REAL MATRIX PENCILS AND SPECTRAL BRAIDS

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Abstract. For a pair (A, B) of complex $n \times n$ matrices consider the real pencil $T_{A,B}(t) = \cos t\, A + \sin t\, B$. If $T_{A,B}(t)$ has simple spectrum for every t, its eigenvalues trace a closed n-strand braid, the *spectral braid* $\beta_{A,B}$. The basic observation of this paper is the linearization

$$A_+ = \tfrac{1}{2}(A - iB), \qquad A_- = \tfrac{1}{2}(A + iB), \qquad T_{A,B}(t) = e^{it}\big(A_+ + e^{-2it} A_-\big).$$

Thus a real pencil is controlled by the complex pencil $A_+ + \tau A_-$ along the unit circle. If μ denotes the braid monodromy of this complex pencil along $|\tau| = 1$, then, up to conjugacy,

$$\beta_{A,B} = \Delta_n^2 \mu^{-2}.$$

Moreover μ is quasipositive of band length k, where k is the number of zeros of the spectral discriminant in the unit disk, counted with multiplicity. When A_- is regular semisimple, μ has a complementary quasipositive factor of length $n(n-1) - k$ in a factorization of the full twist. Consequently

$$\mathbf{e}(\beta_{A,B}) = n(n-1) - 2k,$$

and every even integer in $[-n(n-1), n(n-1)]$ occurs. In particular the avoiding-level-crossing locus has at least $n^2 - n + 1$ connected components; it is also open semialgebraic, hence has only finitely many components and only finitely many spectral braid classes for fixed n.

For $n = 3$ we recover explicit realizations of the nine previously known classes and prove that four further classes occur. One exact example has $\mu \sim \sigma_1^3$ and $\beta \sim \Delta^2\sigma_1^{-6}$; a Routh count and an exact resultant certificate identify its monodromy without numerical continuation. A second, completely explicit block pencil has $\mu \sim \sigma_1^4$ and $\beta = \Delta^2\sigma_1^{-8}$. Together with orientation reversal these give at least thirteen conjugacy classes, and hence at least thirteen connected components, for $n = 3$. We also give a genus bound for the part of a smooth spectral curve lying over the disk and formulate a geometric prefix conjecture which, in degree three, predicts that the thirteen classes are exhaustive. Finally, for a generic configuration of $N \geq 3$ points on $\mathbb{CP}^1$, we correct the topology of the induced wall-and-chamber stratification of de Sitter 3-space, compute its complete face vector, and refine the vertex and edge counts by the numbers of points on the two sides of the corresponding circle.

1. Introduction

1.1. The problem.

1.1. The problem. Let $\mathrm{Mat}_n = \mathrm{Mat}_n(\mathbb{C})$ and let $(A, B) \in \mathrm{Mat}_n \times \mathrm{Mat}_n$. The associated *real pencil* is the loop

$$T_{A,B}(t) = \cos t\, A + \sin t\, B, \qquad t \in \mathbb{R}.$$

It is 2π-periodic and satisfies

$$\operatorname{Spec} T_{A,B}(t + \pi) = -\operatorname{Spec} T_{A,B}(t). \tag{1}$$

We say that (A, B) is *avoiding level crossing*, and write $(A, B) \in \mathcal{ALC}_n$, if $T_{A,B}(t)$ has simple spectrum for every real t. For such a pair, $t \mapsto \operatorname{Spec} T_{A,B}(t)$ is a loop in the unordered configuration space of n distinct points of $\mathbb{C}$ and therefore determines a conjugacy class

$$\beta_{A,B} \in \mathcal{B}_n/\sim,$$

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called the *spectral braid*. Conjugacy, rather than a based braid, is the natural invariant because the loop has no distinguished labelling or basepoint [3, Ch. 1].

Proposition 1.1. *The set $\mathcal{ALC}_n \subset \mathrm{Mat}_n(\mathbb{C})^2$ is an open semialgebraic set. In particular it has finitely many connected components, and only finitely many spectral braid conjugacy classes occur for fixed n.*

Proof. Write $(x, y) = (\cos t, \sin t)$. The discriminant of $\det(\zeta I - xA - yB)$ is a polynomial in x, y and in the real and imaginary parts of the entries of A, B. The complement of $\mathcal{ALC}_n$ is the projection to (A, B) of the real algebraic set cut out by $x^2 + y^2 = 1$ and by the real and imaginary parts of that discriminant. It is therefore semialgebraic by the Tarski–Seidenberg theorem [1]. Since the parameter circle is compact, nonvanishing of the discriminant on it is stable under small perturbations, so $\mathcal{ALC}_n$ is open. A semialgebraic set has finitely many connected components [1]. Finally the spectral braid is locally constant on $\mathcal{ALC}_n$. □

Problem 1.2. Which conjugacy classes in $\mathcal{B}_n$ arise as spectral braids, and how are they distributed among the connected components of $\mathcal{ALC}_n$?

The matrix-pencil version of this problem was posed in the broader Hurwitz-space framework of Libgober–Shapiro [7]. The spectral braid is locally constant on $\mathcal{ALC}_n$, so it is an invariant of connected components; it is not known whether it separates them.

1.2. **Main results.** The first observation is elementary but decisive.

Theorem 1.3 (Linearization)**.** *For $(A, B) \in \mathrm{Mat}_n^2$ set*

$$A_+ = \tfrac{1}{2}(A - iB), \qquad A_- = \tfrac{1}{2}(A + iB).$$

Then $A = A_+ + A_-$, $B = i(A_+ - A_-)$, and

$$T_{A,B}(t) = e^{it}A_+ + e^{-it}A_- = e^{it}\big(A_+ + e^{-2it}A_-\big). \tag{2}$$

Hence

$$\operatorname{Spec} T_{A,B}(t) = e^{it}\operatorname{Spec}\big(A_+ + \tau A_-\big), \qquad \tau = e^{-2it}. \tag{3}$$

The map $(A, B) \mapsto (A_+, A_-)$ is a complex-linear change of coordinates on Mat_n^2, and $(A, B) \in \mathcal{ALC}_n$ if and only if $A_+ + \tau A_-$ has simple spectrum for every $|\tau| = 1$.

Let

$$F(\zeta, \tau) = \det(\zeta I - A_+ - \tau A_-), \qquad D(\tau) = \operatorname{Disc}_\zeta F(\zeta, \tau),$$

and let $k = k(A, B)$ be the number of zeros of D in the open unit disk, counted with multiplicity. Let $\mu = \mu_{A,B}$ be the braid traced by $\operatorname{Spec}(A_+ + \tau A_-)$ as τ runs once counterclockwise around the unit circle. Recall that a *band* is a conjugate of a standard generator σ_i and that a braid is quasipositive of band length k if it is a product of k bands.

Theorem 1.4 (Factorization)**.** *Let $(A, B) \in \mathcal{ALC}_n$. Then, up to conjugacy,*

$$\boxed{\beta_{A,B} = \Delta_n^2 \mu_{A,B}^{-2}},$$

where Δ_n^2 is the full twist. The braid $\mu_{A,B}$ is quasipositive of band length k. If A_- is regular semisimple, then $\deg D = n(n-1)$ and one may choose a representative μ' of the conjugacy class of $\mu_{A,B}$ and a quasipositive braid ν of band length $n(n-1) - k$ such that

$$\mu'\nu = \Delta_n^2.$$

Thus μ is a prefix, up to Hurwitz choices and conjugacy, of the braid monodromy factorization of the full twist associated with the spectral curve. Quasipositivity is the key restriction.

