AROUND WATANABE – YOSIDA'S CONJECTURE ON THE MINIMAL HILBERT–KUNZ MULTIPLICITY

IGOR PAK, BORIS SHAPIRO, ILYA SMIRNOV, AND KEN-ICHI YOSHIDA

Abstract. This paper attempts to settle a conjecture by K. Watanabe and K. Yoshida ([13]) ...

1. Introduction

Let us start with a brief history of the problem.

Hilbert–Kunz multiplicity is a multiplicity theory native to positive characteristic: it is a numerical invariant that asymptotically measures how far is the Frobenius endomorphism from being flat. Its origin lays in the work of Kunz [9, 10]. In particular, [9] shows that flatness of the Frobenius characterizes regular rings and [10] extensively studies the sequence whose limit is the Hilbert–Kunz multiplicity. Kunz did not pursue the theory further since he thought that this limit may not exists, due to a computation mistake in an example. This mistake was corrected and the invariants was defined by Monsky [11], whose motivation was in the Iwasawa theory. The name Hilbert–Kunz multiplicity, chosen by Monsky, honors Kunz's contribution and similarity to the classical Hilbert–Samuel multiplicity, which can also be defined as a limit of a sequence.

Thus, from its very origins we see the two major forces motivating the development of the Hilbert–Kunz theory: its use as a tool of understanding singularities and comparison with the Hilbert–Samuel theory. For example, a classical theory of Nagata ([**?**]) shows that a local ring R is regular if and only if it is unmixed and its multiplicity is 1. The unmixed condition means that the completion $\hat{R}$ has no embedded components; it is necessary since lower-dimensional components do not contribute to the limit. In a parallel result, Watanabe and Yoshida showed in [13] that Hilbert–Kunz multiplicity detects regular rings: a local ring R is regular if and only if it is unmixed and $\mathrm{e_{HK}}(R) = 1$.

However, there are many major differences between two theories of multiplicity. One important difference is that Hilbert–Kunz multiplicity need not be an integer, in particular, it attains values between in $(1, 2)$. We still do not know much about the set of values of Hilbert–Kunz multiplicity. A very basic and natural question is to determine what is the next value after 1, i.e., what is the minimal Hilbert–Kunz multiplicity of an unmixed singular ring.

This question was raised by Blickle and Enescu in [2] who also made a first step and showed that $m_{p,d} := \inf\{\mathrm{e_{HK}}(R) \mid R \text{ is an unmixed local ring }, \dim R = d, \operatorname{char} R = p\} \geq 1 + \frac{1}{d!p^d} > 1$. It is worth noting that another result [2] gave a restriction on singularity of rings with very small Hilbert–Kunz multiplicity: after an improvement in [1] we get that any ring with

$$\mathrm{e_{HK}}(R) \leq 1 + \max\left\{\frac{1}{d!}, \frac{1}{\mathrm{e}(R)}\right\}$$

will be normal, Gorenstein, and F-regular.

The result of Blickle and Enescu does not tell us whether $m_{p,d}$ is attained. In [**?**] Watanabe and Yoshida conjectured that $m_{p,d} = \mathrm{e_{HK}}(F_p[[x_0, \ldots, x_d]]/(x_0^2 + \cdots + x_d^2))$ if $p > 2$ and that this value should characterizes $x_0^2 + \cdots + x_d^2$ (Conjecture 2.2). They proved the conjecture in dimension

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at most 4. Aberbach and Enescu extended this to dimension ≤ 6 ([**?**]). Enescu and Shimomoto proved the conjecture for complete intersections ([**?**]).

An important subtlety is that a closed formula for $\mathrm{e}_{\mathrm{HK}}(F_p[[x_0,\ldots,x_d]]/(x_0^2+\cdots+x_d^2))$ is not known. In her thesis [**?**], Han developed an algorithm for computing Hilbert–Kunz multiplicity of diagonal hypersurfaces, which was published in a simplified form in [8]. The algorithm gives us a way to compute the formula for given p, d, but an actual closed formula was presented only for $d \leq 4$. However, Gessel and Monsky [7] observed that the algorithm connects Euler's numbers to Hilbert–Kunz multiplicity: $\lim_{p\to\infty} \mathrm{e}_{\mathrm{HK}}(F_p[[x_0,\ldots,x_d]]/(x_0^2+\cdots+x_d^2)) - 1$ is a coefficient a_d at x^d of the power series $\sec x + \tan x$. Thus another part of the Watanabe–Yoshida conjecture asserts that $m_{p,d} \geq a_d$, giving also a characteristic-free bound. A much weaker bound was observed in [1]: $m_{p,d} \geq 1 + \frac{1}{d(d!(d-1)+1)^d}$.

Various other lower bounds on Hilbert–Kunz multiplicity are known ([**?**, **?**, **?**, **?**, **?**, **?**])

An unpublished work of the forth author ([**?**]) extended the Watanabe–Yoshida conjecture to characteristic 2, by using the simple (A_1) singularity, and gave a stronger bound (this appeared in [**?**]).

1.1. **Results.** Let us start with the formulation of the relevant conjecture in elementary terms as explained in Yoshida's notes [**?**] postponing the explanation of its connection to the Hilbert-Kunz multiplicity till the next section.

Consider two sequences of matrices $\{T_k\}_{k=1}^{\infty}$ and $\{N_k\}_{k=1}^{\infty}$ given by:

$$T_1 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 2 & 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 \end{pmatrix}, \quad \cdots$$

and

$$N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \cdots$$

In what follows, dor a matrix M, we use $[M]_{1,1}$ to denote its top-left entry, i.e the matrix element in position $(1,1)$.

Conjecture 1.1 (Watanabe – Yoshida)**.** For positive integers $k \geq 1$ and $d \geq 2$, define

$$h_k(d) := \frac{[T_k^{d+1}]_{1,1} - (2k+1)^d}{(2k+1)^d - [N_k^{d+1}]_{1,1}}. \tag{1}$$

Then, for any fixed $d \geq 2$, the sequence $\{h_k(d)\}_{k=1}^{\infty}$ is strictly decreasing in k.

It is known that Conjecture 1.1 implies the following claim.

Conjecture 1.2. (i) For $d = 2m-1$ and any positive integer k, one has $h_k(d) < \frac{2^m}{2^m-1}$.

