[1]

ON THE BOUNDARY OF TOTALLY POSITIVITE UPPER TRIANGULAR MATRICES

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Abstract. Let $\mathbf{B}_+ \subset \mathbf{GL_n}(\mathbf{R})$ denote the subgroup of upper triangular $n \times n$-matrices with positive entries on the main diagonal. A matrix $M \in \mathbf{B}_+$ is called totally positive if the determinants of all its minors not containing a row or column lying completely under the main diagonal are positive. We give a simple determinantal equation for the boundary of all positive upper triangular matrices in $\mathbf{B}_+$.

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Let Δ_i denote the determinant of the $i \times i$-minor of a matrix $M \in \mathbf{B}_+ \subset \mathbf{GL_n}(\mathbf{R})$ consisting of the first i rows and the last i columns. Consider a hypersurface H in $\mathbf{B}_+$ given by the equation $H : \{\Delta_1 \times \cdots \times \Delta_{n-1} = 0\}$ and the semigroup $\mathbf{B}_p$ consisting of all totally positive matrices in $\mathbf{B}_+$.

Definition. A matrix $M \in \mathbf{GL_n}(\mathbf{R})$ ($M \in \mathbf{B}_+$ resp.) is called totally nonnegative if all determinants of all its minors are nonnegative.

The main result of this note is the following

1. Theorem. If some totally nonnegative upper triangular matrix M contains a minor $M\binom{i_1,\dots,i_l}{j_1,\dots,j_l}$ with no row or column under the main diagonal such that its determinant vanishes then at least one of Δ_i vanishes also.

2. Corollary. The semigroup $\mathbf{B}_p$ is one of the connected components of the complement $\mathbf{B}_+ \setminus H$ and H is precisely the equation of the analytic continuation of the boundary of $\mathbf{B}_p$.

The proof of the above theorem is based on two classical statements, see [GK], pp. 113 and 297.

3. Definition. Let $M\binom{i_1,\dots,i_l}{j_1,\dots,j_l}$ denote the $l \times l$-minor in M consisting of elements belonging to the rows with the numbers $i_1, \dots, i_l$ and to the columns with the numbers $j_1, \dots, j_l$. The number $\chi = \sum_{m=1}^{l}(i_m - i_{m-1} - 1)$ is called the row dispersion and the number $\mu = \sum_{m=1}^{l}(j_m - j_{m-1} - 1)$ is called the column dispersion of the minor $M\binom{i_1,\dots,i_l}{j_1,\dots,j_l}$. A minor is called tight if $\chi = \mu = 0$.

4. Proposition. For any totally nonnegative M and $p = 1, \dots, n$

$$\det M \le \det M\binom{1,\dots p}{1,\dots,p} \times \det M\binom{p+1,\dots n}{p+1,\dots,n}. \tag{1}$$

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5. Proposition. For any $n \times (n+1)$-matrix M

$$\det M\binom{1,\dots,n}{1,\dots,n} \times \det M\binom{1,\dots,n-1}{2,\dots,n-1,n+1} + \det M\binom{1,\dots n}{2,\dots,n+1} \times \det M\binom{1,\dots,n-1}{1,\dots n-1} =$$

$$= \det M\binom{1,\dots,n}{1,\dots n-1,n+1} \times \det M\binom{1,\dots n-1}{2,\dots n}. \tag{2}$$

Proof of theorem 1. Let us show that if some totally nonnegative upper triangular matrix M contains a minor $M\binom{i_1,\dots,i_l}{j_1,\dots,j_l}$ with no row or column under the main diagonal such that its determinant vanishes then there exists at least one i such that $\Delta_i = 0$. Let us first show that this is true for all tight minors, namely if $\det M\binom{i,\dots,i+l}{j,\dots,j+l}$=0 then $\Delta_{n-i+j} = 0$. (We assume that $j > i$ since we consider the upper triangular case.)