Let $\mathbf{e} : \mathcal{B}_n \to \mathbb{Z}$ denote the exponent-sum homomorphism.

Theorem 1.5. *For $(A,B) \in \mathcal{ALC}_n$,*

$$\mathbf{e}(\beta_{A,B}) = n(n-1) - 2k.$$

The image of $\mathbf{e}\circ\beta$ on $\mathcal{ALC}_n$ is exactly the set of even integers E with $|E| \le n(n-1)$. Consequently $\mathcal{ALC}_n$ has at least $n^2 - n + 1$ connected components.

For $n = 3$, the factorization organizes the nine previously known classes by the integer $k \in \{0, \ldots, 6\}$. Those nine classes are all realized by explicit cyclic, $2+1$, or diagonal pencils. They do not exhaust the possibilities.

Theorem 1.6. *Let*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = i \begin{pmatrix} 0 & 3 & 0 \\ 0 & 3 & 1 \\ 2 & 0 & -1 \end{pmatrix}.$$

Then $(A,B) \in \mathcal{ALC}_3$ and

$$\beta_{A,B} \sim \Delta^2 \sigma_1^{-6}.$$

This braid is pure, has exponent sum 0, and has pairwise linking numbers $(-2,1,1)$ up to relabelling. Replacing B by $-B$ realizes the inverse class $\Delta^{-2}\sigma_1^{6}$, whose linking numbers are $(2,-1,-1)$. The two classes are not conjugate to one another or to any of the nine previously known classes. Hence at least eleven conjugacy classes occur for $n = 3$.

The proof of Theorem 1.6 is exact. Its two algebraic certificates are a Routh count for the sextic discriminant and a resultant calculation showing that, over the closed unit disk in the τ-plane, two eigenvalues remain in the half-plane $\Re\zeta < 1/2$ and the third remains in $\Re\zeta > 1/2$. The three interior branch points therefore belong to the same two-sheeted part of the spectral covering, forcing $\mu \sim \sigma_1^3$.

There is a second new phenomenon at $k = 4$, for which the computation is closed form.

Theorem 1.7. *Let*

$$A = \begin{pmatrix} 1-i & 1 & 0 \\ -4 & -1-i & 0 \\ 0 & 0 & i \end{pmatrix}, \qquad B = \begin{pmatrix} 1-i & i & 0 \\ -4i & 1+i & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then $(A,B) \in \mathcal{ALC}_3$, the spectral discriminant is $D(\tau) = 4\tau^4(\tau^2 - 4)$, and the interior monodromy is $\mu \sim \sigma_1^4$. Consequently

$$\beta_{A,B} = \Delta^2 \sigma_1^{-8},$$

with pairwise linking numbers $(-3,1,1)$ up to relabelling. Replacing B by $-B$ realizes the non-conjugate inverse class $\Delta^{-2}\sigma_1^{8}$.

Corollary 1.8. *At least thirteen distinct conjugacy classes occur as spectral braids for $n = 3$. In particular $\mathcal{ALC}_3$ has at least thirteen connected components.*

For smooth spectral curves there is also a useful topological restriction. If S is the part of the curve over the closed unit disk, m is the number of connected components of S, and c is the number of cycles of the boundary permutation, then

$$\frac{2m - c - n + k}{2} \le \frac{(n-1)(n-2)}{2}.$$

The precise statement is Proposition 4.2. It constrains possible factorizations of the interior monodromy but, by itself, does not classify them.

Motivated by Theorem 1.4 and by computations in $\mathcal{B}_3$, we conjecture that the realizable monodromies are exactly the prefixes that can be cut out by circles from geometric braid-monodromy factorizations of spectral curves; see Conjecture 4.6. In degree three this leads to the more concrete Conjecture 4.5 that the thirteen classes above are all.

The last part of the paper concerns the fibrewise geometry. For a fixed complex two-plane $L \subset \mathrm{Mat}_n$, a nondegenerate real two-plane $P \subset L$ with $P \otimes_{\mathbb{R}} \mathbb{C} = L$ projectivizes to a generalized circle in $\mathbb{P}(L) \simeq \mathbb{CP}^1$. The ALC condition says that this circle avoids the discriminant configuration. Oriented generalized circles form de Sitter 3-space, and points of the discriminant give walls. For a generic configuration $\mathcal{A}$ of $N \geq 3$ points we prove the following correction and enumeration.

Theorem 1.9. *Let $|\mathcal{A}| = N \geq 3$ and suppose no four points of $\mathcal{A}$ are concyclic. Every stratum of $(dS_3, \mathcal{W}_{\mathcal{A}})$ is contractible except for*

- *the two extreme chambers, each homotopy equivalent to $\mathbb{CP}^1 \setminus \mathcal{A} \simeq \bigvee_{N-1} S^1$;*
- *exactly $2N$ two-dimensional strata, namely those containing one point of $\mathcal{A}$ on the circle and all remaining points on the same side, each of which is homotopy equivalent to S^1.*

Theorem 1.10. *Under the same assumptions, if f_j denotes the number of j-dimensional strata, then*

$$f_0 = \tfrac{1}{3}N(N-1)(N-2), \qquad f_1 = N(N-1)(N-2),$$

$$f_2 = N(N^2-3N+4), \qquad f_3 = \tfrac{1}{3}(N^3-3N^2+8N) = 2\sum_{j=0}^{3}\binom{N-1}{j}.$$

We also refine f_0 and f_1 by the numbers of points on the two sides of the circle; see Theorem 6.1.

1.3. Related work.

Quasipositivity of braid monodromies of algebraic functions goes back to Rudolph [10]; the Hurwitz-factorization viewpoint is standard in braid monodromy, see for example Moishezon–Teicher [8]. The case of 3-braids and their Nielsen–Thurston types is classical [9]. Real matrix pencils and the associated braid questions were discussed in [7]. The de Sitter model for oriented circles is classical [4, 6], and the chamber count is a special case of Cover's theorem [5].

2. The linearization and the dictionary

2.1. Proof of Theorem 1.3.

With $A_\pm = \frac{1}{2}(A \mp iB)$,

$$e^{it}A_+ + e^{-it}A_- = \tfrac{e^{it}+e^{-it}}{2}A + i\,\tfrac{e^{-it}-e^{it}}{2}B = \cos t\,A + \sin t\,B,$$

which is (2); factoring out e^{it} gives the second form, and (3) follows because $\mathrm{Spec}(zM) = z\,\mathrm{Spec}\,M$. The inverse of $(A,B) \mapsto (A_+, A_-)$ is $(A_+, A_-) \mapsto (A_+ + A_-,\ i(A_+ - A_-))$, so the map is a linear automorphism. Finally, as t runs over $[0,\pi)$ the parameter $\tau = e^{-2it}$ runs once over the unit circle, and by (3) the spectrum of $T_{A,B}(t)$ is a nonzero multiple of that of $A_+ + \tau A_-$; so simplicity for all t is equivalent to simplicity for all $|\tau| = 1$. □

Remark 2.1. The relation (1) is now transparent: $t \mapsto t + \pi$ fixes $\tau = e^{-2it}$ and multiplies the rotation factor e^{it} by -1. In particular the spectral braid is the square of the "half-loop" braid only in the weak sense that the τ-circle is traversed twice; see Theorem 1.4.