(ii) For $d = 2m$ and any positive integer k, one has $h_k(d) < \frac{2^m+1}{2^m}$.

Remark 1.3. Observe that it was shown by Gessel–Monsky in [7] that, for any fixed positive integer d, there exists $L_d := \lim_{k\to\infty} h_k(d)$ and, moreover,

$$1 + \sum_{d=1}^{\infty} L_d x^d = \sec(x) + \tan(x).$$

Additionally, Watanabe and Yoshida conjectured that, for every k, $h_k(d) > L_d$.

In this paper, we attempt to settle Conjecture 1.1 and, to start with, we reinterpret the right-hand side of (1) in terms of Erhard polynomials of appropriate integer polytopes. The structure of the paper is as follows...

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2. Background and preliminaries

2.1. Hilbert–Kunz multiplicity.

Definition 2.1. Let $(R, \mathfrak{m})$ be a local ring of positive characteristic $p > 0$. Then the *Hilbert–Kunz multiplicity* of R is defined as the limit

$$\mathrm{e_{HK}}(R) := \lim_{e\to\infty} \frac{\lambda(R/\mathfrak{m}^{[p^e]})}{p^{e \dim R}},$$

where $\mathfrak{m}^{[p^e]}$ is the ideal generated by all p^e-th powers of elements in $\mathfrak{m}$.

Hilbert–Kunz multiplicity originates from the work of E. Kunz ([9, 10]) and was defined by P. Monsky ([11]) who showed that the above limit always exists. Hilbert–Kunz multiplicity is a native multiplicity theory for Frobenius. WHAT DOES THIS MEAN? It detects whether a ring is singular ([13]) and satisfies a number of good properties ([**?**]).

On the other hand, the actual values of Hilbert–Kunz multiplicity are still very mysterious as they need not to be integer or even rational ([3]). In that respect, Watanabe and Yoshida wanted to strengthen their theorem WHAT THEOREM? and made the following claims.

Conjecture 2.2. Let $(R, \mathfrak{m})$ be a formally unmixed, non-regular local ring of dimension $d \geq 1$ and characteristic $p > 2$. Denote by $\mathfrak{f}$ be the residue field of R and define the simple (A_1)-singularity $A_{p,d}$ as

$$A_{p,d} := \mathfrak{f}[[x_0, \ldots, x_d]]/(x_1^2 + \ldots + x_d^2).$$

Then

(1) $\mathrm{e_{HK}}(A) \geq \mathrm{e_{HK}}(A_{p,d}) \geq 1 + c_d$, where c_d is the Euler number (EXPLAIN)
(2) If $\mathfrak{f}$ is algebraically closed, i.e. $\mathfrak{f} = \bar{\mathfrak{f}}$ and if $\mathrm{e_{HK}}(A) = \mathrm{e_{HK}}(A_{p,d})$, then $\widehat{A} \cong A_{p,d}$. Here $\mathrm{e_{HK}}(A)$ is the completion of A???

Conjecture 2.3. Let $(R, \mathfrak{m})$ be a formally unmixed non-regular local ring of dimension $d \geq 1$ such that the residue field $\mathfrak{f}$ is algebraically closed and m be a positive integer. Then the following holds.

(1) If $d = 2m - 1$, then either $\widehat{A} \cong A_{p,d}$ or $\mathrm{e_{HK}}(A) > \frac{2^m}{2^m - 1}$.
(2) If $d = 2m$, then either $\widehat{A} \cong A_{p,d}$ or $\mathrm{e_{HK}}(A) > \frac{2^m + 1}{2^m}$.

2.2. **Han–Monsky algorithm for calculation of Hilbert–Kunz multiplicity.** In their illuminating paper [8] Han and Monsky provided an approach to the calculation of Hilbert–Kunz multiplicity for special local rings. Their key idea is based on a concept of *representation ring.*

Namely, for a fixed field $\mathfrak{f}$, consider the subcategory $Mod_T(\mathfrak{f})$ of finitely generated $\mathfrak{f}[T]$-modules with nilpotent action of an operator T. One can impose a natural equivalence relation on pairs (M, N) of such modules given by: $(M_1, N_1) \equiv (M_2, N_2)$ if and only $M_1 \oplus N_2 \cong M_2 \oplus N_1$. The equivalence class of a pair (M, N) will be denoted as $M - N$.

Definition 2.4. The representation ring $\Gamma_{\mathfrak{f}}$ consists of the equivalence classes $M - N$ with the operations of direct sum and tensor product:

$$(M_1 - N_1) \times (M_2 - N_2) := [(M_1 \otimes_{\mathfrak{f}} M_2) \oplus (N_1 \otimes_{\mathfrak{f}} N_2)] - [(M_1 \otimes_{\mathfrak{f}} N_2) \oplus (N_1 \otimes_{\mathfrak{f}} M_2)] \,.$$

The equivalence class $\mathfrak{f} - 0$ is the identity element of $\Gamma_{\mathfrak{f}}$.

The standard examples of elements in $\Gamma_{\mathfrak{f}}$ are provided by $\delta_i = \mathfrak{f}[T]/(T^i) - 0$, $i = 1, \ldots$?? Since $\mathfrak{f}[T]$ is a PID, then $\Gamma_{\mathfrak{f}}$ is a free $\mathbb{Z}$-module with a basis $\{\delta_i\}$. The function $D_{\mathfrak{f}} \colon \Gamma_{\mathfrak{f}} \to \mathbb{Z}$ is then defined as the sum of the coordinates in this basis. We may associate $M \in Mod_T(\mathfrak{f})$ with its image $M - 0 \in \Gamma_{\mathfrak{f}}$. If $M = \oplus_{i=1}^{\infty} \mathfrak{f}[T]/(T^i)^{\oplus a_i}$, then we define

$$D_{\mathfrak{f}}(M) := \dim_{\mathfrak{f}} M/TM = \sum_{i=1}^{\infty} a_i.$$

Theorem 2.5 (Han–Monsky)**.** *Given a field $\mathfrak{f}$ of positive characteristic $p > 0$, define*

$$R := \mathfrak{f}[[x_0, \ldots, x_d]]/(x_0^{n_0} + \cdots + x_d^{n_d}).$$

For a fixed integer $e \geq 0$ and each $i = 0, \ldots, d$, define the remainder r_i determined by $p^e = n_i a_i + r_i$, $0 \leq r_i < d_i$. In the above notation,

$$\dim_{\mathfrak{f}} R/(x_0^{p^e}, \ldots, x_d^{p^e}) = D_{\mathfrak{f}} \prod_{i=0}^{d} \left((n_i - r_i)\delta_{a_i} + r_i \delta_{a_i+1}\right).$$

