspaces:*

$$\mathbb{K}[x_1,\dots,x_n] = \langle R\rangle \oplus \bigoplus_{I\in\Sigma} p_I\, \mathbb{K}[x_I]\ \langle R_I\rangle\,.$$

Proposition 9.3 immediately imply Theorem 5.2, which says that the monomials in R linearly span the algebra $\mathcal{B} = \mathbb{K}[x_1,\dots,x_n]/\langle p_I \mid I\in\Sigma\rangle$.

Lemma 9.4. *Suppose that a polynomial $p \in \mathbb{K}[x_I]$ is an I-deformation of a monomial $m \in \mathbb{K}[x_I]$, see* (4)*. Then for any polynomial $f \in \mathbb{K}[x_I]$ there exists and unique polynomial $g \in \mathbb{K}[x_I]$ such that the difference $m\cdot f - p\cdot g$ contains no monomials divisible by m. The map $f \mapsto g$ is one-to-one.*

Proof. According to the generosity condition (4) the polynomial $m\cdot f$, as well as any other polynomial in $\mathbb{K}[x_I]$, can be written uniquely in the form $m\cdot f = p\cdot g + r$, where $g \in \mathbb{K}[x_I]$ and r is in the linear span $\langle R_m\rangle$ of monomials in $\mathbb{K}[x_I]$ not divisible by m. This defines the map $f \mapsto g$.

On the other hand, for any $g \in \mathbb{K}[x_I]$ there exist unique $f \in \mathbb{K}[x_I]$ and $r \in \langle R_m\rangle$ such that $p\cdot g = m\cdot f - r$. Thus the map $f \mapsto g$ is invertible. The statement of the lemma follows. □

Proof of Proposition 9.3. Pick any linear ordering $I_1,\dots,I_N$ of the set Σ compatible with the inclusion relation, i.e, the inclusion $I_s \subset I_t$ implies that $s \le t$. Let $\Sigma_{(s)} = \{I_1,\dots,I_s\}$ and $\Sigma^{(s)} = \{I_s,\dots,I_N\}$ be initial and terminal intervals of Σ.

We will prove by induction on $N-s$ that the polynomial ring $\mathbb{K}[x_1,\dots,x_n]$ decomposes into the direct sum of subspaces

$$\mathbb{K}[x_1,\dots,x_n] = \langle R\rangle \oplus \bigoplus_{I'\in\Sigma_{(s)}} m_{I'}\, \mathbb{K}[x_{I'}]\ \langle R_{I'}\rangle \oplus \bigoplus_{I''\in\Sigma^{(s+1)}} p_{I''}\, \mathbb{K}[x_{I''}]\ \langle R_{I''}\rangle\,. \tag{13}$$

If $s = N$ then (13) is true according to Proposition 9.1. This gives the base of induction.

Assume that the induction hypothesis that (13) holds for some s and derive the same statement for $s-1$. Let $I = I_s$. For a polynomial $\phi \in \mathbb{K}[x_1, \dots, x_n]$, write its unique presentation

$$\phi = r + \sum_{I' \in \Sigma_{s-1}} m_{I'} \cdot f_{I'} \cdot r_{I'} + m_I \cdot f_I \cdot r_I + \sum_{I'' \in \Sigma^{s+1}} p_{I''} \cdot f_{I''} \cdot r_{I''}, \tag{14}$$

where $r \in \langle R \rangle$ and $f_J \in \mathbb{K}[x_J]$ and $r_J \in R_J$, for any $J \in \Sigma$.

Let $\tilde{f}_I \in \mathbb{K}[x_I]$ be the unique polynomial, provided by Lemma 9.4, such that the difference $d = m_I \cdot f_I - p_I \cdot \tilde{f} \in \mathbb{K}[x_I]$ contains no monomials divisible by m_I. Let $\psi \in \mathbb{K}[x_1, \dots, x_n]$ be the polynomial obtained from ϕ by keeping all terms in (14) except for $m_I \cdot f_I \cdot r_I$ which we substitute by the term $p_I \cdot \tilde{f}_I \cdot r_I$. Then $\phi - \psi = d \cdot r_I$. Pick any monomial e in d. Remind that, according to (12), $M_J^{\max}$ is the set of all monomials in $m_J \, \mathbb{K}[x_J] \, \langle R_J \rangle$. If $e \cdot r_I \notin M_J^{\max}$ for all $J \in \Sigma$, then $e \cdot r_I \in \langle R \rangle$. Otherwise, suppose that $e \cdot r_I \in M_J^{\max}$ for some J. If $J \not\subset I$, then $e \cdot r_I \in M_J^{\max} \subset M_J$ implies that $r_I \in M_{J/I}$, which is impossible. Thus $J \subset I$. Also $J \neq I$ because e is not divisible by m_I. This shows that

$$\phi - \psi \in \langle R \rangle \oplus \bigoplus_{J \subsetneq I} \langle M_J^{\max} \rangle .$$