2.2. **The spectral curve.** Homogenizing gives

$$\det(zI - uA_+ - vA_-),$$

a ternary form of degree n with z^n-coefficient 1. Thus the affine curve $F(\zeta, \tau) = 0$ is the part, in the chart $u = 1$, of a degree-n plane spectral curve, and the map $(\zeta, \tau) \mapsto \tau$ is projection from the point $[1 : 0 : 0]$. This places the construction in the classical theory of linear determinantal representations of plane curves; see Beauville [2] for the smooth case and related existence questions. No converse assertion about arbitrary singular or nonreduced plane curves is needed below.

If A_- is regular semisimple, with distinct eigenvalues $a_1, \ldots, a_n$, then for $|\tau| \to \infty$ the eigenvalues of $A_+ + \tau A_-$ are $\tau a_j + O(1)$. Hence

$$\mathrm{Disc}_\zeta \det(\zeta I - A_+ - \tau A_-) = \tau^{n(n-1)} \prod_{i<j} (a_i - a_j)^2 + O(\tau^{n(n-1)-1}),$$

so $\deg D = n(n - 1)$ exactly.

2.3. **Changing the circle in the base.** The linearization is most naturally homogeneous. An ordered complex pencil is $uA_+ + vA_-$ with base $[u : v] \in \mathbb{CP}^1$, and the real pencil corresponds to the standard oriented circle $|v/u| = 1$. A projective change of base coordinates replaces (A_+, A_-) by another ordered pair and carries the standard circle to an arbitrary oriented generalized circle in $\mathbb{CP}^1$. Thus every oriented circle avoiding the branch locus can be used as the real locus of a suitable choice of coordinates on the same homogeneous pencil.

For an ordinary Euclidean circle $C = \{\tau : c_0 + r\tau',\ |\tau'| = 1\}$ this is already visible from

$$A_+ + \tau A_- = (A_+ + c_0 A_-) + \tau'(rA_-).$$

The following is the form of the dictionary used later.

Corollary 2.2. *The map $(A, B) \mapsto (A_+, A_-)$ identifies ordered real pencils with ordered complex matrix pencils equipped with the standard oriented circle in the base. After a projective change of base coordinates, any oriented generalized circle may replace the standard one. If a circle is deformed through oriented circles that avoid the branch locus, the resulting spectral braid remains in the same conjugacy class. Reversing the orientation corresponds to $(A, B) \mapsto (A, -B)$ and sends β to β^{-1}.*

For a fixed complex two-dimensional subspace $L \subset \mathrm{Mat}_n$, this yields the fibrewise picture used in Sections 5–6: projectivized real forms $P \subset L$ with $P \otimes_{\mathbb{R}} \mathbb{C} = L$ give generalized circles in $\mathbb{P}(L)$, and the ALC condition is exactly avoidance of the discriminant configuration. Degenerate real spans, which do not complexify to L, are not part of this fibrewise parametrization.

3. The spectral braid

Throughout this section $(A, B) \in \mathcal{ALC}_n$, $G(\tau) = \mathrm{Spec}(A_+ + \tau A_-)$ for $|\tau| = 1$, and $\mu \in \mathcal{B}_n$ is the class of the loop $\theta \mapsto G(e^{i\theta})$, $\theta \in [0, 2\pi]$, based at $G(1) = \mathrm{Spec}\, A$.

3.1. **The factorization.**

Proof of the first part of Theorem 1.4. Define

$$\Phi : S^1 \times S^1 \longrightarrow \mathrm{Conf}_n(\mathbb{C})/S_n, \qquad \Phi(e^{is}, e^{i\theta}) = e^{is} \cdot G(e^{i\theta}).$$

This is well defined and continuous: $G(e^{i\theta})$ consists of n distinct points because $(A, B) \in \mathcal{ALC}_n$, and multiplication by e^{is} preserves distinctness. Take $(1, 1)$ as basepoint, so $\Phi(1, 1) = \mathrm{Spec}\, A$. Since $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ is abelian and Φ_* is a homomorphism,

$$\Phi_*(p, q) = \Phi_*(1, 0)^p \cdot \Phi_*(0, 1)^q \qquad \text{for all } (p, q) \in \mathbb{Z}^2.$$

Now $\Phi_*(0,1)=\mu$ by definition, while $\Phi_*(1,0)$ is the class of the loop $s\mapsto e^{is}\,\mathrm{Spec}\,A$, that is, of a rigid rotation of a configuration by the angle 2π; this is the classical description of the full twist Δ_n^2, the generator of the centre of $\mathcal{B}_n$ [3, §4.2]. By (3) the spectral loop $t\mapsto \mathrm{Spec}\,T_{A,B}(t)$, $t\in[0,2\pi]$, equals $t\mapsto\Phi(e^{it},e^{-2it})$, whose class in $\pi_1(S^1\times S^1)$ is $(1,-2)$. Hence

$$\beta_{A,B}=\Phi_*(1,-2)=\Delta_n^2\,\mu^{-2},$$

as claimed; the identity holds on the nose for the given basepoint and therefore up to conjugacy in general. ☐

Note that Δ_n^2 is central, so no ordering ambiguity arises.

3.2. Quasipositivity and the band count.

Lemma 3.1. *Let $f(\zeta,\tau)$ be monic of degree n in ζ with coefficients holomorphic on a neighbourhood of the closed unit disc $\bar{\mathbb{D}}$, and suppose $\mathrm{Disc}_\zeta f\neq 0$ on $|\tau|=1$. Then the braid traced by the roots of $f(\cdot,\tau)$ as τ traverses $|\tau|=1$ counterclockwise is quasipositive of band length equal to the number k of zeros of $\mathrm{Disc}_\zeta f$ in $\mathbb{D}$, counted with multiplicity.*

Proof. This is Rudolph's theorem on braid monodromy of algebraic functions [10]; we recall the argument. First assume all zeros of $\mathrm{Disc}_\zeta f$ in $\mathbb{D}$ are simple. Choose disjoint discs $U_1,\dots,U_k\subset\mathbb{D}$ around them and a system of arcs from the basepoint, so that the boundary braid is the ordered product of the local monodromies. Near a simple zero τ_j exactly two roots collide, and after an analytic change of coordinates they are $\pm\sqrt{\tau-\tau_j}\,(1+o(1))$; a small counterclockwise loop therefore exchanges them by a *positive* half-twist along an embedded arc, i.e. the local monodromy is a band. Hence the boundary braid is a product of k bands. In general, perturb f to f_ε so that all zeros of the discriminant in $\mathbb{D}$ become simple. For small ε the discriminant has no zeros on $|\tau|=1$, so by Rouché the number of zeros in $\mathbb{D}$ counted with multiplicity is unchanged, and the boundary braid is unchanged up to isotopy. ☐

Applying Lemma 3.1 to $f=F$ gives the quasipositivity assertion of Theorem 1.4, with k the number of branch points in the unit disc.

Proof of the last part of Theorem 1.4. Assume A_- regular semisimple; by §2.2, $\deg D=n(n-1)$, so D has exactly $n(n-1)$ zeros in $\mathbb{C}$ with multiplicity. Choose R so large that all of them lie in $|\tau|<R$. For $|\tau|=R$ the eigenvalues are $\tau a_j+O(1)$ with the a_j distinct, so after a homotopy the boundary braid of the circle $|\tau|=R$ is that of the rigid rotation $\tau\mapsto\tau\{a_j\}$, namely Δ_n^2. Applying Lemma 3.1 to $|\tau|=R$ and choosing the system of arcs so that the branch points inside $\mathbb{D}$ come first, we get $\Delta_n^2=\mu'\nu$ with μ' conjugate to μ, μ' of band length k and ν of band length $n(n-1)-k$, both quasipositive. ☐

3.3. The abelianization.

Proof of Theorem 1.5. Each band has exponent sum 1, while $\mathbf{e}(\Delta_n^2)=n(n-1)$. Theorem 1.4 therefore gives

$$\mathbf{e}(\beta_{A,B})=n(n-1)-2k.$$

Since $\deg D\le n(n-1)$, every spectral braid has even exponent sum in the stated interval.