Proof. Let us sketch the idea of the proof. For a positive integer k, consider $R[x]/(x^{p^e})$ as a $\mathfrak{f}[T]$-module with the action $Tf := x^k f$. Denote this module as M_k. If $p^e = ak + r$, then observe that the submodule $\mathfrak{f}[T]x^i \subset M$ is isomorphic to $\mathfrak{f}[T]/(T^{a+1})$ for $0 \leq i < r$ and to $\mathfrak{f}[T]/(T^a)$ for $r \leq< i < k$. Clearly, this gives a direct sum decomposition of M_k as an $\mathfrak{f}[T]$-module.

Furthermore, $M_{n_0} \otimes_{\mathfrak{f}} \cdots \otimes_{\mathfrak{f}} M_{n_d} \cong \mathfrak{f}[x_0, \ldots, x_d]/(x_0^{p^e}, \ldots, x_d^{p^e})$ with $T \mapsto x_0^{n_0} + \cdots + x_d^{n_d}$. The statement then follows from the fact that, for any T-module M, $D_{\mathfrak{f}}(M) = \dim_{\mathfrak{f}} M/TM$. □

Using Theorem 2.5, Han and Monsky were able to provide a way of computing Hilbert–Kunz multiplicity for $\mathfrak{f}[[x_0, \ldots, x_d]]/(x_0^{n_0} + \cdots + x_d^{n_d})$. In particular, their theorem gives the following recipe.

Corollary 2.6. *Let $\mathfrak{f}$ be a field of characteristic $p > 2$ and $a = (p-1)/2$. Then*

$$\mathrm{e}_{\mathrm{HK}}(\mathfrak{f}[[x_0, \ldots, x_d]]/(x_0^2 + \cdots + x_d^2)) = 1 + \frac{D_{\mathfrak{f}}((\delta_a + \delta_{a+1})^{d+1}) - p^d}{p^d - (-1)^{a(d+1)} D_{\mathfrak{f}}((\delta_{a+1} - \delta_a)^{d+1})}.$$

2.3. **Linear algebra approach.** In order to compute the Hilbert–Kunz multiplicity, we need to understand the multiplication in the representation ring $\Gamma_{\mathfrak{f}}$. It will be more convenient instead of $\{\delta_i\}$ to use another basis of $\Gamma_{\mathfrak{f}}$. We set $\lambda_0 = \delta_1$ and $\lambda_i = (-1)^i(\delta_{i+1} - \delta_i)$, so that $\delta_i = \lambda_0 - \lambda_1 + \cdots + (-1)^{i-1}\lambda_{i-1}$. Then, by definition, $D_{\mathfrak{f}}(M - N)$ is the coefficient at λ_0 of the decomposition of $M - N$ in the basis $\{\lambda_i\}$.

Theorem 2.7 (Han–Monsky)**.** *Let $\mathfrak{f}$ be a field of characteristic $p > 0$ and $0 \leq i < j \leq p-1$. Then*

(1) *if* $i+j \le p-1$ *then* $\lambda_i\lambda_j = \sum_{j-i}^{j+i} \lambda_k$.
(2) *if* $i+j \ge p$ *then* $\lambda_i\lambda_j = \lambda_{p-1-i}\lambda_{p-1-j}$.

Example 2.8. The multiplication matrices of λ_1 and λ_2 in the basis $\{\lambda_i\}_{i=0}^{p-1}$ are given by:

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \text{ and } \lambda_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix}.$$

Combining this with Theorem 2.5 we get the following.

Corollary 2.9. *Let* $\mathfrak{k}$ *be a field of characteristic* $p > 0$ *and* $R = \mathfrak{k}[[x_0, \dots, x_d]]/(x_0^{n_0} + \cdots + x_d^{n_d})$. *For a fixed positive integer* n, *write* $p = an + r$ *with* $0 \le r < m$ *and set* $b = m - r$. *Consider the* $p \times p$ *matrix*

$$M_n = \begin{bmatrix} n & \cdots & n & r & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & b & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & r & \ddots & \ddots & \ddots & & \vdots \\ n & \ddots & \ddots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ r & b & r & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & r & b & r & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & n \\ \vdots & \ddots & \ddots & \ddots & r & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & r & n & \cdots & \cdots & n \end{bmatrix},$$

where the corner of 'n' has size a *and the middle rectangles have* r *and* b *alternating.*

Then $\dim_{\mathfrak{k}} R/(x_0^p, \dots, x_d^p) = \prod_{i=0}^{d} ((n_i - r_i)\delta_{a_i} + r_i\delta_{a_i+1})$ *is the* $(1,1)$*-entry of the matrix product* $M_{n_0}M_{n_1}\cdots M_{n_d}$.

Proof. We note that $(n-r)\delta_a + r\delta_{a+1} = n(\lambda_0 - \lambda_1 + \cdots + (-1)^{a-1}\lambda_{a-1}) + r(-1)^a\lambda_a$ and the matrix of the multiplication by this element in the basis of $\{\lambda_i\}_{i=0}^{p-1}$ is M_n with alternating signs:

$$\begin{bmatrix} n & -n & \cdots & (-1)^{a-1}n & (-1)^a r & 0 & \cdots & \cdots & 0 \\ -n & n & \cdots & (-1)^a r & (-1)^{a+1} b & (-1)^{a+2} r & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Since the signs only depend on the parity and are identical in the matrices, the product of these matrices is again an alternating sign version of $M_{n_0}M_{n_1}\cdots M_{n_d}$. □

Corollary 2.10. *For an integer* $a \ge 1$, *consider square matrices* T_a *and* N_a *of size* $(a+1+a)$

$$T_a = \begin{bmatrix} 2 & \cdots & 2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 2 & \ddots & \ddots & \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\ 0 & \ddots & \ddots & \vdots & \ddots & \ddots & 2 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 2 & \cdots & 2 \end{bmatrix} \text{ and } N_a = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\ 0 & \ddots & \ddots & \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Let k be a field of characteristic $p > 2$. If $a = (p-1)/2$, then

$$\mathrm{e}_{\mathrm{HK}}(k[[x_0,\ldots,x_d]]/(x_0^2+\cdots+x_d^2)) = 1 + \frac{[T_a^{d+1}]_{(1,1)} - p^d}{p^d - [N_a^{d+1}]_{(1,1)}}. \tag{2}$$

3. Hilbert–Kunz multiplicity via the Ehrhart theory

In this section, we will reinterpret the entries of (2) in Corollary 2.10 as the number of integer points in certain polytopes.

