# MATCHING ALGEBRAS AND THEIR DUNKL SUBALGEBRAS

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Abstract. For a finite simple graph $G$, we consider the square-zero face algebra of its matching complex, whose basis is indexed by matchings of $G$, and study the subalgebra generated by signed incidence sums, which we call Dunkl matching elements. We identify each graded component of this subalgebra with a space spanned by matching-restricted products of the graphic linear forms $x_i - x_j$. For the complete graph these spaces are two-row Specht modules; this gives the Catalan-triangle Hilbert series and a presentation of the Dunkl matching algebra by linear and quadratic relations. For arbitrary graphs we formulate the saturation problem of deciding when the selected matching Specht generators span the full two-row Specht module. We show that saturation in degree $k$ forces $k$-connectivity, that saturation in degree 2 is equivalent to 2-connectivity, and that $k$-linked graphs are saturated in degree $k$. We also prove saturation in every possible degree whenever the complement of $G$ is a matching, settling in particular the complete multipartite family with all parts of size 2. We show that the Dunkl matching algebra equals the full matching algebra exactly for forests, and give an explicit Hilbert series formula for unicyclic graphs; this yields minimal matching-equivalent graphs with different Dunkl Hilbert series. Finally, we clarify a related Postnikov–Shapiro quotient: depending on the ambient algebra, its Hilbert series is either the central zonotopal specialization of the Tutte polynomial or a regraded all-terminal reliability polynomial.

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## 1. Introduction

Let $G = (V, E)$ be a finite simple graph. The matching generating polynomial of $G$ records the numbers of sets of pairwise vertex-disjoint edges. In this note we lift this elementary invariant to a graded algebra

$$\mathcal{M}(G) = \mathbf{k}[u_e : e \in E]/(u_e u_f : e \cap f \neq \varnothing).$$

Its distinguished linear forms are the signed incidence sums

$$\theta_v = \sum_{e=(v,w)} u_e - \sum_{e=(w,v)} u_e,$$

where an orientation of $G$ has been chosen. We denote by $\mathcal{D}(G)$ the subalgebra generated by the $\theta_v$ and call it the *Dunkl matching algebra*.

The main observation is that $\mathcal{D}(G)$ has a useful dual polynomial model. For a matching $M$, set

$$p_M(x) = \prod_{\{i,j\} \in M} (x_i - x_j)$$

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(each edge of $M$ appears only once in the product; this is well-defined up to sign). Then $\dim \mathcal{D}(G)_k$ is the dimension of the span of the polynomials $p_M$ over all $k$-matchings of $G$. Thus the Hilbert function is a rank refinement of the matching polynomial. For $K_n$, the products $p_M$ are the usual Specht polynomials of shape $(n-k,k)$, and their span is the irreducible Specht module of that shape.

Products of arbitrary subfamilies of a vector configuration have a rich theory related to Tutte polynomials; see Berget [4] and the zonotopal literature [1, 11]. Our restriction to vertex-disjoint subfamilies is not matroidal: it remembers the incidence of edges at vertices. It also differs from graph Specht modules and their relationship with matching polytopes studied by Liu [16]. The construction lies naturally between these two subjects.

Several nearby constructions attach graded objects to matchings but use a different algebraic mechanism. Numata [18] studies an Artinian Gorenstein algebra obtained by apolarity from a weighted matching generating function and proves a strong Lefschetz property. Li [14] proves strong equivariant log-concavity for the graded permutation representation on the matchings of an arbitrary graph. For perfect matchings of complete graphs, Filmus and Lindzey [10] develop harmonic polynomial models involving Specht polynomials and matching-inclusion matrices. Bedratyuk's graph algebra [3], whose basis is indexed by all simple graphs on a fixed vertex set, also carries Boolean up/down operators and two-row Specht modules. None of these is the incidence-generated subalgebra $\mathcal{D}(G) \subseteq \mathcal{M}(G)$ studied here.

An older but more distant use of algebraic language for graph invariants is Biggs's "algebra of graph types" [5]. Biggs packages a graph by the isomorphism types of its blocks and studies subgraph-counting transforms on the resulting space of graph types. Thus his construction is not an edge-variable algebra attached to one fixed graph; we mention it to distinguish the terminology and the general philosophy.

There is one direct overlap at the level of the abstract complete-graph presentation. Jonsson Kling, Lundqvist, Mohammadi, Orth, and Sáenz-de-Cabezón [13] study the more general almost complete intersections

$$\mathbf{k}[z_1,\ldots,z_n]/(z_1^2,\ldots,z_n^2,(z_1+\cdots+z_n)^q).$$

The algebra appearing below for $\mathcal{D}(K_n)$ is their case $q=1$. The contribution here is therefore not the abstract quotient itself, but its realization inside a matching algebra, its matching-product duality, and the resulting saturation problem for arbitrary graphs.

The adjective "Dunkl" is motivational only. Unlike the Fomin–Kirillov algebra [9], our matching algebra is commutative and kills every product of two incident edge generators.

**AI use statement.** The bulk of this paper was written by GPT-5.6 based on old notes (which contained more questions than proofs) and some tactical prompting; in particular, GPT-5.6 contributed all the proofs in Section 5. We have read and verified the proofs up until that of Theorem 5.16, editing them for clarity and style; we still consider the current form to be improvable, so more edits are likely to come.

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**Notations.** Throughout, $\mathbf{k}$ is a field of characteristic zero. This assumption is mainly used for polarization and for the standard representation theory of the symmetric group.

## 2. The matching algebra

Let $G = (V, E)$ be a finite simple graph. Its edges are 2-element sets of vertices; thus, two edges $e, f$ satisfy $e \cap f \neq \varnothing$ if and only if they have a common endpoint (this allows $e = f$). Two edges $e$ and $f$ are said to be *incident* if $e \cap f \neq \varnothing$.

**Definition 2.1.** The *matching algebra* of $G$ is the commutative $\mathbf{k}$-algebra

$$\mathcal{M}(G) := \mathbf{k}[u_e : e \in E]/\big(u_e u_f : e \cap f \neq \varnothing\big).$$

It is graded by degree, where every generator $u_e$ has degree 1.

For a matching $M \subseteq E$, write

$$u_M := \prod_{e \in M} u_e, \qquad \text{in particular } u_\varnothing := 1.$$

**Proposition 2.2.** *The elements $u_M$, indexed by the matchings of $G$, form a homogeneous $\mathbf{k}$-basis of $\mathcal{M}(G)$. Consequently,*

$$\operatorname{Hilb}(\mathcal{M}(G), t) = \sum_{k \geq 0} m_k(G) t^k,$$

*where $m_k(G)$ is the number of $k$-matchings of $G$.*

*Proof.* The defining ideal is monomial. Its standard monomials are square-free products of generators no two of which correspond to edges with a common endpoint; these are exactly the $u_M$. □

Thus, for the complete graph $K_n$, we have

$$\operatorname{Hilb}(\mathcal{M}(K_n), t) = \sum_{0 \leq k \leq n/2} \frac{n!}{2^k k!(n-2k)!} t^k,$$

and the complete bipartite graph $K_{a,b}$ satisfies

$$\operatorname{Hilb}(\mathcal{M}(K_{a,b}), t) = \sum_{k \geq 0} k! \binom{a}{k} \binom{b}{k} t^k.$$

In particular, $\dim \mathcal{M}(K_n)$ is the number of involutions in $S_n$.