It remains to realize every k. Choose a generic pair with A_- regular semisimple and with $N=n(n-1)$ simple branch points $\tau_1,\dots,\tau_N$. Replacing A_+ by A_++cA_- translates the branch points by $-c$. For generic $c\in\mathbb{C}$ the finitely many distances $|\tau_j-c|$ are pairwise distinct. After making such a translation, replace A_- by rA_- with $r>0$. This divides all branch points by r. As r increases from 0 to ∞, they cross the unit circle one at a time; away from those finitely many values the pair lies in $\mathcal{ALC}_n$. Hence every $k=0,1,\dots,N$ occurs.

For the positive extreme one may also take $A = S$, $B = iS$, where S is the cyclic shift. Then $A_+ = S$, $A_- = 0$, $\mu = 1$ and $\beta = \Delta_n^2$; replacing B by $-iS$ gives Δ_n^{-2}. Finally $\mathbf{e} \circ \beta$ is locally constant on $\mathcal{ALC}_n$ and assumes $N+1$ distinct values, so $\mathcal{ALC}_n$ has at least $N+1 = n^2 - n + 1$ connected components. $\square$

3.4. **Small cases and normal forms.**

Corollary 3.2 ($n = 2$)**.** *For $n = 2$ the spectral braid is one of σ_1^2, 1, σ_1^{-2}, and each occurs.*

Proof. Here $n(n-1) = 2$ and $\mathcal{B}_2 = \langle \sigma_1 \rangle \cong \mathbb{Z}$, so a quasipositive braid of band length k is σ_1^k; thus $\beta = \Delta^2 \mu^{-2} = \sigma_1^{2-2k}$ with $k \in \{0, 1, 2\}$. $\square$

Example 3.3 (Cyclic pencils)**.** Let $C_{d,s}$ be the weighted cyclic shift whose characteristic polynomial is $z^d - \ell_+^s \ell_-^{d-s}$, where $\ell_\pm = e^{\pm it}$. In the linearized picture $F(\zeta, \tau) = \zeta^d - \tau^{d-s}$, so $G(\tau)$ consists of the d-th roots of τ^{d-s}; as τ goes once around, this configuration rotates by $2\pi(d-s)/d$, whence $\mu = \delta_d^{d-s}$ where $\delta_d = \sigma_1 \cdots \sigma_{d-1}$. Since $\delta_d^d = \Delta_d^2$,

$$\beta = \Delta_d^2 \delta_d^{-2(d-s)} = \delta_d^d \delta_d^{-2(d-s)} = \delta_d^{2s-d},$$

recovering the known normal form. Indeed $D(\tau) = c_d\, \tau^{(d-s)(d-1)}$ for a nonzero constant c_d, so $k = (d-s)(d-1)$, in agreement with $\mathbf{e}(\beta) = d(d-1) - 2k = (2s-d)(d-1)$.

3.5. **Separated blocks and linking numbers.** The factorization of Theorem 1.4 also computes linking numbers, which is what we shall use to certify the counterexample in §4. Recall that for a pure braid β on n strands the pairwise linking numbers $\mathrm{lk}_\beta(i,j)$ are defined, and that $\mathrm{lk}_{\Delta_n^2}(i,j) = 1$ for all $i \neq j$.

Lemma 3.4. *Suppose that on a neighbourhood of $\bar{\mathbb{D}}$ the polynomial F factors as $F = F_1 F_2$, with F_i monic in ζ of degrees n_i and holomorphic in τ. Then μ preserves the corresponding partition of the strands. If Z is the number of zeros in $\mathbb{D}$, counted with multiplicity, of $\mathrm{Res}_\zeta(F_1, F_2)$, then the* total *linking number in $\beta_{A,B}$ between strands belonging to different groups is*

$$n_1 n_2 - 2Z.$$

In particular, if the resultant is nowhere zero on $\bar{\mathbb{D}}$, the total intergroup linking is $n_1 n_2$.

Proof. Along $|\tau| = 1$ the winding number of

$$\mathrm{Res}_\zeta(F_1, F_2)(\tau) = \prod_{a,b} (\lambda_a(\tau) - \lambda_b(\tau))$$

is Z by the argument principle, where the product ranges over roots of F_1 and F_2. Therefore the total intergroup linking in μ^2 is $2Z$. The full twist contributes 1 to every cross-group pair, hence $n_1 n_2$ in total, and $\beta = \Delta^2 \mu^{-2}$ gives the formula. $\square$

Corollary 3.5. *If $T = T_1 \oplus T_2$ is a direct sum of real pencils of sizes n_1, n_2 whose spectra remain disjoint, then $\mathbf{e}(\beta_T) = \mathbf{e}(\beta_{T_1}) + \mathbf{e}(\beta_{T_2}) + 2n_1 n_2 - 4Z$ with Z as above.*

4. The case $n = 3$

4.1. **The nine earlier classes and explicit realizations.** For $n = 3$, write

$$\Delta = \sigma_1 \sigma_2 \sigma_1, \qquad \delta = \sigma_1 \sigma_2, \qquad \delta^3 = \Delta^2.$$

The nine previously known classes can be organized by the number k of branch points in the unit disk as follows.

k	μ	$\beta = \Delta^2\mu^{-2}$
0	1	Δ^2
1	σ_1	$\Delta^2\sigma_1^{-2}$
2	σ_1^2 or δ	$\Delta^2\sigma_1^{-4}$ or δ
3	Δ	1
4	δ^2 or $\Delta^2\sigma_1^{-2}$	δ^{-1} or $\Delta^{-2}\sigma_1^4$
5	$\Delta^2\sigma_1^{-1}$	$\Delta^{-2}\sigma_1^2$
6	Δ^2	Δ^{-2}

The exponents in the middle column are important: every displayed μ has exponent sum exactly k.

For completeness we record explicit realizations, which were present in the earlier version of the project. Put $\ell_\pm = e^{\pm it}$. The connected three-sheeted cyclic pencils

$$z^3 - \ell_+^s \ell_-^{3-s}, \qquad s = 1, 2,$$

realize δ^{-1} and δ. The three $2+1$ classes are realized by

$$(z - R\ell_+)(z^2 - \ell_+\ell_-), \qquad z(z^2 - \ell_+\ell_-), \qquad (z - R\ell_-)(z^2 - \ell_+\ell_-), \qquad R > 1.$$

The middle case has the matrix realization

$$T_0(\alpha, \beta) = \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{\beta}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \det(zI - T_0) = z(z^2 - \alpha^2 - \beta^2).$$

Finally, diagonal pencils with eigenvalues $m_j = a_j\ell_+ + b_j\ell_-$ realize the four pure classes. One may take

$\#\{$positive pairwise windings$\}$	$((a_1, b_1), (a_2, b_2), (a_3, b_3))$
0	$((0,0), (0,1), (0,2))$
1	$((0,0), (2,0), (0,3))$
2	$((0,0), (3,0), (0,2))$
3	$((0,0), (1,0), (2,0))$.