Therefore, ϕ can be written as

$$\phi = \tilde{r} + \sum_{I' \in \Sigma_{s-1}} m_{I'} \cdot \tilde{f}_{I'} \cdot \tilde{r}_{I'} + p_I \cdot \tilde{f}_I \cdot r_I + \sum_{I'' \in \Sigma^{s+1}} p_{I''} \cdot f_{I''} \cdot r_{I''}, \tag{15}$$

where $\tilde{r} \in \langle R \rangle$, $\tilde{f}_{I'} \in \mathbb{K}[x_{I'}]$, $\tilde{r}_{I'} \in R_{I'}$, and $f_{I''}$ and $r_{I''}$ are the same as before.

Notice that all steps in the transformation of the presentation (14) to the presentation (15) are invertible. Also if $p_I \cdot \tilde{f}_I \cdot r_I = 0$ then all summands in (14) and (15) coincide. So, if at least the one of the summands in the presentation (15) of $\phi = 0$ is nonzero, then we can also find a nonzero presentation of the form (14) for $\phi = 0$, which is impossible by the induction hypothesis. This shows that the presentation (15) of ϕ is unique.

This proves (13). For $s = 0$ we obtain the claim of Proposition 9.3. □

Finally we can put everything together and prove Theorems 3.1 and 3.2.

Proof of Theorems 3.1 and 3.2. For a graph G, let $\mathcal{A}_G$, $\mathcal{B}_G$, and $\mathcal{C}_G$ be the algebras defined in Sections 2, 3, and 8. Then we have the following termwise inequalities of Hilbert series

$$\operatorname{Hilb} \mathcal{A}_G \geq \operatorname{Hilb} \mathcal{B}_G \geq \operatorname{Hilb} \mathcal{C}_G. \tag{16}$$

The first inequality follows from Theorem 5.2 because $\mathcal{I}_G$ is a monotone monomial ideal and, by Lemma 5.1, $\mathcal{J}_G$ is its deformation. The second inequality follows from Lemma 8.2 that says that $\mathcal{C}_G$ is a quotient of $\mathcal{B}_G$. Theorem 2.1 claims that $\dim \mathcal{A}_G = N_G$ is the number of spanning trees of the graph G. On the other hand, by Theorem 8.1, $\dim \mathcal{C}_G = N_G$. Thus all inequalities in (16) are actually equalities. Moreover, by Theorem 8.1, the dimensions of k-th graded components are equal to

$$\dim \mathcal{A}_G^k = \dim \mathcal{B}_G^k = \dim \mathcal{C}_G^k = N_G^{|G|-n-k},$$

the number of spanning trees of G with external activity $|G| - n - k$. □

Corollary 9.5. *The algebras $\mathcal{B}_G$ and $\mathcal{C}_G$ are isomorphic.*

10. Algebras related to forests

Definitions of the algebras $\mathcal{B}_G$ and $\mathcal{C}_G$ and the proof of Theorem 8.1 are similar to constructions from [PSS1]. Let us briefly review main results of [PSS1].

Let G be a graph on the vertices $0, \dots, n$. Let $\hat{\mathcal{J}}_G$ be the ideal in $\mathbb{K}[x_1, \dots, x_n]$ generated by the polynomials

$$\hat{p}_I = \left(\sum_{i \in I} x_i \right)^{D_I + 1},$$

where I ranges over all nonempty subsets in $\{1, \dots, n\}$ and the number D_I is the same as in Section 3, cf. (3). Let $\hat{\mathcal{B}}_G = \mathbb{K}[x_1, \dots, x_n]/\hat{\mathcal{J}}_G$.

Let $\hat{\Phi}_G$ be the commutative algebra generated by the variables $\hat{\phi}_e$, $e \in G$, with the defining relations:

$$(\hat{\phi}_e)^2 = 0, \qquad \text{for any edge } e.$$

And let $\hat{\mathcal{C}}_G$ be the subalgebra of $\hat{\Phi}_G$ generated by the elements

$$\hat{X}_i = \sum_{e \in G} e_{i,e} \hat{\phi}_e,$$

for $i = 1, \dots, n$, cf. Section 8.