Equivalently, if $\Delta(G)$ denotes the matching complex of $G$ (the independence complex of the line graph $L(G)$), then $\mathcal{M}(G)$ is the Artinian monomial quotient obtained from its Stanley–Reisner ring $\mathbf{k}[\Delta(G)]$ by adjoining the squares $u_e^2$ to the Stanley–Reisner ideal. Thus $\mathcal{M}(G)$ itself is not a Stanley–Reisner ring in the usual sense. Matching complexes and their face rings are classical; see, for example, Björner–Lovász–Vrećica–Živaljević [6] and Jonsson [12, Chapter 5]. Ferrarello and Fröberg [8] explicitly relate the Hilbert series of the ordinary Stanley–Reisner ring of a matching complex to graph-subgraph polynomials.

More directly, Dao and Nair [7] associate to an arbitrary simplicial complex $\Delta$ the square-zero face algebra

$$A(\Delta) := \mathbf{k}[x_1, \ldots, x_m]/(I_\Delta, x_1^2, \ldots, x_m^2),$$

and study its Lefschetz properties. Thus our ambient algebra is exactly $\mathcal{M}(G) = A(\Delta(G))$. Such algebras are also a special case of the quadratic monomial quotients studied by Migliore, Nagel, and Schenck [17]. What is special here is therefore not the ambient square-zero face algebra but the distinguished incidence-generated subalgebra $\mathcal{D}(G)$.

**Example 2.3.** Let $G = P_4$ be the path $1 - 2 - 3 - 4$. Then

$$\mathcal{M}(P_4) = \mathbf{k}[u_{12}, u_{23}, u_{34}]/(u_{12}^2, u_{23}^2, u_{34}^2, u_{12}u_{23}, u_{23}u_{34}).$$

Thus

$$1, \quad u_{12}, \quad u_{23}, \quad u_{34}, \quad u_{12}u_{34}$$

is a basis. The only nonzero product of two edge generators is $u_{12}u_{34}$, so

$$\mathrm{Hilb}(\mathcal{M}(P_4), t) = 1 + 3t + t^2, \qquad (u_{12} + u_{34})^2 = 2u_{12}u_{34}.$$

By contrast, for the triangle $K_3$ every two edges meet, so $\mathcal{M}(K_3)$ has basis $1, u_{12}, u_{13}, u_{23}$ and its positive-degree ideal has square zero.

## 3. Dunkl matching elements

Fix an orientation of $G$. If $e$ is oriented from $v$ to $w$, let $\varepsilon_{v,e} = 1$, $\varepsilon_{w,e} = -1$, and let all other incidence coefficients $\varepsilon_{t,e}$ (with $t \in V \setminus \{v, w\}$) be zero.

**Definition 3.1.** For $v \in V$, define

$$\theta_v = \sum_{e \in E} \varepsilon_{v,e} u_e.$$

The *Dunkl matching algebra* is the graded subalgebra

$$\mathcal{D}(G) := \mathbf{k}[\theta_v : v \in V] \subseteq \mathcal{M}(G).$$

Reversing the orientation of an edge amounts to replacing the corresponding generator $u_e$ by $-u_e$; hence the isomorphism type of $\mathcal{D}(G)$ is orientation-independent. We have

$$\theta_v^2 = 0, \qquad \sum_{v \in C} \theta_v = 0 \tag{1}$$

for every vertex $v$ and every connected component $C$ of $G$.

We also fix the polynomial notation used in this section. If an edge is oriented as $e = (i, j)$, put $p_e := x_i - x_j$; for a matching $M$, define

$$p_M := \prod_{e \in M} p_e = \prod_{e=(i,j) \in M} (x_i - x_j), \qquad p_\varnothing := 1. \tag{2}$$

Reversing the orientation of an edge changes the corresponding $p_M$ only by a sign, so all spans considered below are orientation-independent.

Consider the $\mathbf{k}$-vector space $\mathbf{k}^V$ with basis $V$; its standard basis vectors will be denoted by $e_v$. Let $S^k(\mathbf{k}^V)$ denote its $k$th symmetric power. Multiplication gives a linear map

$$\mu_k : S^k(\mathbf{k}^V) \longrightarrow \mathcal{M}(G)_k, \qquad e_{v_1} \cdots e_{v_k} \longmapsto \theta_{v_1} \cdots \theta_{v_k},$$

whose image is $\mathcal{D}(G)_k$.

**Theorem 3.2** (Matching-product realization)**.** *For every $k \geq 0$,*

$$\dim \mathcal{D}(G)_k = \dim \operatorname{span}_{\mathbf{k}}\{p_M : M \in \operatorname{Match}_k(G)\}.$$

*More precisely, under the polarization identification $S^k(\mathbf{k}^V)^* \cong \mathbf{k}[x_v : v \in V]_k$, the transpose $\mu_k^*$ sends the functional dual to $u_M$ to $k!p_M$.*

*Proof.* For a $k$-matching $M$, let $u_M^* \in \mathcal{M}(G)_k^*$ denote coefficient extraction at $u_M$. This is the $\mathbf{k}$-linear map sending each polynomial in $\mathcal{M}(G)_k$ to its $u_M$-coefficient (which is well-defined, since $u_M$ is not in the monomial ideal factored out in $\mathcal{M}(G)$). If $a = (a_v)_{v \in V} \in \mathbf{k}^V$, then

$$\sum_v a_v \theta_v = \sum_{e=(i,j) \text{ is an oriented edge}} (a_i - a_j) u_e$$

(keep in mind that $G$ is oriented, so that each edge appears only with one orientation). Since products of incident edges vanish, we obtain

$$\begin{aligned} \left(\sum_v a_v \theta_v\right)^k &= \sum_{\substack{e_1=(i_1,j_1),\ \ldots,\ e_k=(i_k,j_k) \\ \text{are } k \text{ disjoint oriented edges}}} (a_{i_1} - a_{j_1}) \cdots (a_{i_k} - a_{j_k}) u_{e_1} \cdots u_{e_k} \\ &= k! \sum_{N \text{ is a } k\text{-matching}} \left( \prod_{\{i,j\} \in N} (a_i - a_j) \right) u_N \end{aligned}$$

(because each $k$-matching $N$ can be written as $\{e_1, e_2, \ldots, e_k\}$ in exactly $k!$ many ways, corresponding to the $k!$ possible orderings of its $k$ edges). Therefore,

$$u_M^* \left( \left( \sum_v a_v \theta_v \right)^k \right) = k! \prod_{\{i,j\} \in M} (a_i - a_j) = k! p_M(a). \tag{3}$$

In characteristic zero, polarization identifies $S^k(\mathbf{k}^V)^*$ with the space of homogeneous degree-$k$ polynomials on $\mathbf{k}^V$: a functional $\lambda$ corresponds to

$$a \longmapsto \lambda \left( \left( \sum_v a_v e_v \right)^k \right).$$

Under this identification, (3) says exactly that

$$\mu_k^*(u_M^*) = k! p_M.$$

Thus, if

$$W_k := \operatorname{span}\{p_M : M \in \operatorname{Match}_k(G)\},$$

then $\operatorname{im} \mu_k^* = W_k$ (since the $u_M^*$ span $\mathcal{M}(G)_k^*$). Hence,

$$\dim W_k = \dim \operatorname{im} \mu_k^* = \operatorname{rank} \mu_k^* = \operatorname{rank} \mu_k = \dim \operatorname{im} \mu_k = \dim \mathcal{D}(G)_k.$$

The proof is now complete, but let us also rewrite the last argument in a more natural way. The transpose description given above induces a canonical perfect pairing

$$\mathcal{D}(G)_k \times W_k \longrightarrow \mathbf{k}.$$

Indeed, for any linear map $T : X \to Y$, the natural pairing

$$\operatorname{im} T \times \operatorname{im} T^* \to \mathbf{k}, \qquad (Tx, T^*\varphi) \mapsto \varphi(Tx) = (T^*\varphi)(x)$$

between $\operatorname{im} T$ and $\operatorname{im} T^*$ is perfect, since

$$\operatorname{im} T \cong X/\ker T, \qquad \operatorname{im} T^* = (\ker T)^{\perp} \cong (X/\ker T)^*.$$