Their exponent sums are respectively $-6, -2, 2, 6$. Together these families realize all nine entries in the table.

Proposition 4.1. *Assume A_- is regular semisimple. If $k \in \{0, 1, n(n-1)-1, n(n-1)\}$, then $\beta_{A,B}$ is determined by k up to conjugacy: it is respectively*

$$\Delta_n^2, \qquad \Delta_n^2\sigma_1^{-2}, \qquad \Delta_n^{-2}\sigma_1^2, \qquad \Delta_n^{-2}.$$

Proof. For $k = 0$, quasipositivity of band length 0 gives $\mu = 1$. For $k = 1$, μ is a single band, and every band is conjugate to σ_1; centrality of Δ_n^2 gives the second class. Since A_- is regular semisimple, Theorem 1.4 supplies the complementary quasipositive factor. If $k = n(n-1)$ that factor has length 0, so $\mu = \Delta_n^2$; if $k = n(n-1) - 1$ it has length 1. The two resulting classes are the inverses of the first two, as displayed. □

4.2. **A genus bound over the disk.** The prefix condition of Theorem 1.4 has a topological refinement when the spectral curve is smooth.

Proposition 4.2. *Assume that the projective spectral curve $\bar{\Gamma}$ is smooth. Let $\pi : \bar{\Gamma} \to \mathbb{CP}^1$ be the projection to the pencil parameter and put $S = \pi^{-1}(\bar{\mathbb{D}})$. Let m be the number of connected* *components of S, let c be the number of cycles of the permutation underlying μ, and let G be the sum of the genera of the components of S. Then*

$$\chi(S) = n - k, \qquad G = \frac{2m - c - n + k}{2} \leq g(\bar{\Gamma}) = \frac{(n-1)(n-2)}{2}.$$

Proof. The restriction $\pi : S \to \bar{\mathbb{D}}$ is an n-sheeted branched cover. For a smooth spectral curve the order of the discriminant is the total ramification multiplicity, hence Riemann–Hurwitz gives $\chi(S) = n-k$. The boundary ∂S is the covering of the unit circle determined by the permutation of μ, so it has c components. Since $\chi(S) = 2m - 2G - c$, the displayed formula for G follows. Finally the components of S are disjoint subsurfaces of the smooth plane curve $\bar{\Gamma}$, whose genus is $(n-1)(n-2)/2$. □

Corollary 4.3. *Assume, in addition, that $k > 0$, all branch points in the disk are simple, and that their local monodromies exchange the same pair of sheets. Then*

$$k \leq (n-1)(n-2) + 2 \quad \text{if } k \text{ is even}, \qquad k \leq (n-1)(n-2) + 1 \quad \text{if } k \text{ is odd.}$$

For $n = 3$ this gives $k \leq 4$ in the even case and $k \leq 3$ in the odd case.

Proof. The monodromy group over the disk has one orbit of size two and $n-2$ singleton orbits, so $m = n-1$. The boundary permutation is the identity for k even and a transposition for k odd, hence $c = n$ or $n-1$, respectively. Substitution in Proposition 4.2 gives $G = (k-2)/2$ in the even case and $G = (k-1)/2$ in the odd case. □

The hypothesis on the local monodromies is essential: the conjugacy class of the product μ alone does not determine the number m of components of the covering over the disk. Thus Corollary 4.3 is an obstruction to a particular type of factorization, not a classification of all braids with the same boundary conjugacy class.

4.3. The first new inverse pair.

Proof of Theorem 1.6. For the displayed matrices one finds

$$A_+ = \tfrac{1}{2}\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 0 & -1 \end{pmatrix}, \qquad A_- = \tfrac{1}{2}\begin{pmatrix} 1 & -3 & 1 \\ 2 & -4 & -1 \\ -2 & 0 & 1 \end{pmatrix},$$

and

$$\begin{aligned} F(\zeta, \tau) = \zeta^3 + (\tau - 1)\zeta^2 + \Big(\frac{\tau^2}{4} + \tau - \frac{9}{4}\Big)\zeta \\ + \frac{3\tau^3}{2} - \frac{9\tau^2}{4} + \frac{3\tau}{2} - \frac{3}{4}. \end{aligned}$$

Its discriminant is

$$D(\tau) = -\frac{1}{16}Q(\tau), \tag{4}$$

where

$$\begin{aligned} Q(\tau) = 960\tau^6 - 3504\tau^5 + 6767\tau^4 - 8192\tau^3 \\ + 5618\tau^2 - 1680\tau - 33. \end{aligned} \tag{5}$$

We first count its roots relative to the unit circle exactly. Under the Cayley transform $\tau = (1+s)/(1-s)$,

$$(1-s)^6 Q\Big(\frac{1+s}{1-s}\Big) = 16R(s),$$

with

$$R(s) = 1668s^6 + 972s^5 + 179s^4 + 954s^3 + 11s^2 + 60s - 4.$$

The first column of the Routh table of R is

$$1668,\ 972,\ -\frac{13123}{9},\ \frac{11714850}{13123},\ \frac{156611}{92975},\ \frac{48710388}{22373},\ -4.$$

It has three sign changes and no zero entry or zero row. Hence R has exactly three roots in the right half-plane and three in the left half-plane, with none on the imaginary axis. Since the Cayley transform sends the open unit disk to the left half-plane, Q has exactly three roots in $|\tau| < 1$ and none on $|\tau| = 1$. Thus $(A, B) \in \mathcal{ALC}_3$ and $k = 3$.

It remains to identify the monodromy, and for this we prove an exact separation of the three eigenvalues over $\bar{\mathbb{D}}$. Put $\zeta = \frac{1}{2} + iy$ and parametrize the unit circle, except for $\tau = -1$, by

$$\tau = \frac{1+ix}{1-ix}, \qquad x, y \in \mathbb{R}.$$

Write

$$(1-ix)^3 F\left(\frac{1}{2} + iy, \frac{1+ix}{1-ix}\right) = U(x, y) + iV(x, y),$$

with $U, V \in \mathbb{Q}[x, y]$. Direct elimination gives

$$\operatorname{Res}_x(U, V) = -\frac{4y^2+1}{256}\, P(y^2), \tag{6}$$

where

$$P(Y) = 4096Y^8 + 59392Y^7 + 385792Y^6 + 1298432Y^5 + 2425456Y^4 + 2620936Y^3 + 1547701Y^2 + 275457Y + 37044.$$

All coefficients of P are positive, so the resultant never vanishes for real y. At the omitted point $\tau = -1$,

$$\Re F\left(\frac{1}{2} + iy, -1\right) = \frac{y^2}{2} - \frac{63}{8}, \qquad \Im F\left(\frac{1}{2} + iy, -1\right) = -y^3 - \frac{17}{4}y,$$

which also cannot vanish simultaneously. Hence

$$F(\zeta, \tau) \neq 0 \qquad (\Re\zeta = \tfrac{1}{2},\ |\tau| = 1). \tag{7}$$

For fixed y, the number of zeros in $|\tau| < 1$ of $\tau \mapsto F(\frac{1}{2} + iy, \tau)$ can therefore not change as y varies. For $|\frac{1}{2} + iy| \geq 4$ and $|\tau| = 1$,

$$\left|F(\zeta, \tau) - \zeta^3\right| \leq 2|\zeta|^2 + \frac{7}{2}|\zeta| + 6 < |\zeta|^3,$$

so Rouché's theorem shows that this polynomial in τ has no zero in the unit disk. Consequently it has none for any real y. Thus

$$F(\zeta, \tau) \neq 0 \qquad (\Re\zeta = \tfrac{1}{2},\ |\tau| \leq 1). \tag{8}$$

At $\tau = 0$,

$$F(\zeta, 0) = \frac{1}{4}(2\zeta + 1)(2\zeta^2 - 3\zeta - 3),$$

whose roots are $-\frac{1}{2}$, $(3 - \sqrt{33})/4$, and $(3 + \sqrt{33})/4$. Hence two roots lie in $\Re\zeta < 1/2$ and one lies in $\Re\zeta > 1/2$. By (8) the same $2 + 1$ separation holds throughout the closed unit disk.