A *forest* is a graph without cycles. The connected components of a forest are trees. A *subforest* in a graph G is a subgraph $F \subset G$ without cycles. Fix a linear order of edges of G. An edge $e \in G \setminus F$ is called *externally active* for a forest F if there exists a cycle C in G such that e is the minimal element of C and $(C \setminus \{e\}) \subset F$. The *external activity* of F is the number of externally active edges for F.

Theorem 10.1. [PSS1] *The algebras $\hat{\mathcal{B}}_G$ and $\hat{\mathcal{C}}_G$ are isomorphic to each other. Their dimension is equal to the number of subforests in the graph G.*

The dimension $\dim \hat{\mathcal{B}}_G^k$ of the k-th graded component of the algebra $\hat{\mathcal{B}}_G$ equals the number of subforests F of G with external activity $|G| - |F| - k$.

In [PSS2] we investigated the algebra $\hat{\mathcal{B}}_G$ for the graph $G = K_{n+1}$. Let $\hat{\mathcal{I}}_n = \langle \hat{m}_I \rangle$ and $\hat{\mathcal{J}}_n = \langle \hat{p}_I \rangle$ be two ideals in the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ generated by the monomials $\hat{m}_I$ and the polynomials $\hat{p}_I$, correspondingly, given by

$$\hat{m}_I = (x_{i_1} \cdots x_{i_r})^{n-r+1} x_{i_1},$$

$$\hat{p}_I = (x_{i_1} + \cdots + x_{i_r})^{r(n-r+1)+1},$$

where $I = \{i_1 < \cdots < i_r\}$, ranges over nonempty subsets of $\{1, \dots, n\}$, cf. Subsection 4.1. Notice that $\hat{\mathcal{I}}_n$ is a monotone monomial ideal and $\hat{\mathcal{J}}_n$ is its deformation. Let $\hat{\mathcal{A}}_n = \mathbb{K}[x_1, \dots, x_n]/\hat{\mathcal{I}}_n$ and $\hat{\mathcal{B}}_n = \mathbb{K}[x_1, \dots, x_n]/\hat{\mathcal{J}}_n$.

Let us say that a nonnegative integer sequence $b = (b_1, \dots, b_n)$ is an *almost parking function* of size n if the monomial $x^b = x_1^{b_1} \cdots x_n^{b_n}$ does not belong to the ideal $\hat{\mathcal{I}}_n$. Clearly the class of almost parking functions includes usual parking functions.

For a forest F on the vertices $0, \dots, n$, an *inversion* is a pair of vertices labelled i and j such that $i > j$ and the vertex i belong to the path in F that joins the vertex j with the minimal vertex in its connected component.

Theorem 10.2. [PSS1, PSS2] *The algebras $\hat{\mathcal{A}}_n$ and $\hat{\mathcal{B}}_n$ have the same Hilbert series. The dimension of these algebras is equal to the number of forests on $n+1$ vertices.*

Moreover, the dimension $\dim \hat{\mathcal{A}}_n^k = \dim \hat{\mathcal{B}}_n^k$ of the k-th graded components of the algebras $\hat{\mathcal{A}}_n$ and $\hat{\mathcal{B}}_n$ is equal to

(A) *the number of almost parking functions b of size n such that $\sum_{i=1}^n b_i = k$;*
(B) *the number of forests on $n+1$ vertices with external activity $\binom{n}{2} - k$;*
(C) *the number of forests on $n+1$ vertices with $\binom{n}{2} - k$ inversions.*

The images of the monomials x^b, where b ranges over almost parking functions of size n, form linear bases in both algebras $\hat{\mathcal{A}}_n$ and $\hat{\mathcal{B}}_n$.

Theorem 10.2, first stated in [PSS2], follows from results of [PSS1]. The algebra $\hat{\mathcal{B}}_n$ is the algebra generated by curvature forms on the complete flag manifold. It was introduced in an attempt to lift Schubert calculus on the level of differential forms, see [PSS1, PSS2, ShSh]. This example related to Schubert calculus was our original motivation.

11. ρ-ALGEBRAS AND ρ-PARKING FUNCTIONS

In this section we discuss a special class of monotone monomial ideals and their deformations.