Definition 3.1. For $d \geq 1$, define the d-dimensional *Fibonacci polytope* $\mathfrak{F}_d$ as given by:

$$x_i + x_{i+1} \leq 1, \quad x_i \in [0,1], \quad i = 1,\ldots,d$$

and the *extended d-dimensional Fibonacci polytope* $\mathfrak{E}\mathfrak{F}_d$ as given by:

$$|x_i| + |x_{i+1}| \leq 1,, \quad x_i \in [-1,1], \quad i = 1,\ldots,d.$$

Remark 3.2. The vertices in the Fibonacci polytope $\mathfrak{F}_d$ are in $1-1$-correspondence with the binary sequence of length d with no consecutive ones. The number of these vertices forms the famous Fibonacci sequence $\{a_d\}$ where $a_1 = 2$, $a_2 = 3$ and $a_d = a_{d-1} + a_{d-2}$. It is well-known that

$$a_d = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^d - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^d.$$

Observe that every integer point in $\mathfrak{F}_d$ is one of its vertices.

The vertices in the extended Fibonacci polytope $\mathfrak{E}\mathfrak{F}_d$ are ??? Their number is ???

The numbers of integer points in the extended Fibonacci polytope $\mathfrak{E}\mathfrak{F}_d$ form the Jacobsthal sequence (OEIS A001045): $3, 5, 11, 21, 43, 85, \ldots$. To see this, one can verify that this number satisfies the linear recurrence $p(n) = p(n-1) + 2p(n-2)$ with the standard initial conditions. Namely, if we increase the dimension by adding x_{d+1}, then any integer point can be extended by $x_{d+1} = 0$. However, if $x_d = 0$, then such point can be also extended by ± 1.

The Fibonacci polytope is affinely equivalent to the order polytope of the zigzag poset ([**?**, Section 3.8]), i.e., the set of points in $[0,1]^d$ such that $x_1 \geq x_2 \leq x_3 \geq \cdots x_d$. This equivalence also works for the dilations: the map

$$\begin{cases} y_i = k - x_i & \text{if } i \text{ is odd} \\ y_i = x_i & \text{if } i \text{ is even} \end{cases}$$

gives a bijection on integral points.

3.1. **Results.** As a warm-up exercise, consider a $k \times k$-matrix M_k triangular with respect to the opposite diagonal

$$M_k = \begin{bmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & & \iddots & 0 \\ \vdots & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Then, by definition of the matrix product, we may interpret

$$[M_k^{d+1}]_{(1,1)} = \sum_{1 \leq i_1,\ldots,i_d \leq k} m_{1,i_1} \cdots m_{i_d,1}.$$

Of course, a collection of indices $(i_1,\ldots,i_d)$ will give a nonzero product if and only if $m_{i_n,i_{n+1}} = 1$ for all $1 \leq n \leq d$. However, due to the shape of M_k, this means that $i_n + i_{n+1} \leq k+1$ and,

therefore, $(i_1-1,\dots,i_d-1)$ is an integer point in the $(k-1)$-st dilation of the d-dimensional Fibonacci polytope.

We may generalize this observation further.

Theorem 3.3. *If $P_d(k)$ denotes the Ehrhart polynomial of the d-dimensional Fibonacci polytope, then $[T_k^{d+1}]_{(1,1)} = (2k+1)^d + 2^d P_d(k-1)$.*

Proof. We consider $(2k+1)^d$ as the $(1,1)$ entry of the matrix B^{d+1}, where $b_{ij}=1$ for $1\le i,j\le 2k+1$, and recall that

$$[T_k^{d+1}]_{(1,1)} = \sum_{1\le i_1,\dots,i_d\le k} a_{1,i_1}a_{i_1,i_2}\cdots a_{i_d,1}.$$

For convenience, we label the above product of elements in T_k by the set of indices $(i_0,\dots,i_{d+1})$ where $i_0,i_{d+1}=1$, and, similarly, we label a similar product in B by the set of indices $(j_0,\dots,j_{d+1})$. We will give a procedure that will assign to a nonzero product of elements $a_{i_0,i_1}a_{i_1,i_2}\cdots a_{i_d,i_{d+1}}$ a set of $d+1$-tuples $(j_0,\dots,j_{d+1})$. Suppose that $a_{i_n,i_{n+1}}=1$ and $a_{i_m,i_{m+1}}=2$ for $0\le m<n$. We produce a set of $d+1$-tuples $(j_0,\dots,j_{d+1})$ as follows:

(1) $j_{d+1}=j_0=1$.
(2) if $m<n$ and $a_{i_m,i_{m+1}}=2$, then we put two values for j_{m+1}: i_{m+1} and $2k+1-i_{m+1}$,
(3) if $m<n$ and $a_{i_m,i_{m+1}}=1$, then $j_{m+1}=i_{m+1}$,
(4) if $m>n$ and $a_{i_m,i_{m+1}}=2$, then we put two values for j_m: i_m and $2k+1-i_m$,
(5) if $m<n$ and $a_{i_m,i_{m+1}}=1$, then $j_m=i_m$.

If $a_{1,i_1}\dots a_{i_d,1}=2^l$, with $l<d$, this procedure produces 2^l d-tuples, so their sum is $a_{1,i_1}a_{i_1,i_2}\cdots a_{i_d,1}$. This operation is reversible: given a product $\prod_{0\le m\le d} b_{i_m,i_{m+1}}$, we seek for the first index m such that (i_m,i_{m+1}) belongs to the middle rhombus of T_k and modify accordingly the preceding and succeeding indices. Note that there are no conflicts, because if $a_{i,j}=2$ then $a_{i,2k+1-j}=a_{2k+1-i,j}=0$ due to the shape of the matrix T_k and if $a_{i,j}=2$ then $a_{i,2k+1-j}=a_{2k+1-i,j}=1$.