All three interior branch points therefore involve the two left-hand sheets. Along the unit circle the monodromy lies in the natural $\mathcal{B}_2$ subgroup, and by quasipositivity and $k = 3$ it is σ_1^3 up to conjugacy. The factorization theorem gives

$$\beta_{A,B} \sim \Delta^2 \sigma_1^{-6}.$$

In μ^2 the left pair has linking number 3 and the other two pairwise linking numbers are 0; subtracting from the full twist gives the three linking numbers $(-2, 1, 1)$ for β.

Among the nine classes in §4.1, the only class with exponent sum 0 and trivial permutation is the identity, so the present class is new. Moreover $T_{A,-B}(t) = T_{A,B}(-t)$, hence $(A, -B)$ realizes β^{-1}, with linking numbers $(2, -1, -1)$. Conjugation of a pure braid can only permute the pairwise linking numbers; the multisets $\{-2, 1, 1\}$ and $\{2, -1, -1\}$ differ. Thus the inverse is a second new class and is not conjugate to β. □

Remark 4.4. The proof above removes the numerical gap in the earlier draft. Numerical eigenvalue tracking was useful for discovering the example, but neither the ALC property nor the braid identification now depends on floating-point continuation.

4.4. **A second new inverse pair in closed form.**

Proof of Theorem 1.7. From $A_\pm = \frac{1}{2}(A \mp iB)$ one obtains

$$A_+ = \begin{pmatrix} -i & 1 \\ -4 & -i \end{pmatrix} \oplus (i), \qquad A_- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus (0).$$

Hence

$$F(\zeta, \tau) = \big((\zeta + i)^2 - \tau^2 + 4\big)(\zeta - i).$$

The eigenvalues of $A_+ + \tau A_-$ are

$$\zeta_\pm(\tau) = -i \pm w(\tau), \qquad w(\tau) = \sqrt{\tau^2 - 4}, \qquad \zeta_3 = i.$$

The discriminant of the quadratic factor is $4(\tau^2 - 4)$, while its resultant with $\zeta - i$ is $-\tau^2$. Therefore

$$D(\tau) = 4\tau^4(\tau^2 - 4).$$

No zero lies on $|\tau| = 1$, so $(A, B) \in \mathcal{ALC}_3$, and $k = 4$.

On $|\tau| \le 1$ choose the analytic square-root branch with $w(0) = 2i$. The set $\{\tau^2 - 4 : |\tau| \le 1\}$ lies in the disk $|z + 4| \le 1$, on which $|\sqrt{z}| \ge \sqrt{3}$. Integrating the derivative of this branch along the segment from -4 to $\tau^2 - 4$ gives

$$|w(\tau) - 2i| \le \frac{1}{2\sqrt{3}} < 0.29.$$

Thus ζ_+ and the constant eigenvalue $\zeta_3 = i$ stay in the disk of radius 0.29 about i, while ζ_- stays in the disjoint disk of radius 0.29 about $-3i$. Consequently the boundary monodromy is supported on the pair (ζ_+, ζ_3).

Moreover

$$(w - 2i)(w + 2i) = w^2 + 4 = \tau^2,$$

and $w + 2i$ has no zero on $|\tau| \le 1$. Hence $w - 2i$ has a zero of order exactly two at $\tau = 0$ and no other zero in the disk. Its winding number on the unit circle is therefore 2, so the two-strand braid is $\mu \sim \sigma_1^4$. Theorem 1.4 gives

$$\beta_{A,B} = \Delta^2 \sigma_1^{-8}.$$

Since $\mu^2 = \sigma_1^8$ has linking number 4 on this pair and zero on the other two pairs, the linking numbers of β are $(-3, 1, 1)$ up to relabelling. The inverse pencil $(A, -B)$ has linking multiset $\{3, -1, -1\}$, so it is not conjugate to β. □

The curve in Theorem 1.7 is reducible, so Proposition 4.2 does not apply to it literally. Nevertheless the example reaches the value $k = 4$ singled out by Corollary 4.3 for a smooth cubic with a common-pair simple branching pattern. The important point for the present paper is independent of that comparison: the boundary braid is computed completely in closed form.

4.5. **Thirteen realized classes and two conjectures.** Combining the nine classes in §4.1 with Theorems 1.6 and 1.7 and their inverses gives

$$\Delta^2\sigma_1^{-2j} \quad (0 \le j \le 4), \qquad \Delta^{-2}\sigma_1^{2j} \quad (0 \le j \le 4), \qquad \delta, \quad \delta^{-1}, \quad 1.$$

These thirteen classes are pairwise distinct. Indeed the two new classes with exponent sum 0 are distinguished from the identity by their nonzero linking multisets $\{-2, 1, 1\}$ and $\{2, -1, -1\}$. At exponent sums -2 and 2, the new pure classes have linking multisets $\{-3, 1, 1\}$ and $\{3, -1, -1\}$; these differ from the pure classes already present in the table, whose corresponding multisets are $\{-1, 1, 1\}$ and $\{1, -1, -1\}$, while $\delta^{\pm 1}$ are non-pure. This proves Corollary 1.8.

Conjecture 4.5. *For $n = 3$, the thirteen classes displayed above are all spectral braid conjugacy classes.*

To formulate a broader conjecture, call a factorization

$$\Delta_n^2 = b_1 \cdots b_{n(n-1)}$$

geometric if it is a braid-monodromy factorization of the projection of a smooth degree-n spectral curve $\det(\zeta I - uA_+ - vA_-) = 0$ with A_- regular semisimple and simple finite branching. Different choices of vanishing paths act by Hurwitz moves.

Conjecture 4.6 (Geometric-prefix completeness)**.** *A braid occurs as an interior spectral monodromy $\mu_{A,B}$ if and only if it is conjugate to a prefix $b_1 \cdots b_k$ of a geometric factorization, after allowing Hurwitz moves and varying the spectral curve. Equivalently, the spectral braid classes are precisely*

$$\Delta_n^2\mu^{-2}$$

with μ in this geometric prefix set.

Theorem 1.4 supplies the prefix condition for generic smooth pencils; the converse is the geometric content of the conjecture, because a Hurwitz prefix is cut out by a topological disk in the base, whereas a real pencil selects a generalized circle. Random exploration of six-band Hurwitz factorizations in $\mathcal{B}_3$ found the thirteen classes above and no others. This is evidence for Conjecture 4.5, not an exhaustive computation.

4.6. **What the algebra does and does not give.** The conditions in Theorem 1.4 do not by themselves classify the possible 3-braids. In particular, quasipositive prefixes of the required length can already be pseudo-Anosov.