In the present case (where $X = S^k(\mathbf{k}^V)$ and $Y = \mathcal{M}(G)_k$ and $T = \mu_k$), the pairing has the concrete form

$$\langle d, p_M \rangle = \frac{1}{k!}[u_M]d \qquad (d \in \mathcal{D}(G)_k,\ M \in \mathrm{Match}_k(G)). \tag{4}$$

Thus $\mathcal{D}(G)_k$ and $W_k$ are in perfect duality, and in particular $\dim \mathcal{D}(G)_k = \dim W_k$, as claimed. $\square$

We write

$$P_k(G) := \operatorname{span}\{p_M : M \in \mathrm{Match}_k(G)\}.$$

The theorem says that $\mathrm{Hilb}(\mathcal{D}(G), t) = \sum_k \dim P_k(G) t^k$. Although $m_k(G)$ counts the chosen generators of $P_k(G)$, their linear relations contain additional information about $G$.

**Example 3.3** (Small Dunkl matching algebras)**.** Orient every displayed edge from the smaller endpoint to the larger one.

For the path $P_4$, we have

$$\theta_1 = u_{12}, \qquad \theta_2 = -u_{12} + u_{23}, \qquad \theta_3 = -u_{23} + u_{34}, \qquad \theta_4 = -u_{34}.$$

Hence

$$u_{12} = \theta_1, \qquad u_{23} = \theta_1 + \theta_2, \qquad u_{34} = -\theta_4,$$

so every edge generator already lies in $\mathcal{D}(P_4)$. Therefore

$$\mathcal{D}(P_4) = \mathcal{M}(P_4), \qquad \mathrm{Hilb}(\mathcal{D}(P_4), t) = 1 + 3t + t^2.$$

(The same argument shows $\mathcal{D}(G) = \mathcal{M}(G)$ for every forest $G$, since the incidence matrix of a forest has rank $|E|$.)

For the triangle $K_3$,

$$\theta_1 = u_{12} + u_{13}, \qquad \theta_2 = -u_{12} + u_{23}, \qquad \theta_3 = -u_{13} - u_{23}.$$

Thus $\theta_1 + \theta_2 + \theta_3 = 0$, and all products of two $\theta_i$ vanish because every pair of edges of $K_3$ is incident. Consequently

$$\mathcal{D}(K_3) = \mathbf{k} \oplus \operatorname{span}\{\theta_1, \theta_2\}, \qquad \mathrm{Hilb}(\mathcal{D}(K_3), t) = 1 + 2t,$$

so here $\mathcal{D}(K_3)$ is a proper subalgebra of $\mathcal{M}(K_3)$.

The first nontrivial degree-2 example is $K_4$. Put

$$m_1 = u_{12}u_{34}, \qquad m_2 = u_{13}u_{24}, \qquad m_3 = u_{14}u_{23}.$$

These are the three degree-2 basis elements of $\mathcal{M}(K_4)$. Direct multiplication gives

$$\theta_1\theta_2 = m_2 + m_3, \qquad \theta_1\theta_3 = m_1 - m_3, \qquad \theta_2\theta_3 = -m_1 - m_2.$$

The three displayed vectors have one linear relation and span a 2-dimensional space; since $\theta_4 = -(\theta_1 + \theta_2 + \theta_3)$ and $\theta_i^2 = 0$, they span all of $\mathcal{D}(K_4)_2$. Hence

$$\mathrm{Hilb}(\mathcal{D}(K_4), t) = 1 + 3t + 2t^2.$$

This is the smallest example in which the matching algebra has more degree-2 basis elements than the Dunkl matching algebra has degree-2 dimensions.

**Proposition 3.4.** *The following properties hold.*

1. *If $G = G_1 \sqcup G_2$, then $\mathcal{D}(G) \cong \mathcal{D}(G_1) \otimes \mathcal{D}(G_2)$.*
2. *$\mathcal{D}(G)_k = 0$ for $k > \nu(G)$, where $\nu(G)$ is the matching number of $G$ (that is, the size of a largest matching of $G$).*
3. *$\dim \mathcal{D}(G)_1 = |V| - c(G)$, where $c(G)$ is the number of connected components of $G$.*
4. *If $H$ is a spanning subgraph of $G$, then*

$$P_k(H) \subseteq P_k(G), \qquad \dim \mathcal{D}(H)_k \leq \dim \mathcal{D}(G)_k.$$

*Proof.* The first assertion follows from the disjoint sets of variables and incidence sums. The second follows from Theorem 3.2. For the third, $P_1(G)$ is the span of the graphic vectors $x_i - x_j$ and therefore has the rank of the graphic matroid. The last assertion is immediate. □

## 4. Complete graphs and two-row Specht modules

Consider the complete graph $K_n$. Let $0 \leq k \leq \lfloor n/2 \rfloor$. From a $k$-matching $M$ of $K_n$, form a Young tableau of shape $(n-k, k)$ by placing the two endpoints of each edge in a column of height 2 and the unmatched vertices in the remaining boxes of the first row. The order of the columns does not matter, and reversing the order within a two-box column changes the associated Specht polynomial only by sign. Thus that Specht polynomial is $p_M$ up to sign. Note that $P_k(K_n)$ is a representation of the symmetric group $S_n$, since $K_n$ has $S_n$-symmetry.

**Theorem 4.1.** *As an $S_n$-module,*

$$P_k(K_n) \cong S^{(n-k,k)}.$$

*Consequently,*

$$\mathrm{Hilb}(\mathcal{D}(K_n), t) = \sum_{0 \leq k \leq n/2} \left( \binom{n}{k} - \binom{n}{k-1} \right) t^k.$$

*Proof.* The span of the Specht polynomials of all tableaux of a fixed shape is the Specht module of that shape. For the two-row shape $(n-k, k)$, its columns of height 2 form a $k$-matching and its singleton columns are the unmatched vertices. Thus these tableaux give precisely the polynomials defining $P_k(K_n)$. The hook-length formula gives

$$f^{(n-k,k)} = \binom{n}{k} - \binom{n}{k-1} = \frac{n-2k+1}{n-k+1} \binom{n}{k}.$$

□

The coefficients of $\mathrm{Hilb}(\mathcal{D}(K_n), t)$ form the Catalan triangle and count standard Young tableaux of the fixed shape $(n-k, k)$.

*Remark* 4.2 (An exterior analogue). Bergeron, Chan, Soltani, and Zabrocki [2] study a strikingly parallel construction in the exterior algebra

$$R_n = \bigwedge(\xi_1, \ldots, \xi_n).$$

For a noncrossing pairing $C = ((i_1, j_1), \dots, (i_k, j_k))$, ordered so that $j_1 < \cdots < j_k$, they set

$$\Delta_C := (\xi_{j_1} - \xi_{i_1}) \cdots (\xi_{j_k} - \xi_{i_k}).$$

Their Proposition 2.10 shows that the elements $\Delta_{C(\alpha)}$ indexed by ballot sequences are linearly independent, and their Corollary 4.5 shows that they form a basis of the degree-$k$ exterior quasisymmetric harmonic space $EQH_{n,k}$. Thus

$$\dim EQH_{n,k} = f^{(n-k,k)} = \dim P_k(K_n).$$

There is in fact a very explicit vector-space isomorphism between these two spaces. The same ballot sequences give noncrossing matchings $C(\alpha)$, and the commutative polynomials $p_{C(\alpha)}$ are linearly independent: with the usual lexicographic order, the monomial obtained by choosing the right endpoint of every pair is a distinguished extremal monomial and has different support for different $\alpha$. Since there are $f^{(n-k,k)}$ such ballot sequences, these polynomials form a basis of $P_k(K_n)$. Hence there is a unique isomorphism

$$\Psi_k : P_k(K_n) \longrightarrow EQH_{n,k}, \qquad p_{C(\alpha)} \longmapsto \Delta_{C(\alpha)}. \tag{5}$$

So, at the level of graded vector spaces with their preferred noncrossing bases, the commutative and exterior pictures are very close.