In the case of $\prod a_{i_m,i_{m+1}}=2^{d+1}$, we do not have enough elements in B^{d+1} as the procedure above (always setting j_{m+1} to be either i_{m+1} or $2k+1-i_{m+1}$) will produce only 2^d products. Since all possible $(d+1)$-tuples can be obtained as the output of the procedure, the claim now follows after observing that $P_d(k-1)$ is also the number of $(i_1,\dots,i_d)$ such that $a_{1,i_0}\cdots a_{i_d,1}=2^{d+1}$. □

Definition 3.4. Let σ be an alternating permutation in S_n. Its *permutation statistics* $\mathrm{swap}(\sigma)$ is the number of $i<n$ such that $\sigma^{-1}(i)<\sigma^{-1}(i+1)-1$. Equivalently, $\mathrm{swap}(\sigma)$ is the number of $i<n$ such that i is to the left of $i+1$ and swapping i and $i+1$ in σ yields another alternating permutation.

By Coons and Sullivant ([6]) the Ehrhart series of the order polytope of the zigzag poset is

$$\frac{\sum_\sigma t^{\mathrm{swap}(\sigma)}}{(1-t)^{d+1}}.$$

over alternating permutations. If we let $s_d(m)=\#\{\sigma \mid \mathrm{swap}(\sigma)=m\}$, then by the Ehrhart theory

$$P_d(k)=\#(mO\cap\mathbb{Z}^d)=\sum_{m=0}^{d-2} s_d(m)\binom{k+d-m}{d}.$$

The volume of the Fibonacci polytope is the Euler number a_d (see for example, [**?**, Section 3.8]).

Corollary 3.5. *$d!\left(T(k,d+1)-(2k+1)^d\right)/2^d$ is a polynomial in k of degree d with integer coefficients:*

$$a_dk^d+\left(a_d\frac{d^2+d}{2}-d\sum_{m=0}^{d-2}s_d(m)(m+1)\right)+\cdots$$

Furthermore, the sum of all coefficients is $d!$.

Proof. By the theorem $d!\left(T(k,d+1)-(2k+1)^d\right)/2^d = d!P_d(k-1)$, so it is a polynomial with integer coefficients. Note that $P_d(0)=1$, which shows that the sum of the coefficients is $d!$. Furthermore, using that $\left(\sum_{i=0}^{d-1}(d-m-i)\right) = \left(d(d-m)-\frac{d(d-1)}{2}\right) = \left(\frac{d^2+d}{2}-dm\right)$, the generating series of $P_d(k-1)$ will be shifted by 1, and that the leading coefficient of the Ehrhart polynomial, $\sum s_d(m)$, is the volume a_d, we may write the first coefficients. □

Thus we may give a combinatorial interpretation of the quantity that we seek.

Corollary 3.6. *Let $F_d(k)$ and $E_d(k)$ be the Ehrhart polynomials of the Fibonacci and extended Fibonacci polytopes. If $\mathfrak{k}$ is a field of characteristic $p>2$ and $k=(p-1)/2$, then*

$$\mathrm{e_{HK}}(\mathfrak{k}[[x_0,\dots,x_d]]/(x_0^2+\cdots+x_d^2)) = 1+\frac{2^dF_d(k-1)}{(2k+1)^d-E_{d-2}(k)}$$

Proof. By Corollary 2.10,

$$\mathrm{e_{HK}}(\mathfrak{k}[[x_0,\dots,x_d]]/(x_0^2+\cdots x_d^2)) = 1+\frac{[T_k^{d+1}]_{(1,1)}-(2k+1)^d}{(2k+1)^d-[N_k^{d+1}]_{(1,1)}}.$$

By Theorem 3.3 the numerator has the required form. As for the denominator, we interpret

$$[N_k^{d+1}]_{(1,1)} = \sum_{-k\le i_1,\dots,i_d\le k} a_{1,k+1+i_1}\cdots{}_{k+1+i_d,1}\,.$$

Since we need all $a_{i,j}\neq 0$, we must have that $i_1=i_2=0$ and the remaining indices must form integer points in a k-dilation of the extended Fibonacci polytope. □

This combinatorial interpretation gives us monotonicity of Hilbert–Kunz multiplicity as dimension increases. The significance of this observation is that it demonstrates that $\mathrm{e_{HK}}(R/(L)) > \mathrm{e_{HK}}(R)$ for *every* linear form L, showing that there are no possible superficial-like elements for Hilbert–Kunz multiplicity (Recall, that $\mathrm{e}(R)=\mathrm{e}(R/(L))$ if L is superficial.).

Corollary 3.7. *For any field $\mathfrak{k}$ of characteristic $p>2$ and any d, we have*

$$\mathrm{e_{HK}}(\mathfrak{k}[[x_0,\dots,x_d]]/(x_0^2+\cdots+x_d^2)) > \mathrm{e_{HK}}(\mathfrak{k}[[x_0,\dots,x_{d+1}]]/(x_0^2+\cdots+x_{d+1}^2)).$$

Proof. We will use the formula in Corollary 3.6.