Proposition 4.7. *In $\mathcal{B}_3$ there are products of two bands that are pseudo-Anosov. For example*

$$\mu_0 = \sigma_1\big(\sigma_2^4\sigma_1\sigma_2^{-4}\big)$$

has image $\begin{pmatrix} -11 & -2 \\ -16 & -3 \end{pmatrix}$ under the reduced Burau representation at $t = -1$, and this matrix has trace -14.

Proof. Use

$$\sigma_1 \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_2 \longmapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The displayed matrix is obtained by direct multiplication. Modulo the centre, $\mathcal{B}_3$ is $PSL_2(\mathbb{Z})$, and the hyperbolic elements, equivalently those with absolute trace greater than 2, are the pseudo-Anosov classes [9]. □

Since $\beta = \Delta^2\mu^{-2}$ and the full twist is central, β and μ have the same Nielsen–Thurston type except that periodic orders may change; in particular one is pseudo-Anosov if and only if the other is. This motivates the remaining geometric question.

Conjecture 4.8. *For $n = 3$, no spectral braid is pseudo-Anosov.*

Our exploratory computations sampled chambers for several random families of pencils and found no pseudo-Anosov example. The two classes of Theorem 1.7 are close to a reducible spectral curve and were not suggested reliably by such random sampling; they were found by a deliberate construction. Thus the experiments are useful for discovery but provide no upper bound on the set of possible classes.

5. The wall-and-chamber structure of de Sitter 3-space

Let $\mathcal{A} = \{a_1, \dots, a_N\} \subset S^2 \cong \mathbb{CP}^1$ be a finite configuration. By Corollary 2.2, the fibrewise ALC problem asks for the stratification of the space of oriented circles by the walls of circles passing through the points of $\mathcal{A}$.

5.1. Circles as null hyperplane sections.

An oriented circle in S^2 can be written

$$C_{u,c} = \{x \in S^2 : \langle u, x\rangle = c\}, \qquad u \in S^2, \quad c \in (-1, 1),$$

with orientation determined by u. Equivalently, use the de Sitter quadric

$$\Omega = \{(u, c) \in \mathbb{R}^{3,1} : |u|^2 - c^2 = 1\},$$

with Lorentz form $\langle (u, c), (v, d)\rangle_L = \langle u, v\rangle - cd$. Radial normalization identifies Ω with $S^2 \times (-1, 1)$.

Proposition 5.1. *For $a \in S^2$, the wall of oriented circles through a is*

$$\mathcal{W}_a = \Omega \cap H_a, \qquad H_a = \{(u, c) : \langle u, a\rangle - c = 0\}.$$

The Lorentz normal $(a, 1)$ is null and

$$H_a \cap \{(u, c) : |u| \le |c|\} = \mathbb{R}(a, 1).$$

Moreover four walls $H_{a_1}, \dots, H_{a_4}$ have a common nonzero point if and only if $a_1, \dots, a_4$ are concyclic. Thus, for distinct points, the induced central arrangement is in general position precisely when no four points are concyclic.

Proof. The first identity is immediate. Since $\langle (a, 1), (a, 1)\rangle_L = |a|^2 - 1 = 0$, the Lorentz normal is null. If $(u, c) \in H_a$ and $|u| \le |c|$, then

$$|c| = |\langle u, a\rangle| \le |u| \le |c|,$$

so equality holds throughout and $u = ca$; the converse is clear. Finally the four hyperplanes have a nonzero common solution exactly when

$$\det \begin{pmatrix} a_1 & -1 \\ a_2 & -1 \\ a_3 & -1 \\ a_4 & -1 \end{pmatrix} = 0,$$

which says that the four points lie in an affine plane, equivalently on a circle of S^2. □

A stratum is encoded by the sign vector

$$L^0 = \{i : \langle u, a_i\rangle = c\}, \qquad L^\pm = \{i : \langle u, a_i\rangle \gtrless c\}.$$

Under the no-four-concyclicity assumption, $|L^0| \le 3$ and the stratum has dimension $3 - |L^0|$.

5.2. **Homotopy types.**

Proof of Theorem 1.9. Use the normalization $|u| = 1$, $c \in (-1, 1)$.

If $L^0 = \emptyset$ and both L^+ and L^- are nonempty, the possible u form an intersection of S^2 with an open convex cone defined by the inequalities $\langle u, a_j - a_k\rangle < 0$ for $j \in L^-$, $k \in L^+$. This spherical cell is contractible, and the admissible values of c form a nonempty open interval over every u; the chamber is therefore contractible. If $L^+ = \emptyset$, the fibre is

$$\big(\max_j \langle u, a_j\rangle, 1\big),$$

which is nonempty exactly for $u \notin \mathcal{A}$. Hence the chamber is homeomorphic to $(S^2 \setminus \mathcal{A}) \times (0, 1)$. The case $L^- = \emptyset$ is symmetric. These are the two extreme chambers.

Now let $L^0 = \{i\}$. Then $c = \langle u, a_i\rangle$, and after removing the degenerate values $u = \pm a_i$ the stratum is an open convex spherical polygon. The point $u = a_i$ belongs to the corresponding polygon exactly when $L^+ = \emptyset$, and $u = -a_i$ belongs exactly when $L^- = \emptyset$. Thus the stratum is contractible when both $L^\pm$ are nonempty, while in either extreme case it is a disk-like open polygon with one interior point removed and hence has the homotopy type of S^1. There are N choices of i for each of the two extreme signs, giving exactly $2N$ such strata.

Finally suppose $|L^0| \geq 2$. A puncture would require $|\langle u, a_i\rangle| = 1$, hence $u = \pm a_i$ for every $i \in L^0$, impossible for distinct points. For $N \geq 3$, each one-dimensional stratum is an open arc and each zero-dimensional stratum is a point. Hence all these strata are contractible. $\square$

Remark 5.2. The restriction $N \geq 3$ is essential. For $N = 1$ the unique wall is $S^2 \setminus \{\pm a_1\}$, a cylinder. For $N = 2$ the intersection $\mathcal{W}_{a_1} \cap \mathcal{W}_{a_2}$ is an entire circle, so there is one additional noncontractible one-dimensional stratum. These small cases are easily treated separately.

5.3. **The face vector.**

Proof of Theorem 1.10. A vertex is an oriented circle through three prescribed points. Every triple determines one unoriented circle and hence two orientations, so

$$f_0 = 2\binom{N}{3} = \frac{1}{3}N(N-1)(N-2).$$

Fix a pair $\{a_i, a_j\}$. Oriented circles through the pair form a circle. Each of the other $N - 2$ points gives two distinct vertices on this circle, and no two such vertices coincide because no four points are concyclic. Thus the circle is divided into $2(N - 2)$ open edges. Multiplying by $\binom{N}{2}$ gives

$$f_1 = \binom{N}{2} 2(N-2) = N(N-1)(N-2).$$

Fix a_i. Inside the three-dimensional space H_{a_i}, the other $N - 1$ hyperplanes restrict to a central arrangement of $N - 1$ planes in general position. Such an arrangement has

$$2\left(1 + (N-2) + \binom{N-2}{2}\right) = N^2 - 3N + 4$$

regions [5]. The Lorentz form restricted to H_{a_i} is positive semidefinite with null line $\mathbb{R}(a_i, 1)$, so every three-dimensional conical region contains spacelike vectors and meets Ω in exactly one connected two-dimensional stratum; if it contains a null ray, that intersection is one of the punctured strata described above. Therefore

$$f_2 = N(N^2 - 3N + 4).$$

Finally, chambers correspond to sign patterns of the affine function $x \mapsto \langle u, x\rangle - c$ on the N points. Since no four points are coplanar, Cover's theorem gives

$$f_3 = 2\sum_{j=0}^{3}\binom{N-1}{j} = \frac{1}{3}(N^3 - 3N^2 + 8N).$$

Every nonconstant separable sign pattern is realized by a plane meeting the unit sphere, because it has points on both sides; the two constant sign patterns are realized by choosing c just inside ± 1. Thus all of Cover's regions correspond to genuine oriented circles. □

Corollary 5.3. *For $N \geq 3$,*

$$f_0 - f_1 + f_2 - f_3 = 0,$$

and the compactly supported Euler characteristics of the strata in Theorem 1.9 sum to $\chi_c(dS_3) = -2$.