They are nevertheless genuinely different if one asks to match the *whole* family of matching products. There is an obvious linear isomorphism from the squarefree degree-$k$ part of $\mathbf{k}[x_1, \dots, x_n]$ to $\bigwedge^k \mathbf{k}^n$, sending $x_{a_1} \cdots x_{a_k}$ (with $a_1 < \cdots < a_k$) to $\xi_{a_1} \cdots \xi_{a_k}$. But this map does not send $p_M$ to the corresponding exterior product: when the chosen endpoints are reordered, anticommutativity contributes signs which depend on the chosen term of the expansion.

There is also a representation-theoretic way to see the obstruction. Let

$$W = \operatorname{span}\{\xi_i - \xi_j : 1 \le i, j \le n\},$$

the standard $(n-1)$-dimensional representation of $S_n$. For a $k$-matching $M$, let $\Delta_M$ denote the exterior product of the $k$ corresponding differences, in any fixed order. These $k$ differences are linearly independent, so $\Delta_M \neq 0$. The $S_n$-orbit of any one such $\Delta_M$ consists, up to sign, of all exterior matching products. Since

$$\bigwedge^k W \cong S^{(n-k,1^k)}$$

is irreducible in characteristic zero, it follows that

$$\operatorname{span}\{\Delta_M : M \in \operatorname{Match}_k(K_n)\} = \bigwedge^k W \cong S^{(n-k,1^k)}. \tag{6}$$

Thus the complete commutative matching family spans the two-row module $S^{(n-k,k)}$, whereas the complete exterior matching family spans the hook module $S^{(n-k,1^k)}$. Their dimensions are respectively

$$\binom{n}{k} - \binom{n}{k-1} \qquad \text{and} \qquad \binom{n-1}{k},$$

which agree only for $k = 0, 1$.

Already for $n = 4$ and $k = 2$, this is visible in the Garnir relation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$$

in commuting variables. The three corresponding exterior products are linearly independent (they span $\bigwedge^2 W$, which has dimension 3). Consequently there cannot be a single linear isomorphism which sends $p_M$ to $\Delta_M$ for every matching $M$. This also clarifies Remark 2.11 of [2]: the ballot/noncrossing pieces are linearly isomorphic as in (5), but their natural ambient $S_n$-module and algebra structures are different. Even the equality of Hilbert series does not extend to a graded-algebra isomorphism with the exterior coinvariant quotient: every degree-one element of an exterior algebra (and hence of its quotients) has square zero, whereas for $n \geq 4$ the element $z_1 + z_2 \in \mathcal{D}(K_n)$ has square $2z_1z_2 \neq 0$ under the presentation of Theorem 4.3.

**Theorem 4.3.** *There is an isomorphism of graded algebras*

$$\mathcal{D}(K_n) \cong \mathbf{k}[z_1, \dots, z_n]/(z_1 + \cdots + z_n, z_1^2, \dots, z_n^2), \qquad z_i \longmapsto \theta_i.$$

As an abstract commutative algebra, the quotient in Theorem 4.3 is the case $q = 1$ of the family

$$\mathbf{k}[z_1, \dots, z_n]/(z_1^2, \dots, z_n^2, (z_1 + \cdots + z_n)^q)$$

studied in detail by Jonsson Kling et al. [13]; among other things, they determine Gröbner bases and Hilbert series for the general family and give another proof of the strong Lefschetz property of the squarefree algebra. We nevertheless include the short Boolean-lattice argument needed for the present case.

*Proof.* The relations in the displayed ideal hold by (1), so there is a surjection

$$\Phi : \mathbf{k}[z_1, \dots, z_n]/(z_1 + \cdots + z_n, z_1^2, \dots, z_n^2) \to \mathcal{D}(K_n)$$

that sends each $z_i$ to $\theta_i$. We must show that this surjection $\Phi$ is an isomorphism. It clearly suffices to show that its domain and its codomain have equal Hilbert functions. Put

$$A := \mathbf{k}[z_1, \dots, z_n]/(z_1^2, \dots, z_n^2) \qquad \text{and} \qquad L := z_1 + \cdots + z_n \in A.$$

Note that the $\mathbf{k}$-vector space $A$ has dimension $2^n$ and a basis consisting of the (residue classes of the) squarefree monomials

$$z_S := \prod_{i \in S} z_i \qquad \text{for all } S \subseteq [n].$$

We must compute the Hilbert function of $A/(L)$. This boils down to determining $\dim(A/(L))_k$ for all $k$. This is rather well-known; indeed, the maximal-rank statement below is the classical Boolean up-operator phenomenon underlying the strong Lefschetz property of this squarefree complete intersection; see also Stanley [20] and [13, §3.1]. We give the elementary argument here for the sake of completeness. We claim that multiplication by $L$ – that is, the $\mathbf{k}$-linear map

$$U_r : A_r \longrightarrow A_{r+1}, \qquad f \longmapsto Lf$$

– has maximal rank in every degree (i.e., is injective for $r < n/2$ and surjective for $r \geq n/2$). It is enough to prove this over $\mathbb{R}$. Indeed, in the squarefree monomial bases, the matrix of $U_r$ has only 0's and 1's, so a maximal minor which is nonzero over $\mathbb{R}$ is a nonzero integer and therefore remains nonzero over every field of characteristic zero.

The maps $U_r : A_r \to A_{r+1}$ for all $r$ are the graded pieces of a $\mathbf{k}$-linear map $U : A \to A$, which we call the *up-operator* since it raises degrees by 1. Define the *down-operator* $D : A \to A$ as the $\mathbf{k}$-linear map given by

$$D(z_S) := \sum_{i \in S} z_{S \setminus \{i\}} \qquad \text{for each } S \subseteq [n].$$

This operator decreases degrees by 1; that is, $D(A_r) \subseteq A_{r-1}$ for each $r$.

Give the vector space $A$ the inner product for which the squarefree monomials $z_S$ are orthonormal. Then $D$ is the adjoint of $U$. A direct calculation gives, on $A_r$, the equality

$$DU - UD = (n - 2r)\, \mathrm{id}_{A_r}\,. \tag{7}$$

Indeed, for $|S| = r$, the mixed terms $z_{S \setminus \{i\} \cup \{j\}}$ occur once in both $DU(z_S)$ and $UD(z_S)$, while $z_S$ occurs respectively $n - r$ and $r$ times.

If $r < n/2$ and $0 \neq f \in A_r$, then the adjointness of $D$ and $U$ yields

$$\begin{aligned}
\|Uf\|^2 = \langle DUf, f\rangle &= \langle UDf + (n-2r)f, f\rangle \qquad (\text{since } DU - UD = (n-2r)\,\mathrm{id}_{A_r}) \\
&= \langle UDf, f\rangle + (n-2r)\|f\|^2 \\
&= \|Df\|^2 + (n-2r)\|f\|^2 \qquad (\text{again since } D \text{ and } U \text{ are adjoint}) \\
&> 0 \qquad (\text{by positive definiteness and since } n - 2r > 0)\,.
\end{aligned}$$

Thus $U_r$ is injective on the lower half of $A$ (that is, for $r < n/2$). On the other hand, the transpose of $U_r$ is $D : A_{r+1} \to A_r$, and under the complement identification $z_S \leftrightarrow z_{[n]\setminus S}$ this down-operator becomes $U_{n-r-1}$. Hence the injectivity just proved implies surjectivity of $U_r$ on the upper half (i.e., for $r \geq n/2$). This proves the claim.