First, we observe that $F_{d+1}(k)<(k+1)F_d(k)$. This follows from the definition: we add x_{d+1} such that $x_d+x_{d+1}\le k$, so there are at most $(k+1)$ options. Hence

$$\frac{2^{d+1}F_{d+1}(k-1)}{(2k+1)^{d+1}} < \frac{(2k)2^dF_d(k-1)}{(2k+1)^{d+1}} < \frac{F_d(k-1)}{(2k+1)^d}.$$

Similarly, $E_{d+1}(k)<(2k+1)E_d(k)$ and, therefore, $\frac{E_{d-1}(k)}{(2k+1)^{d+1}} < \frac{(2k+1)E_{d-2}(k)}{(2k+1)^{d+1}}$. Thus

$$\begin{aligned}\mathrm{e_{HK}}(\mathfrak{k}[[x_0,\dots,x_{d+1}]]/(x_0^2+\cdots+x_{d+1}^2)) &= 1+\frac{2^{d+1}F_{d+1}(k-1)}{(2k+1)^{d+1}-E_{d-1}(k)} = 1+\frac{2^{d+1}F_{d+1}(k-1)/(2k+1)^{d+1}}{1-E_{d-1}(k)/(2k+1)^{d+1}}\\ &< 1+\frac{2^dF_d(k-1)/(2k+1)^d}{1-E_{d-2}(k)/(2k+1)^d} = \mathrm{e_{HK}}(\mathfrak{k}[[x_0,\dots,x_d]]/(x_0^2+\cdots+x_d^2)).\end{aligned}$$

□

3.2. **An approach to monotonicity.** Let us rewrite

$$\frac{T_d(k)-(2k+1)^{d-1}}{(2k+1)^{d-1}-N_d(k)}=\frac{T_d(k)/(2k+1)^{d-1}-1}{1-N_d(k)/(2k+1)^{d-1}}.$$

We need to effectively determine monotonicity of $\frac{N_d(k)}{(2k+1)^{d-1}}$ and $\frac{T_d(k)}{(2k+1)^{d-1}}$.

One way to approach is by making substitution $k=x-\frac{1}{2}$ and studying the coefficients of the resulting polynomial. Namely, if

$$\frac{N_d(k)}{(2k+1)^{d-1}}=\frac{1}{2^{d-1}}\sum_{i=2}^{d-1}a_ix^{-i},$$

we know that $a_2>0$, so for large x this is a decreasing function. In fact, BY EXPERIMENT, it seems that all $a_i\geq 0$.

Similarly,

$$\frac{T_d(k)}{(2k+1)^{d-1}}=\frac{1}{2^{d-1}}\sum_{i=0}^{d-1}b_ix^{-i}.$$

BY EXPERIMENT, IT SEEMS THAT $b_1=0$ and $b_2>0$ for $d\geq 7$ (actually we have only even coefficients). Thus, this is also a decreasing function for large k, thus the total fraction is also decreasing! BUT we need a bound on how large is k.

The following might be helpful: Recall that, for $|x|<1$, $\frac{1}{1-x}=1+x+o(x^2)$, so since $T_d(k)$ is a polynomial of degree k^{d-3} we may further rewrite for large (IN practice should be all) k

$$\frac{T_d(k)-(2k+1)^{d-1}}{(2k+1)^{d-1}-N_d(k)}=\left(\frac{T_d(k)}{(2k+1)^{d-1}}-1\right)\left(1+\frac{N_d(k)}{(2k+1)^{d-1}}\right).$$

Question: can we also explain why $T_d(x-1/2)$ will be either have only odd or even powers?

3.3. **Induction?** We can try an induction formula. We may think about elements in $[-k-1,k+1]^d/[-k,k]^d$ as elements that have at least one coordinate equal to $\pm k$ and sum over the number of such coordinates getting a familiar formula

$$[-k-1,k+1]^d=\sum_{i=0}^{d}\binom{d}{i}2^i[-k,k]^{d-i}.$$

Similar reasoning can be applied to the zigzag reincarnation of the Fibonacci, which we think about as points in $[0,k-1]^d$ such that $x_1\geq x_2\leq\cdots$. Similarly, we choose k indices that are equal to k and the remaining indices form zigzag polytopes of smaller dimension

$$P_d(k)=P_d(k-1)+\sum_{i=1}^{d}\sum_{d_1+\cdots+d_i=d-i}\prod_{j=1}^{i}P_{d_j}(k-1).$$

A similar formula may also work for extended Fibonacci.

4. Further developments

Motivated by Theorem 3.3, we want to present results and questions about the combinatorial structure of more general type of a matrix.

4.1. **Matrix questions.** Consider a family of $(2k+1)\times(2k+1)$ matrices

$$Q(q,k) = \begin{bmatrix} q & \cdots & q & 1 & 0 & \cdots & 0 \\ \vdots & \iddots & & \vdots & & \iddots & \vdots \\ q & & & \vdots & & & 0 \\ 1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\ 0 & & & \vdots & & & q \\ \vdots & \iddots & & \vdots & & \iddots & \vdots \\ 0 & \cdots & 0 & 1 & q & \cdots & q \end{bmatrix}.$$

Thus, the matrices in Corollary 2.10 are obtained by specializing $q = 0, 2$. This corollary motivates us to ask the following question.

Question 4.1. Compute

$$\lim_{k\to\infty} \frac{[Q(q,k)^{d+1}]_{(1,1)}}{(2k+1)^d}.$$

It can be shown that $[Q(q,k)^d]_{1,1}$ is also a polynomial in q, k (add reasoning below LATER), with a remarkable leading coefficient. Let $A(n,i)$ be the number of permutations of $[n]$ having $i-1$ alternating descents, $1 \le i \le n$ (sequence A145876 in OEIS). Then the coefficient at k^{d-1} is

$$\frac{1}{(d-1)!}\sum_{i=1}^{d-1} A(d-1,i)q^{i+1}2^{\max(d-2-i,0)}.$$

For example, at $d=5$ we will get $\frac{1}{24}(20q^2+14q^3+7q^4+5q^5)$.

More generally, it would be good to find the limit for $x_0^m + \ldots + x_d^m$.

Question 4.2. For $m > r > 0$ mutually prime, we may consider a family of $am+r \times am+r$-matrices $M(m,a,r)$ with the shape given in Corollary 2.9. If m is fixed, what is the limit

$$\lim_{a\to\infty} \frac{[M(m,a,r)^d]_{1,1}}{(ma+r)^d} = ?$$

Due to the Hilbert–Kunz connections (ADD) the limit is likely independent of r!

5. Volumes

Observe that the ratio $a_d/d!$ is the volume of the convex polyhedron defined as the set of $(x_1,\ldots,x_n)$ in $[0,1]^n$ such that $x_i + x_{i+1} <= 1$ for every $1 \le i \le n-1$. We are going to expand on this idea.

5.1. **Carving a hypercube.** Consider the hypercube $C = [-.5,.5]^d$ with coordinate system $x_1,\ldots,x_d$ and the partition of the hypercube C given by the hyperplane arrangement $x_i = x_{i+1}$ for $i = 1,\ldots,d_{i-1}$. We will describe the volumes and the number of resulting pieces.