Proof. The first identity follows by substitution. Ordinary open j-cells contribute $(-1)^j$, each punctured two-stratum is homeomorphic to $S^1 \times \mathbb{R}$ and has $\chi_c = 0$, while each extreme chamber is $(S^2 \setminus \mathcal{A}) \times \mathbb{R}$ up to homeomorphism and contributes $-(2-N) = N-2$. Hence the total is

$$f_0 - f_1 + (f_2 - 2N) - \big(f_3 - 2\big) + 2(N-2) = -2,$$

as required by $dS_3 \cong S^2 \times \mathbb{R}$. If the $2N$ punctured two-strata were incorrectly counted as cells, the answer would be larger by $2N$. □

Corollary 5.4. *Let $L \subset \mathrm{Mat}_n$ be a generic complex two-plane whose projective spectral discriminant consists of $N = n(n-1)$ simple points with no four concyclic. Then the number of chambers of oriented circles avoiding the discriminant is*

$$R(N) = \frac{1}{3}(N^3 - 3N^2 + 8N),$$

or, after substitution,

$$R\big(n(n-1)\big) = \frac{1}{3}\big(n^3(n-1)^3 - 3n^2(n-1)^2 + 8n(n-1)\big).$$

6. Refined counts by side type

For a vertex let (a, b) denote the numbers of points of $\mathcal{A}$ on the negative and positive sides of the oriented circle; thus $a + b = N - 3$. Write $V_{a,b}$ for the set of such vertices. Likewise let $E_{a,b}$ be the set of edges with $a + b = N - 2$.

Theorem 6.1. *For every configuration of $N \geq 3$ distinct points on S^2 with no four concyclic,*

$$|V_{a,b}| = 2(a+1)(b+1), \qquad a + b = N - 3,$$

and

$$|E_{a,b}| = 6ab + 3a + 3b, \qquad a + b = N - 2.$$

Proof. It is convenient to count oriented k-facets. Let $v_k(N)$ be the number of oriented circles through three points for which exactly k of the remaining $N - 3$ points lie on the positive side. Because the N points lie on the strictly convex sphere, their convex hull is a simplicial 3-polytope. Its number of triangular facets is $2N - 4$, so

$$v_0(N) = 2N - 4.$$

Delete one point of the configuration and sum $v_k(N-1)$ over the N possible deletions. An oriented triple having k positive points survives with the same value of k when a negative point is deleted, in $N-3-k$ ways. A triple having $k+1$ positive points contributes after one of those $k+1$ points is deleted. Therefore

$$Nv_k(N-1) = (N-3-k)v_k(N) + (k+1)v_{k+1}(N). \tag{9}$$

Starting with $v_0(N) = 2N-4$, this recurrence gives inductively

$$v_k(N) = 2(k+1)(N-k-2).$$

Taking $k=b$ and using $a+b=N-3$ yields

$$|V_{a,b}| = 2(a+1)(b+1).$$

For edges, each vertex is the transverse intersection of three walls and has six incident edge germs. An edge of type (a,b) can end at a vertex of type $(a-1,b)$, by moving one of its negative-side points onto the circle, or at a vertex of type $(a,b-1)$. At each such vertex exactly three of the six germs have type (a,b). Counting endpoints of edges gives, with the convention that an out-of-range V is empty,

$$2|E_{a,b}| = 3|V_{a-1,b}| + 3|V_{a,b-1}|.$$

Substituting the vertex formula gives

$$|E_{a,b}| = 3a(b+1) + 3(a+1)b = 6ab + 3a + 3b.$$

□

Summing the formulas in Theorem 6.1 recovers f_0 and f_1 in Theorem 1.10. The proof also shows directly that these refined counts are independent of the configuration; no concyclicity-flip argument is needed.

7. Open problems

(1) Prove or disprove Conjecture 4.5. Equivalently, determine all possible interior monodromies μ for $n=3$.
(2) Prove or disprove the geometric-prefix Conjecture 4.6. Even for $n=3$, the missing point is to turn the Hurwitz-prefix condition into an exact statement about generalized circles in the base.
(3) Is Conjecture 4.8 true? In particular, can the monodromy of a spectral cubic along a branch-free circle ever be pseudo-Anosov?
(4) Is the spectral braid a complete invariant of connected components of $\mathcal{ALC}_n$? Theorem 1.5 gives the general lower bound n^2-n+1, while Corollary 1.8 gives at least thirteen components for $n=3$.
(5) Proposition 1.1 gives finiteness for fixed n. Find an effective upper bound for the number of connected components or spectral braid classes as a function of n.
(6) How sharp is Proposition 4.2, and which boundary braids force a large genus for every quasipositive surface arising from a spectral curve?
(7) Characterize the quasipositive braids μ that occur as monodromies of spectral curves over circles, taking into account both the Hurwitz factorization and the geometry of circles in the base.
(8) Investigate structured subclasses, for example real symmetric, Hermitian, or other matrix pencils for which additional spectral constraints are present.

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Appendix A. Reproducibility of the calculations

The main statements of the paper no longer depend on numerical eigenvalue tracking. We record here how the exact and exploratory computations may be reproduced.

The exact 3×3 certificate. Starting from the matrices in Theorem 1.6, exact symbolic arithmetic over $\mathbb{Q}$ gives the polynomial F and the sextic Q in (4)–(5). The Cayley substitution $\tau = (1+s)/(1-s)$ gives the polynomial R displayed in the proof; the ordinary Routh algorithm produces the stated first column. The separator calculation is obtained by substituting $\zeta = \frac{1}{2} + iy$, $\tau = (1+ix)/(1-ix)$, taking real and imaginary parts after multiplication by $(1-ix)^3$, and eliminating x. The resulting exact resultant is (6). These operations involve only integer arithmetic and polynomial gcd/resultant operations.

Numerical checks and exploration. For discovery and independent checking, a loop of matrices can be sampled at a fine mesh and eigenvalues between successive samples matched by minimum maximal displacement. For a pure loop, pairwise linking numbers can then be computed from

$$\mathrm{lk}(i,j) = \frac{1}{2\pi} \oint d \arg(\lambda_i - \lambda_j).$$

For the example of Theorem 1.6 this gives $(-2, 1, 1)$ to machine precision, in agreement with the exact proof.

The chamber experiments mentioned after Conjecture 4.8 were performed by computing branch points, stereographically projecting them to S^2, and testing sign patterns by linear separation. Separately, we explored the Hurwitz orbit of six-band factorizations of Δ^2 in $\mathcal{B}_3$ and recorded the conjugacy classes of prefixes using the reduced Burau representation at $t = -1$ together with exponent sum. This exploration found the thirteen classes listed in §4.5 and no others, but it was not exhaustive and is not used as a proof. The example of Theorem 1.7 requires no computer calculation.

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