Now $\dim A_k = \binom{n}{k}$, and the degree-$k$ component of $A/(L)$ is the cokernel of $U_{k-1} : A_{k-1} \to A_k$. Therefore

$$\dim(A/(L))_k = \max\left\{\binom{n}{k} - \binom{n}{k-1}, 0\right\}.$$

This agrees with $\dim \mathcal{D}(K_n)_k$ by Theorem 4.1. The surjection is therefore an isomorphism in every degree. ☐

*Remark* 4.4. The characteristic-zero hypothesis matters in Theorem 4.3; the ranks of the Boolean up-operators and the Specht-module dimensions can change in small positive characteristic.

## 5. Arbitrary graphs

We begin with conventions. For a finite vertex set $W$, let $K_W$ denote the complete graph with vertex set $W$. Thus $K_{[n]} = K_n$. The graphs $K_n$ and $K_W$ are isomorphic whenever $|W| = n$; thus, they differ only in the labeling of the vertices. Thus, for the rest of this section, we **assume that the vertex set $V$ of our graph $G$ is $[n] = \{1, 2, \ldots, n\}$**, but we expect the reader to understand that all our results hold equally well for any other complete graph with $n$ vertices.

The spaces $P_k(G)$ can be computed in finite time: enumerate the $k$-matchings, expand their products in the monomial basis of $\mathbf{k}[x_v]_k$, and take their span (which is the column space of a matrix). More conceptually, the map

$$\mathbf{k}\,\mathrm{Match}_k(G) \longrightarrow S^{(n-k,k)}, \qquad M \longmapsto p_M,$$

places $P_k(G)$ inside a fixed two-row Specht module. The graph determines a selected collection of its matching generators. Thus the following problem is naturally a Specht-matroid question: the Specht matroid introduced by Wiltshire-Gordon, Woo, and Zajaczkowska [21] records the linear dependences among the classical Specht generators, while here we ask whether the subcollection selected by the $k$-matchings of $G$ spans.

**Definition 5.1.** Let $0 \le k \le \lfloor n/2 \rfloor$. We say that a graph $G$ on $n$ vertices is *$k$-saturated* if

$$P_k(G) = P_k(K_n).$$

**Problem 5.2.** *Characterize the $k$-saturated graphs, either for a prescribed $k$ or simultaneously for all $k \le \nu(G)$.*

5.1. **The elementary Garnir relations.** The elementary Garnir relations already give useful information. For distinct vertices write $p_{ij} = x_i - x_j$, so that $p_{ji} = -p_{ij}$.

**Lemma 5.3** (Elementary Garnir relations)**.** *Let $N$ be a matching disjoint from the displayed vertices. Then*

$$p_N(p_{ab} + p_{bc} + p_{ca}) = 0, \tag{8}$$

$$p_N(p_{ab}p_{cd} - p_{ac}p_{bd} + p_{ad}p_{bc}) = 0. \tag{9}$$

*Proof.* Both identities follow by expanding $p_{ij} = x_i - x_j$. They are the three-vertex and four-vertex straightening relations for the two-row Specht polynomial model. Multiplication by $p_N$ is legitimate because $N$ is disjoint from all displayed vertices. □

5.2. **$k$-saturation implies $k$-connectivity.** The first general obstruction is connectivity. We say that $G$ is *$k$-connected* if $G$ has more than $k$ vertices and the graph $G - S$ is connected for every set $S \subset V$ with $|S| < k$. (In particular, the graph $G$ is 2-connected if it has at least three vertices and no cut-vertex.)

**Proposition 5.4** (Connectivity obstruction)**.** *Let $1 \le k \le \lfloor n/2 \rfloor$. If $G$ is $k$-saturated, then $G$ is $k$-connected.*

*Proof.* Suppose that $S \subset V$ has size $s < k$ and that $G - S$ has at least two components $C_1, \ldots, C_m$. In the polynomial ring $R = \mathbf{k}[x_v : v \in V]$, define the ideal

$$J = (x_u - x_v : u, v \in C_i \text{ for some } i).$$

The quotient $R/J$ is a polynomial ring with one variable for each $C_i$ and one variable for each vertex of $S$. Hence a linear factor $p_{uv}$ belongs to $J$ exactly when $u$ and $v$ lie in the same component $C_i$.

Every $k$-matching of $G$ has at most $s$ edges incident with $S$. All its remaining edges lie inside the components $C_i$, so

$$p_M \in J^{k-s} \qquad \text{for each } M \in \mathrm{Match}_k(G). \tag{10}$$

Thus $P_k(G) \subseteq J^{k-s}$.

On the other hand, choose vertices $c_1 \in C_1$ and $c_2 \in C_2$. Since $n \geq 2k$ and $s < k$, there are enough further vertices outside $S$ to pair each vertex of $S$ with a distinct vertex outside $S \cup \{c_1, c_2\}$. Together with the edge $c_1c_2$, these give $s+1$ pairwise disjoint edges of $K_n$ whose linear factors do not belong to $J$. Complete them arbitrarily to a $k$-matching $Q$ of $K_n$. We shall show that $p_Q \notin J^{k-s}$. Since $P_k(G) \subseteq J^{k-s}$ by (10), this will show $P_k(K_n) \nsubseteq J^{k-s}$ and hence contradict $k$-saturation.

Choose one vertex $c_i \in C_i$ in each component and use the coordinates

$$z_i := x_{c_i}, \qquad y_u := x_u - x_{c_i} \quad (\text{for } u \in C_i \setminus \{c_i\}),$$

together with the variables $x_s$ for $s \in S$. In these coordinates

$$R = \mathbf{k}[z_1, \dots, z_m,\ x_s\ (s \in S),\ y_u], \qquad J = (y_u).$$

Introduce an auxiliary variable $t$ and define the $R$-algebra morphism $\sigma_t : R \to R[t]$ that fixes the $z_i$ and the $x_s$ and sends every $y_u$ to $ty_u$. If $f \in J^d$, then $\sigma_t(f)$ is divisible by $t^d$.

For a linear factor $p_{uv} = x_u - x_v$, the polynomial $\sigma_t(p_{uv})$ is divisible by $t$ exactly when $u$ and $v$ lie in the same component $C_i$; in that case it is divisible by $t$ exactly once. Otherwise its constant term at $t = 0$ is a nonzero linear form in the $z_i$ and $x_s$. Since the polynomial ring is a domain, the $t$-adic order of a product of such factors is therefore exactly the number of factors that belong to $J$.

The matching $Q$ contains the $s+1$ chosen edges whose factors are not in $J$, so at most $k-s-1$ of its factors belong to $J$. Hence $\sigma_t(p_Q)$ is not divisible by $t^{k-s}$, and therefore

$$p_Q \notin J^{k-s}.$$

This is the contradiction we need to complete our proof.

(The above argument using $\sigma_t$ resembles the use of Rees algebras in the study of ideal powers.) $\square$

For $k \geq 3$, $k$-connectivity is not sufficient.

**Example 5.5.** Let $G$ be the triangular prism. It is 3-connected, but it has only four perfect matchings. Hence

$$\dim P_3(G) \leq 4 < 5 = \binom{6}{3} - \binom{6}{2} = \dim P_3(K_6),$$

so $G$ is not 3-saturated.