Let $\rho = (\rho_1, \dots, \rho_n)$ be a weakly decreasing sequence of nonnegative integers, called a *degree function.* Let $\mathcal{I}_\rho = \langle m_I \rangle$ and $\mathcal{J}_\rho = \langle p_I \rangle$ be the ideals the ring $\mathbb{K}[x_1, \dots, x_n]$ generated by the monomials m_I and the polynomials p_I, correspondingly, given by

$$m_I = (x_{i_1} \cdots x_{i_r})^{\rho_r},$$

$$p_I = (x_{i_1} + \cdots + x_{i_r})^{r \cdot \rho_r},$$

where in both cases $I = \{i_1, \dots, i_r\}$ runs over all nonempty subsets of $\{1, \dots, n\}$. Let $\mathcal{A}_\rho = \mathbb{K}[x_1, \dots, x_n]/\mathcal{I}_\rho$ and $\mathcal{B}_\rho = \mathbb{K}[x_1, \dots, x_n]/\mathcal{J}_\rho$.

Let us say that a nonnegative integer sequence $b = (b_1, \dots, b_n)$ is a *ρ-parking function* if the monomial $x_1^{b_1} \cdots x_n^{b_n}$ does not belong to the ideal $\mathcal{I}_\rho$. More explicitly, this condition can be reformulated as follows. A nonnegative integer sequence $b = (b_1, \dots, b_n)$ is a ρ-parking function if and only if, for $r = 1, \dots, n$, we have

$$\#\{i \mid b_i < \rho_{n-r+1}\} \geq r.$$

This condition can also be formulated in terms of the increasing rearrangement $c_1 \leq \cdots \leq c_n$ of the elements of b as $c_i < \rho_{n+1-i}$. The ρ-parking functions were studied in [PiSt] and in [Yan]. They also appeared under a different name in [PP]. Notice that $(n, \dots, 1)$-parking functions are exactly the usual parking functions of size n.

The monomials x^b, where b ranges over ρ-parking functions, form the standard monomial basis of the algebra $\mathcal{A}_\rho$. Thus the Hilbert series of the algebra $\mathcal{A}_\rho$ equals

$$\operatorname{Hilb} \mathcal{A}_\rho = \sum_b q^{b_1 + \cdots + b_n},$$

where the sum is over ρ-parking functions. The dimension $\dim \mathcal{A}_\rho$ of this algebra is equal to the number of ρ-parking functions.

Theorem 5.2 specializes to the following statement.

Corollary 11.1. *The monomials x^b, where b ranges over ρ-parking functions, linearly span the algebra $\mathcal{B}_\rho$. Thus we have the termwise inequality of Hilbert series:*

$$\operatorname{Hilb}\mathcal{A}_\rho \geq \operatorname{Hilb}\mathcal{B}_\rho.$$

It would be interesting to describe the class of degree functions ρ such that $\operatorname{Hilb}\mathcal{A}_\rho = \operatorname{Hilb}\mathcal{B}_\rho$. If $\rho_r = l + k(n-r)$ is a linear degree function then, according to Corollary 4.3, the Hilbert series of $\mathcal{A}_\rho$ and $\mathcal{B}_\rho$ are equal to each other and

$$\dim\mathcal{A}_\rho = \dim\mathcal{B}_\rho = l\,(l+kn)^{n-1}.$$

Let us say that a degree function ρ is *almost linear* if there exists an integer k such that $\rho_i - \rho_{i+1}$ equals either k or $k+1$, for $i = 1, \ldots, n-1$. Computer experiments show that the equality $\operatorname{Hilb}\mathcal{A}_\rho = \operatorname{Hilb}\mathcal{B}_\rho$ often holds for almost linear degree functions ρ. The table below lists some almost linear degree functions, for which the equality $\operatorname{Hilb}\mathcal{A}_\rho = \operatorname{Hilb}\mathcal{B}_\rho$ holds.

ρ	$\dim\mathcal{A}_\rho$	ρ	$\dim\mathcal{A}_\rho$
$(4,2,1)$	25	$(8,5,3)$	306
$(8,5,1)$	142	$(8,6,3)$	351
$(6,4,3)$	153	$(11,7,2)$	506
$(9,5,2)$	290	$(12,8,3)$	855
$(6,4,3,2)$	632	$(8,7,5,3)$	3021
$(9,6,4,2)$	2512	$(9,7,6,5)$	4925
$(8,6,5,3)$	2643	$(11,8,6,3)$	7587
$(8,6,5,4)$	2832	$(12,9,7,4)$	12460
$(9,8,6,4,2)$	31472	$(10,9,7,5,3)$	65718

On the other hand, the equality of Hilbert series fails for the almost linear degree functions $\rho = (9,6,3,1)$ and $\rho = (9,7,5,4,3)$. We do not know an example when $\operatorname{Hilb}\mathcal{A}_\rho = \operatorname{Hilb}\mathcal{B}_\rho$ and ρ is not almost linear.