First, we note that each chamber is described by $d-1$ inequalities: either $x_i < x_{i+1}$ or $x_i > x_{i+1}$. Thus there are 2^{d-1} pieces. If we have had a total order on variables, such as $x_1 < x_2 < \ldots < x_d$, or a chamber in the partition of all possible hyperplanes $x_i = x_j$, then the volume of the chamber is equal to $1/d!$. By subdividing, we see that the volume of a given chamber is $1/d!$ times the number of permutations satisfying the $d-1$ inequalities describing the chamber. Such sequence of inequalities is known as the *signature* of a permutation.

Example 5.1. If $d=3$ we have 4 parts with the volumes $\mathrm{vol}(x_1 > x_2 > x_3) = \mathrm{vol}(x_1 < x_2 < x_3) = 1/6$ (total orders) and $\mathrm{vol}(x_1 > x_2 < x_3) = \mathrm{vol}(x_1 < x_2 > x_3) = 1/3$.

Carlitz [4] was the first to study the number of permutations with prescribed signature. Unfortunately, there seem to be no easy formula, but, luckily, we will not need it.

Last, we want to remark that for any hyperplane arrangement of the form $x_i = \pm x_{i+1}$ for $i = 1, \dots, d_{i-1}$ and an arbitrary choice of $+$ or $-$, due to the cube symmetries, the partition of the cube does not depend on the signs and has the same volumes as described above.

[12, Example 4.3] showed that the volume of alternating is $\sec(x) + \tan(x)$.

5.2. **Description of the pieces.** We can describe a partition of the cube $[-.5, .5]^{n+1}$ using n-letter words in the alphabet $\{g, l\}$. We will use each letter to describe an inequality depending on the position in the word: the letter g ("greater") on place i gives inequality $x_i + x_{i+1} > 0$ and the letter l ("less") gives inequalities $x_i + x_{i+1} < 0$. For example, the word 'ggl' describes the system

$$\begin{cases} x_1 + x_2 > 0 \\ x_2 + x_3 > 0 \\ x_3 + x_4 < 0, \end{cases}$$

which we may simplify to $x_1 > -x_2 < x_3 < -x_4$. Similarly, each such word will describe a region on the partitions obtained by hyperplanes $x_i = -x_{i+1}$.

Following Chebikin [5] we introduce the following definition.

Definition 5.2. The index i is an alternating descent of a permutation π if either i is odd and $\pi(i) > \pi(i+1)$, or i is even and $\pi(i) < \pi(i+1)$.

We will use $A(n, k)$ to denote the number of permutations of $[n]$ having k alternating descents.

It is easy to see that $A(n, n-1) = A(n, 0)$ is the Eulerian number given by $\sum_n A(n, 0)x^n = \sec(x) + \tan(x)$.

WRITE A PICTURE

For example, the permutation described by $x_1 > -x_2 < x_3 < -x_4$ has alternating descents at $1, 2$.

Lemma 5.3. *The number of alternating descents of a permutation described by a word on $\{g, l\}$ of length n depends only on the number 'g', but not the order. Hence the number of permutations described by all words of length n with k letters 'g' is $A(n+1, k)$.*

Proof. Any two words with the same number of 'g' can be obtained by a sequence of swaps between adjacent 'g' and 'l'. However, gl and lg will define opposite signs $x_i > -x_{i+1} > x_{i+2}$ and $x_i < -x_{i+1} < x_{i+2}$, depending on a position in the word. These two permutations will have the same number of alternating descents by definition.

The last claim then follows by considering a word of the form $g \dots gl \dots l$. □

5.3. **Back to the matrices.** First we recall that

$$[Q(q, k)^{d+1}]_{1,1} = \sum_{1 \le i_{1_i}, \dots, i_{i_d} \le k+1} Q_{1,i_1} Q_{i_1,i_2} \cdots Q_{i_{d-2},i_{d-1}} Q_{i_{d-1},1}.$$

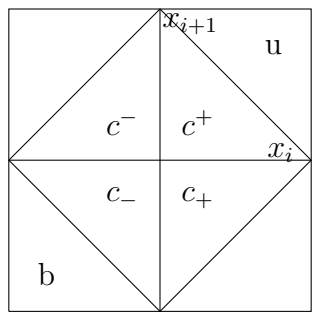
We plot the indices $(i_1, \dots, i_{d-1})$ as integer points on a cube $[1, 2k+1]^{d-1}$. The value of this product is thus a function on the cube. After translating, this becomes a question about integer points in the k-dilutions of certain regions of the cube $[-1, 1]^{d-1}$. By the Ehrhart theory the leading coefficient is given by the volumes of the regions.

Note that the value of each $Q_{i_c,i_{c+1}}$ is given by simple inequalities depending on what region of the matrix Q it is in. After discarding 0, we need to consider three possible regions.

$$Q(q,k) = \begin{bmatrix} q & \cdots & q & 1 & 0 & \cdots & 0 \\ \vdots & \iddots & & \vdots & & \iddots & \vdots \\ q & & & \vdots & & & 0 \\ 1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\ 0 & & & \vdots & & & q \\ \vdots & \iddots & & \vdots & & \iddots & \vdots \\ 0 & \cdots & 0 & 1 & q & \cdots & q \end{bmatrix}.$$

Note that due to the matrix multiplication rules, two consecutive elements may only be in the same or adjacent regions, i.e., cannot move from the top-left to the bottom-right corner immediately. However, a closer look at the matrix multiplication indicates that we need to subdivide the central region of 1s into four pieces.

Since we will be working with a dilation of $[-1,1]^n$, we will similarly define regions for each coordinate. For convenience, we will mirror the picture to have more natural signs for the regions as this will not affect volumes.



Thus, we may use words in the alphabet $E = \{b, u, c^+, c^-, c_+, c_-\}$ to encode regions of the cube $[-1,1]^n$ corresponding to nonzero products. It is easy to say when a word defines a non-empty region:

(1) u, c^+, c^- can be only followed by u, c^+, c_+
(2) b, c_+ and c_- can be only followed by b, c^-, c_-,
(3) the first letter must be either u, c^+, or c^- and the last letter must be either u, c^+, or c_+ (this rule corresponds to the fact that $Q_{1,i_1} \neq 0$ and $Q_{i_n,1} \neq 0$).