5.3. **2-connectivity implies 2-saturation.** In degree 2 the connectivity obstruction is sharp. To prove this, we need some lemmas:

**Lemma 5.6** (Two-row branching sequence)**.** *Let $V$ be a finite set with $|V| \geq 4$, let $v \in V$, and put $V' := V \setminus \{v\}$. Every element of $P_2(K_V)$ is at most linear in $x_v$, so coefficient extraction at $x_v$ defines a map $[x_v] : P_2(K_V) \to P_1(K_{V'})$. There is a short exact sequence*

$$0 \longrightarrow P_2(K_{V'}) \longrightarrow P_2(K_V) \xrightarrow{[x_v]} P_1(K_{V'}) \longrightarrow 0. \tag{11}$$

*Proof.* The first map is the evident inclusion, and its image lies in the kernel. The last map is surjective: for distinct $c, d \in V'$, choose $a \in V' \setminus \{c, d\}$; then, up to sign,

$$[x_v](p_{va}p_{cd}) = p_{cd}.$$

The dimension formula in Theorem 4.1 gives

$$\dim P_2(K_V) - \dim P_2(K_{V'}) = |V| - 2 = \dim P_1(K_{V'}),$$

so the kernel has exactly the dimension of $P_2(K_{V'})$. □

**Lemma 5.7** (A degree-two vertex)**.** *Every finite* 2*-connected graph contains a spanning subgraph that is edge-minimal subject to being* 2*-connected. Every such edge-minimal subgraph has a vertex of degree* 2*.*

*Proof.* The first assertion follows simply by deleting edges as long as 2-connectivity is preserved. For the second we use the standard ear characterization of 2-connected graphs; see [15, Theorem 25.4]. Let $R_1, \dots, R_q$ be an ear decomposition of the edge-minimal graph $H$. If $H$ is a cycle, every vertex has degree 2. Otherwise, the final ear $R_q$ cannot have length one: deleting that one edge would leave the union of the preceding ears, which is still a spanning 2-connected graph. Thus $R_q$ has an internal vertex. Since no later ear exists, every internal vertex of $R_q$ has degree exactly 2 in $H$. □

**Lemma 5.8** (Suppressing a degree-two vertex)**.** *Let $H$ be a* 2*-connected graph with $|V(H)| \geq 4$, and let $v \in V(H)$ have degree* 2*, with neighbors $a, b$. Put $V' := V(H) \setminus \{v\}$ and*

$$H^+ := (H - v) + ab, \qquad H^- := H^+ - ab = (H - v) - ab.$$

*Thus $H^-$ is obtained from $H - v$ by deleting $ab$ if that edge was already present; if $ab \notin E(H - v)$, then simply $H^- = H - v$. Then $H^+$ is* 2*-connected and $H^-$ is connected.*

*Proof.* First consider $H^+$. It has at least three vertices. Let $w \in V'$. If $w \notin \{a, b\}$, then $H - w$ is connected. Given two vertices of $H^+ - w$, choose a path between them in $H - w$. Whenever this path uses $v$, it uses the segment $a - v - b$ (or $b - v - a$), since $v$ has only the two neighbors $a, b$; replacing this segment by the edge $ab$ gives a path in $H^+ - w$. Thus $H^+ - w$ is connected. If $w = a$ (and similarly if $w = b$), then $H - a$ is connected and $v$ is a leaf of $H - a$; deleting that leaf leaves $H^+ - a = (H - a) - v$ connected. Hence deletion of any vertex leaves $H^+$ connected, so $H^+$ is 2-connected.

Now consider $H^-$. Since $H$ is 2-connected, $H - v$ is connected. If $ab \notin E(H - v)$, there is nothing more to prove. Suppose instead that $ab \in E(H - v)$ and that deleting $ab$ disconnects $H - v$. Then $ab$ is a bridge of $H - v$. Let $A$ and $B$ be the two components of $(H - v) - ab$, with $a \in A$ and $b \in B$. If $A \neq \{a\}$, then in $H - a$ the nonempty set $A \setminus \{a\}$ has no edge to the rest of the graph: there is no such edge in $H - v - ab$, and the only edges incident with $v$ are $av$ and $bv$. This contradicts the connectedness of $H - a$. Hence $A = \{a\}$, and similarly $B = \{b\}$. Thus $V(H) = \{a, b, v\}$, contradicting $|V(H)| \geq 4$. Therefore $H^-$ is connected.

For example, if $H = C_4$ and $v$ is any vertex, then its neighbors $a, b$ are not adjacent in $H$. Thus $H - v$ is a three-vertex path, $H^+$ is a triangle, and $H^- = H - v$ is that same path. □

**Theorem 5.9** (Degree-two saturation)**.** *Let $G$ be a graph on $n \geq 4$ vertices. Then*

$$P_2(G) = P_2(K_n) \quad \Longleftrightarrow \quad G \text{ is 2-connected.}$$

*Proof.* The forward implication is Proposition 5.4. Conversely, let $G$ be 2-connected and choose, by Lemma 5.7, a spanning edge-minimal 2-connected subgraph $H \subseteq G$ and a degree-2 vertex $v$ of $H$, with neighbors $a, b$. Put $V' := V \setminus \{v\}$ and

$$H^+ = (H - v) + ab, \qquad H^- = (H - v) - ab.$$

By Lemma 5.8, $H^+$ is 2-connected and $H^-$ is connected. If $|V'| = 3$, then $H^+ = K_{V'}$; otherwise induction on $n$ gives

$$P_2(H^+) = P_2(K_{V'}).$$

We want to prove that $P_2(G) = P_2(K_n)$. Since $P_2(H) \subseteq P_2(G) \subseteq P_2(K_V) = P_2(K_n)$, it is enough to prove $P_2(H) = P_2(K_V)$.

Recall the elementary linear-algebra fact that if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of vector spaces, and if $W \subseteq B$ is a subspace that contains $A$ and surjects onto $C$, then $W = B$. Applying this to the short exact sequence (11) and the subspace $P_2(H)$ of $P_2(K_V)$, we see that it suffices to show that $P_2(H)$ contains $P_2(K_{V'})$ and surjects onto $P_1(K_{V'})$ under $[x_v]$.

We start with the first claim. Since $P_2(H^+) = P_2(K_{V'})$, it is enough to show that every matching polynomial $p_M \in P_2(H^+)$ belongs to $P_2(H)$. So consider a 2-matching $M$ of $H^+$. If $ab \notin M$, then $M$ is already a 2-matching of $H$, and thus $p_M \in P_2(H)$ follows. If $ab \in M$, then we let $cd$ be the other edge of $M$ (so $p_M = p_{ab}p_{cd}$); then, the triangle Garnir relation gives

$$p_{ab}p_{cd} = (p_{av} + p_{vb})p_{cd} = p_{av}p_{cd} + p_{vb}p_{cd},$$

and both terms on the right come from 2-matchings of $H$. Hence

$$P_2(K_{V'}) = P_2(H^+) \subseteq P_2(H).$$

It remains to prove the surjection claim. For every edge $cd$ of the connected graph $H^-$, at least one of $a, b$ is not incident with $cd$. Pairing $cd$ with the corresponding edge $av$ or $bv$ gives a 2-matching of $H$ whose $x_v$-coefficient is, up to sign, $p_{cd}$. Therefore

$$[x_v]P_2(H) \supseteq P_1(H^-) = P_1(K_{V'}),$$

where the last equality follows from the connectivity of $H^-$. That is, $P_2(H)$ surjects onto $P_1(K_{V'})$ under $[x_v]$. The preceding linear-algebra observation now yields $P_2(H) = P_2(K_V)$, and hence $P_2(G) = P_2(K_n)$. $\square$

5.4. ***k*-linkedness implies *k*-saturation.** There is also a simple sufficient condition in every degree.