Proposition 6.2 specializes to an expression for the Hilbert series $\operatorname{Hilb}\mathcal{A}_\rho$ with alternating signs. Actually, in this case it is possible to give a simpler subtraction-free expression for the Hilbert series.

Proposition 11.2. *The Hilbert series of the algebra $\mathcal{A}_\rho$ equals*

$$\operatorname{Hilb}\mathcal{A}_\rho = \sum_a \prod_{i=1}^n \frac{q^{\rho_{n-a_i+1}} - q^{\rho_{n-a_i}}}{1-q}, \tag{17}$$

where the sum is over $(n+1)^{n-1}$ usual parking functions $a = (a_1, \ldots, a_n)$ of size n. Here we assume that $\rho_{n+1} = 0$. Thus the dimension of $\mathcal{A}_\rho$, which is the number of ρ-parking functions, is given by the following polynomial in $\rho_1, \ldots, \rho_n$:

$$\dim\mathcal{A}_\rho = \sum_a \prod_{i=1}^n (\rho_{n-a_i} - \rho_{n-a_i+1}),$$

where again the sum is over usual parking functions of size n.

Proof. For $i = 0, \ldots, n$, let Z_i be the interval of integers $Z_i = [\rho_{n-i+1}, \rho_{n-i}[$, where we assume that $\rho_0 = +\infty$ and $\rho_{n+1} = 0$. Then the set of positive integers is the disjoint union of $Z_0, \ldots, Z_n$. Let $f : b \mapsto a$ be the map that sends a positive integer sequence $b = (b_1, \ldots, b_n)$ to the sequence $a = (a_1, \ldots, a_n)$ such that $b_i \in Z_{a_i}$ for

$i = 1, \dots, n$. Then b is a ρ-parking function if and only if a is a usual parking function of size n. Fix a parking function a of size n. Then

$$\sum_{b:\, f(b)=a} q^{b_1+\cdots+b_n} = \prod_{i=1}^{n} \sum_{b_i \in Z_{a_i}} q^{b_i}.$$

is exactly the summand in (17). □

For example, the Hilbert series of $\mathcal{A}_\rho$, for $n = 2$ and $n = 3$, are given by

$$\operatorname{Hilb} \mathcal{A}_{(\rho_1,\rho_2)}(q) = [\rho_2]^2 + 2q^{\rho_2}[\rho_1 - \rho_2]\,[\rho_2]\,,$$

$$\operatorname{Hilb} \mathcal{A}_{(\rho_1,\rho_2,\rho_3)}(q) = [\rho_3]^3 + 3q^{\rho_3}[\rho_2 - \rho_3]\,[\rho_3]^2 + 3q^{2\rho_3}[\rho_2 - \rho_3]^2\,[\rho_3] + \\ +3q^{\rho_2}[\rho_1 - \rho_2]\,[\rho_3]^2 + 6q^{\rho_2+\rho_3}[\rho_1 - \rho_2]\,[\rho_2 - \rho_3]\,[\rho_3]\,,$$

where $[s] = 1 + q + \cdots + q^{s-1}$ denotes the q-analogue of an integer s.

Finally, we formulate a theorem that gives a combinatorial interpretation of the value of the Hilbert series $\operatorname{Hilb} \mathcal{A}\rho$ at $q = -1$. This theorem follows from results of [PP] on ρ-parking functions.

Theorem 11.3. [PP] *The number* $(-1)^{\rho_1+\cdots+\rho_n-n} \operatorname{Hilb} \mathcal{A}_\rho(-1)$ *equals the number of permutations* $\sigma_1, \dots, \sigma_n$ *of* $1, \dots, n$ *such that*

$$\sigma_1 \vee^{\rho_1} \sigma_2 \vee^{\rho_2} \cdots \vee^{\rho_{n-1}} \sigma_n \vee^{\rho_n} 0\,,$$

where the notation $a \vee^k b$ *means that* $a < b$ *for even* k *and* $a > b$ *for odd* k. *In particular,* $\operatorname{Hilb} \mathcal{A}_\rho(-1)$ *is zero if and only if* ρ_n *is even.*

This theorem basically says that $\operatorname{Hilb} \mathcal{A}_\rho(-1)$ is either 0 or plus/minus the number of permutations with prescribed descent positions.

For the case of usual parking functions of size n, i.e., for $\rho = (n, \dots, 1)$, this theorem amounts to the well-known fact that the value of the inversion polynomial $I_n(-1) = (-1)^{\binom{n}{2}} \operatorname{Hilb} \mathcal{A}_{(n,\dots,1)}(-1)$ is the number of alternating permutations of size n, see [Krew].

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