Indeed using (1) and total nonnegativity of M one gets

$$0 = \det M\binom{i,\dots,i+l}{j,\dots,j+l} \times \det M\binom{1,\dots,i-1}{j-i,\dots,j-1} \geq \det M\binom{1,\dots i+l}{j-i,\dots,j+l} = 0.$$

Applying (1) again one gets

$$0 = \det M\binom{1,\dots,i+l}{j-i,\dots,j+l} \times \det M\binom{i+l+1,\dots,n-j+i}{j+l+1,\dots,n} \geq \Delta_{n-j+i} = 0.$$

Now let $M\binom{i_1,\dots,i_l}{j_1,\dots,j_l}$ be some nontight minor of the minimal size with vanishing determinant for which μ (or χ) is positive. Let us show that there exists another minor of the same size with vanishing determinant and smaller μ (χ resp.). We consider the case $\mu > 0$. Let us add to $M\binom{i_1,\dots,i_l}{j_1,\dots,j_l}$ the intersection of the last column in M with the number $\tilde{j} < j_l$ not included into $M\binom{i_1,\dots,i_l}{j_1,\dots,j_l}$ with the rows $i_1,\dots,i_l$. We apply the identity (2) to this $l \times (l+1)$-matrix. Since by assumption all minors with the size $< l$ are positive and the right-hand side of (2) vanishes then the determinant of a more tight minor $M\binom{i_1,\dots,i_l}{j_1,\dots,\tilde{j}}$ equals to zero. (Its μ is smaller than of the initial minor.) Applying this procedure several times we get $M\binom{i_1,\dots,i_1+l}{j_1,\dots,j_1+l} = 0$ and thus $\Delta_{n-j_1+i_1} = 0$ by the first part of this proof.

6. Definition. Let us consider some semigroup imbedded into a group. The set of all inverse elements to a given semigroup in the containing group is called the inverse semigroup.

7. Corollary. The set $\mathbf{B}_+ \setminus H$ contains 2^{n-1} connected compoments obtained by conjugation of $\mathbf{B}_p$ by all possible diagonal matrices with $\pm 1's$ on the main diagonal. These components are semigroups splitted into inverse pairs. See example on Fig.1.

Question. The domain D given by a system of inequalities $\Delta_i > 0,\ i = 1, \ldots, n$ contains $\mathbf{B}_p$ and $2^{[\frac{n+1}{2}]-1} - 1$ semigroup components mentioned above as well as some other connected components. It will be interesting to find the total number of connected components in D as well as in $\mathbf{B}_+ \setminus H$ and to enumerate and study topology of connected components other than the above semigroups (which are obviously contractible).

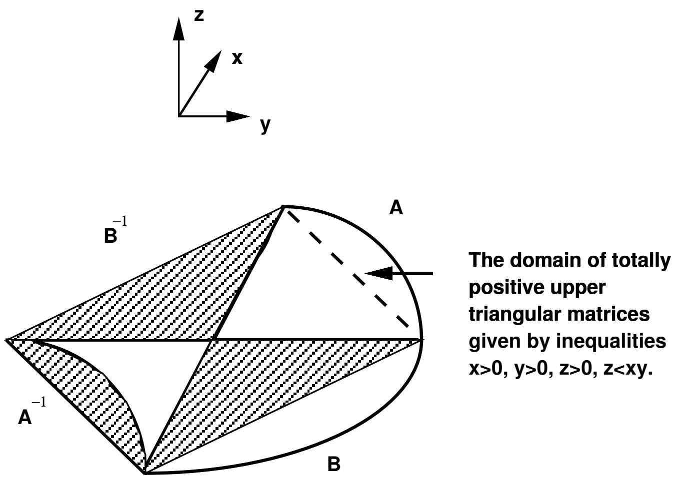

Fig.1. The complement $B_+\backslash H$ in GL_3 to H:{z(z-xy)=0} consists of 6 connected components. Totally positive matrices form one of 2 connected components of the domain D given by inequalities Δ_1=z>0 and Δ_2=xy-z>0.

There are 4 semigroup components on Fig.1 splitted into 2 inverse pairs A and A^{-1}, B and B^{-1}. The component A is the semigroup of totally positive matrices and the rest are its conjugates.

Example. For the 4×4 matrices D consists of 3 connected components containing respectively

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & \frac{-1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{-1}{2} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first two of them are semigroups while the third one is not.

Remark. Theorem 1 was found by the authors while working on some questions in the linear ordinary diferential equations and briefly mentioned in [Sh]. Later talking to F. Brenti the authors were surprised to find out that this result and its corollary have apparently escaped the attention of the specialists although very closely related theorems proved by more or less the same methods are quoted in [An] and especially in [Cr].

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