**Definition 5.10.** A graph $G$ is *$k$-linked* if, whenever $a_1, b_1, \ldots, a_k, b_k$ are distinct vertices, there exist pairwise vertex-disjoint paths $Q_i$ joining $a_i$ to $b_i$ for $1 \leq i \leq k$. (Paths are defined with respect to the undirected graph $G$; they need not conform with the orientation we chose for the edges.)

**Proposition 5.11** (Linkedness criterion)**.** *If $G$ is $k$-linked, then $G$ is $k$-saturated.*

*Proof.* Take an arbitrary $k$-matching $M = \{\{a_1, b_1\}, \ldots, \{a_k, b_k\}\}$ of $K_n$, and choose pairwise vertex-disjoint paths $Q_i$ in $G$ joining $a_i$ to $b_i$. Along each path,

$$x_{a_i} - x_{b_i} = \sum_{uv \in E(Q_i)} \varepsilon_{uv}(x_u - x_v), \qquad \varepsilon_{uv} \in \{\pm 1\}.$$

Multiplying these $k$ identities expresses $p_M$ as a linear combination of products obtained by choosing one edge from each $Q_i$. Since the paths are vertex-disjoint, every such choice is a $k$-matching of $G$. Hence $p_M \in P_k(G)$, and the result follows. □

Thus, for $1 \leq k \leq \lfloor n/2 \rfloor$, we have the useful chain

$$k\text{-linked} \quad \Longrightarrow \quad k\text{-saturated} \quad \Longrightarrow \quad k\text{-connected}.$$

The second implication is an equivalence for $k = 1, 2$, but Example 5.5 shows that it is not an equivalence for $k = 3$.

5.5. **Matching-complement implies $k$-saturation.** The Garnir relations also settle the complete multipartite example mentioned in the introduction, in a somewhat stronger form.

**Theorem 5.12** (Matching-complement saturation). *Let $G$ be a graph on $n \geq 3$ vertices whose complement $\overline{G}$ is a matching. Then $G$ is $k$-saturated for every $0 \leq k \leq \lfloor n/2 \rfloor$.*

*Proof.* Call the edges of $\overline{G}$ *forbidden.* We show, by induction on the number of forbidden edges in a $k$-matching $M$ of $K_n$, that $p_M \in P_k(G)$. There is nothing to prove if $M$ contains no forbidden edge. Suppose that $ab \in M$ is forbidden.

If $2k < n$, choose an unmatched vertex $c$. Since the forbidden edges form a matching, both $ac$ and $cb$ are edges of $G$. The triangle relation gives

$$p_{ab} = p_{ac} + p_{cb}.$$

After multiplication by the factors belonging to $M \setminus \{ab\}$, this expresses $p_M$ as a sum of two matching polynomials with one fewer forbidden edge.

If $2k = n$, then $n \geq 4$ and $k \geq 2$. Choose another edge $cd \in M$. Again the matching condition on $\overline{G}$ implies that all four cross edges $ac, ad, bc, bd$ belong to $G$. The four-vertex Garnir relation gives

$$p_{ab}p_{cd} = p_{ac}p_{bd} - p_{ad}p_{bc}.$$

Multiplying by the remaining factors again decreases the number of forbidden edges. Induction completes the proof. □

**Corollary 5.13** (Multipartite saturation). *Let $r \geq 2$ and let*

$$G = K_{2,\ldots,2}$$

*be the complete $r$-partite graph with all $r$ parts of size 2. Then*

$$P_k(G) = P_k(K_{2r}) \qquad \text{for all } 0 \leq k \leq r,$$

*and therefore*

$$\mathrm{Hilb}(\mathcal{D}(G), t) = \sum_{k=0}^{r} \left( \binom{2r}{k} - \binom{2r}{k-1} \right) t^k.$$

*Proof.* The complement of $K_{2,\dots,2}$ is the perfect matching consisting of the $r$ within-part pairs, so Theorem 5.12 applies. □

*Remark* 5.14. The restriction $r \geq 2$ is necessary: for $r = 1$ the graph has two vertices and no edges, so it is not saturated in degree one.

For $r = 5$, Corollary 5.13 gives

$$1 + 9t + 35t^2 + 75t^3 + 90t^4 + 42t^5.$$

Thus the suggested straightening of forbidden within-part edges is provided entirely by the two elementary Garnir relations (8)–(9); no choice of a noncrossing basis is needed for this family.

5.6. **Matching-equivalent graphs.** The matching numbers of $G$ are the coefficients of the Hilbert series of $\mathcal{M}(G)$, but they do not determine the Hilbert series of $\mathcal{D}(G)$. Before giving minimal counterexamples, let us record the opposite extreme, where no linear dependence among matching products is lost.

**Proposition 5.15** (Forests)**.** *For every graph $G$,*

$$\mathcal{D}(G) = \mathcal{M}(G) \quad \Longleftrightarrow \quad G \text{ is a forest.}$$

*Consequently, if $G$ is a forest, then*

$$\mathrm{Hilb}(\mathcal{D}(G), t) = \sum_{k\geq 0} m_k(G)t^k.$$

*Proof.* The degree-one part $\mathcal{D}(G)_1$ is the row space of an oriented incidence matrix of $G$, whereas $\mathcal{M}(G)_1$ has the edge generators $u_e$ as a basis. Hence

$$\dim \mathcal{D}(G)_1 = |V| - c(G), \qquad \dim \mathcal{M}(G)_1 = |E|.$$

If $G$ is a forest, then $|E| = |V| - c(G)$, so the incidence rows span all of $\mathcal{M}(G)_1$. Thus every generator $u_e$ belongs to $\mathcal{D}(G)$, and therefore $\mathcal{D}(G) = \mathcal{M}(G)$. Conversely, equality of the two algebras implies equality in degree one, hence $|E| = |V| - c(G)$, which is equivalent to every connected component of $G$ being a tree. □

For brevity, write

$$\mathfrak{m}_G(t) := \sum_{k\geq 0} m_k(G)t^k$$

for the matching generating polynomial of $G$. The next result shows explicitly what information beyond $\mathfrak{m}_G(t)$ can enter even when $G$ has only one cycle.

**Theorem 5.16** (Unicyclic graphs)**.** *Let $G$ be a connected unicyclic graph, let $C$ be its unique cycle, and put $F := G - V(C)$. Then, with the convention $m_{-1}(F) = 0$,*

$$\dim P_k(G) = m_k(G) - m_{k-1}(F) \qquad \text{for all } k \geq 0.$$

*Equivalently,*

$$\mathrm{Hilb}(\mathcal{D}(G), t) = \mathfrak{m}_G(t) - t\mathfrak{m}_F(t).$$

*Proof.* Choose orientations of the edges and introduce formal variables $y_e$, $e \in E(G)$. Consider the algebra map

$$\varphi : \mathbf{k}[y_e : e \in E(G)] \longrightarrow \mathbf{k}[x_v : v \in V(G)], \qquad y_{ij} \longmapsto x_i - x_j.$$

Since $G$ is connected and unicyclic, its edge-difference vectors have exactly one linear dependence, namely the signed cycle relation

$$\ell := \sum_{e\in C} \epsilon_e y_e, \qquad \epsilon_e \in \{\pm 1\}.$$

Thus $\ker \varphi = (\ell)$.