Let W_n to denote the set of all such n-letter words. The signature of the word $\sigma(w)$ is the number of 'c's.

For the word uc^+ will describe the following region in the cube $[-1,1]^3$: $x_1 + x_2 > 1$ (from 'u'), $x_2 + x_3 \leq 1$ (from 'c^+'), which, after simplifications, is given by $x_1 > 1 - x_2 \leq x_3$.

Theorem 5.4. *Let $A(n,k)$ be the number of permutations of $[n]$ having k alternating descents. Then*

$$\sum_{w \in W_n, \sigma(w)=k} \mathrm{vol}(w) = \frac{1}{(n+1)!} 2^{\max\{0,k-1\}} A(n+1,k).$$

Proof. We want to express the volume of the region given by a word in W_n. The substitution $x_i' \mapsto -x_i$ does not change the volume of a region or the permutation defined by the word. Thus given a word in W_n, the map $b, u \mapsto u$ and $c^{\pm}, c_{\mp} \to u$ will give a word of the same volume contained in $[0,1]^n$. Furthermore, note that after translating to $[-.5,.5]^n$ we obtain exactly the words in Lemma 5.3. So, the volume of the original word is the same as the volume of its image under the map ϕ that sends $b, u \mapsto g$, and all c to l.

Thus the total volume of words on $\{g, l\}$ with k letters l is $A(n,k)/n!$ by Lemma 5.3. It remains to count $|\phi^{-1}(w)|$, the number of words mapping to w. We want this number to be $2^{\max\{0,k-1\}}$.

For $k = 0$, this is trivial as the word is $u \dots u$ is the only mapping to $g \dots g$. For $k = 1$ and 'l' at the ith spot, we note that a preceding 'g' or the boundary condition, when $i = 1$, requires that $x_i \geq 0$ and, similarly, the succeeding letter will force $x_{i+1} \geq 0$. Thus we must have c^+ in the unique lift.

Now, we may assume that $k \geq 2$ and use induction on the length of the word to describe the number of pieces. First, if we extend a word w to wg, then any option for $v \in \phi^{-1}(w)$ can be only extended to vu due to the boundary conditions. Note that this is a valid extension: if $|w| = n$, then $x_n \geq 0$ was a boundary condition which is now required by 'u'. There cannot be other elements in $\phi^{-1}(wg)$ for the same reason. Thus $|\phi^{-1}(w)| = |\phi^{-1}(wg)|$ does not change in such case.

Second, suppose that we extend a word w of signature $k-1$ and length n to wl. Suppose w was ending on 'g' and write it as $w = vg \dots g$. Then for any lift $\tilde{v}$ of v we have two lifts of w: $\tilde{v}u \dots uc^+$ and $\tilde{v}b \dots bc^-$. Note that all lifts of w have the form $\tilde{v}u \dots u$ due to the boundary conditions, so we must have $2|\phi^{-1}(w)| = |\phi^{-1}(wl)|$.

In the last case, when w is ending on 'l' itself, say, $w = vl$, then for any lift of w has the form $\tilde{v}c^+$ or $\tilde{v}c^-$ due to the boundary condition (the actual sign may depend on $\tilde{v}$). However, in wl it is now not affected by the boundary condition and, for example, the lift $\tilde{v}c^-$ will be extended by both $\tilde{v}c^-c^+$ and $\tilde{v}c_-c^-$. Thus we have $2|\phi^{-1}(w)| = |\phi^{-1}(wl)|$ again. □

Recall, that the alternating Eulerian polynomials are defined by $A_n(x) = \sum_{k=0}^{n-1} A(n,k)x^k$. Chebikin [5] computed the exponential generating function

$$\sum_{n\geq 1} A_n(x)\frac{z^n}{n!} = \frac{\sec((1-x)z) + \tan((1-x)z) - 1}{1 - x\sec((1-x)z) - x\tan((1-x)z)}.$$

Corollary 5.5. *The leading coefficient of* $[Q(q,k)^{d+1}]_{1,1}$ *as a function of* k *(at* k^d*) is* $\frac{q^2}{2(d!)}\left(A(d,0) + A_d(2q)\right)$.

Proof.

$$\sum_{w\in W_n} \mathrm{vol}(w) = \frac{1}{(n+1)!}\sum_{k=0}^{n} 2^{\max\{0,k-1\}} A(n+1,k) = \frac{1}{2(n+1)!}(A_{n+1}(2) + A(n+1,0)).$$

□

5.4. **Lower-dimensional components.** There is an exceptional possibility of $a_{1,k+1}$ which will give us 1 rather than q. (ADD THIS ABOVE) Then, also $a_{k+1,i_2} = 1$, but we don't have other restrictions on i_2, i_3. Similarly, we may have $a_{k+1,1}$ that will force similar restrictions.

Thus, we arrive to the words in the alphabet E with the first two rules and at most one boundary condition from the third rule.

(1) u, c^+, c^- can be only followed by u, c^+, c_+
(2) b, c_+ and c_- can be only followed by b, c^-, c_-,
(3) the first letter must be either u, c^+, or c^- and the last letter must be either u, c^+, or c_+ (this rule corresponds to the fact that $Q_{1,i_1} \neq 0$ and $Q_{i_n,1} \neq 0$).

So, with one restriction we get 4 times (2 for first or last boundary, and 2 for making new boundary condition by symmetry?) words in dimension $d-2$ with q going up to $d-2$.

For N_k we may try to write an inclusion-exclusion for the interaction of the 4 parts.

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Department of Mathematics, UCLA, Los Angeles, CA, 90095, USA
Email address: pak@math.ucla.edu

Department of Mathematics, Stockholm University, SE - 106 91 Stockholm, Sweden
Email address: shapiro@math.su.se

Department of Mathematics, Stockholm University, SE - 106 91 Stockholm, Sweden
Email address: smirnov@math.su.se

Department of Mathematics, College of Humanities and Sciences, Nihon University, 3-25-40 Sakurajosui, Setagaya-Ku, Tokyo 156-8550, Japan
Email address: yoshida@math.chs.nihon-u.ac.jp