Let $U_k$ be the span of the squarefree monomials $y_M := \prod_{e\in M} y_e$ over all $k$-matchings $M$ of $G$. Then $\varphi(U_k) = P_k(G)$, so we must determine $U_k \cap (\ell)$. Because $\ell$ involves only the cycle variables, we may decompose by the monomial in the variables $y_e$ with $e \notin C$. Fix such a monomial $y_N$ occurring in a matching, and suppose that an element of $U_k \cap (\ell)$ has this off-cycle monomial. It has the form

$$y_N \ell q,$$

where $q$ is a homogeneous polynomial in the cycle variables. Every monomial in $U_k$ is squarefree, so $\ell q$ has degree at most 1 in each cycle variable. Fix a cycle edge $e$ and regard the cycle-variable ring as a polynomial ring in $y_e$ over the other variables. Since the coefficient of $y_e$ in $\ell$ is nonzero,

$$\deg_{y_e}(\ell q) = \deg_{y_e}(q) + 1$$

whenever $q \neq 0$. Thus $\deg_{y_e}(q) = 0$ for every $e \in C$. Hence $q$ contains no cycle variable at all; in the fixed off-cycle multidegree under consideration, it is a scalar.

Consequently, every relation in $U_k$ is a scalar multiple of $y_N \ell$. This belongs to $U_k$ precisely when $N$ is a $(k-1)$-matching all of whose vertices lie outside $C$: indeed, every term $y_N y_e$ with $e \in C$ must itself be a matching monomial. Such $N$ are exactly the $(k-1)$-matchings of $F = G - V(C)$. The resulting elements $y_N \ell$ are linearly independent for distinct $N$, since they have distinct off-cycle monomials. Hence

$$\dim \ker(\varphi|_{U_k}) = m_{k-1}(F),$$

and the claimed formulas follow. □

**Proposition 5.17** (Minimal matching-equivalent counterexamples)**.** *The Hilbert series of $\mathcal{D}(G)$ is not determined by the matching numbers $m_k(G)$ (equivalently, by the matching generating polynomial). Among graphs on a fixed number $n$ of vertices, the smallest counterexample occurs for $n = 4$. Even among connected graphs the smallest counterexample occurs already for $n = 5$; moreover the two graphs can be chosen with the same degree sequence.*

*Proof.* For the first assertion, let

$$G_4 = K_{1,3}, \qquad H_4 = K_3 \sqcup K_1.$$

Both graphs have matching numbers

$$(m_0, m_1, m_2) = (1, 3, 0).$$

However, $G_4$ is connected while $H_4$ has two connected components, so Proposition 3.4 gives

$$\mathrm{Hilb}(\mathcal{D}(G_4), t) = 1 + 3t, \qquad \mathrm{Hilb}(\mathcal{D}(H_4), t) = 1 + 2t.$$

There is no counterexample on at most three vertices: in that range there are no matchings of size 2, and the number of edges determines the number of connected components.

For a connected counterexample on five vertices, let $G_5$ be the triangle on $\{1, 2, 3\}$ with the path $3-4-5$ attached, and let $H_5$ be the cycle $1-2-3-4-1$ with the pendant edge $1-5$. Thus

$$E(G_5) = \{12, 23, 13, 34, 45\},$$
$$E(H_5) = \{12, 23, 34, 14, 15\}.$$

Both graphs are connected, both have degree sequence $(3, 2, 2, 2, 1)$, and both have matching generating polynomial

$$\mathfrak{m}_{G_5}(t) = \mathfrak{m}_{H_5}(t) = 1 + 5t + 4t^2.$$

For $G_5$, deleting the vertices of the unique triangle leaves the edge 45, whereas for $H_5$, deleting the vertices of the unique 4-cycle leaves only the isolated vertex 5. Hence Theorem 5.16 gives

$$\mathrm{Hilb}(\mathcal{D}(G_5), t) = (1 + 5t + 4t^2) - t(1 + t) = 1 + 4t + 3t^2,$$

while

$$\mathrm{Hilb}(\mathcal{D}(H_5), t) = (1 + 5t + 4t^2) - t = 1 + 4t + 4t^2.$$

Thus the two graphs have the same matching data and even the same degree sequence, but different Dunkl Hilbert series.

It remains only to justify minimality among connected graphs. For connected graphs on at most three vertices, the Hilbert series is determined for the reason already given. If $G$ is connected on four vertices, then $\dim \mathcal{D}(G)_1 = 3$, and any 2-matching is a perfect matching. The three perfect-matching polynomials of $K_4$ span the two-dimensional space $P_2(K_4)$, with the single four-vertex Garnir relation among them; in particular, any two are linearly independent. Hence

$$\dim P_2(G) = \min\{m_2(G), 2\},$$

which is determined by the matching numbers. Thus no connected counterexample exists before $n = 5$. □

**Corollary 5.18** (Two-connected counterexamples)**.** *The matching numbers do not determine the Hilbert series of $\mathcal{D}(G)$ even among 2-connected graphs. The smallest such examples have six vertices, and may again be chosen with the same degree sequence.*

*Proof.* Given a graph $X$ on $n$ vertices, let $X^*$ be its cone, obtained by adjoining one new vertex adjacent to every vertex of $X$. A $k$-matching of $X^*$ either avoids the new vertex, or consists of a $(k-1)$-matching of $X$ together with an edge from the new vertex to one of the $n - 2k + 2$ vertices left unmatched. Hence

$$m_k(X^*) = m_k(X) + (n - 2k + 2)m_{k-1}(X). \tag{12}$$

In particular, coning preserves equality of matching numbers.

Let $G_5, H_5$ be the two graphs in Proposition 5.17, and add a universal vertex 6 to each. The resulting graphs $G_5^*, H_5^*$ are 2-connected, have the same degree sequence $(5, 4, 3, 3, 3, 2)$, and by (12) have matching numbers

$$(m_0, m_1, m_2, m_3) = (1, 10, 19, 4).$$

By Theorem 5.9, both have $\dim P_1 = 5$ and $\dim P_2 = 9$.

The four perfect matchings of $G_5^*$ give

$$p_{12}p_{34}p_{56}, \quad p_{12}p_{36}p_{45}, \quad p_{13}p_{26}p_{45}, \quad p_{16}p_{23}p_{45}.$$

The last three satisfy the four-vertex Garnir relation

$$p_{12}p_{36}p_{45} - p_{13}p_{26}p_{45} + p_{16}p_{23}p_{45} = 0,$$

while the first polynomial is independent of their span (for instance its $x_1x_3x_6$-coefficient is nonzero and the other three have none). Any two of the last three are independent, so $\dim P_3(G_5^*) = 3$.

The four perfect matchings of $H_5^*$ give

$$p_{12}p_{34}p_{56}, \quad p_{14}p_{23}p_{56}, \quad p_{15}p_{23}p_{46}, \quad p_{15}p_{26}p_{34}.$$

They are independent: the monomials $x_1x_4x_5$, $x_1x_2x_5$, $x_1x_3x_4$, and $x_1x_2x_3$ occur, respectively, in only one of the four displayed polynomials. Hence $\dim P_3(H_5^*) = 4$. Therefore

$$\mathrm{Hilb}(\mathcal{D}(G_5^*), t) = 1 + 5t + 9t^2 + 3t^3, \qquad \mathrm{Hilb}(\mathcal{D}(H_5^*), t) = 1 + 5t + 9t^2 + 4t^3.$$

Finally, a 2-connected graph on at most five vertices has no graded piece above degree 2, and Theorem 5.9 fixes that degree as well. Thus six vertices are minimal in the 2-connected class. □

### 5.7. **Problems.**

**Problem 5.19.** *Describe the kernel*

$$I_G = \ker\left(\mathbf{k}[z_v : v \in V]/(\sum_{v\in C} z_v : C \text{ a component}) \longrightarrow \mathcal{D}(G)\right).$$

*It contains $z_v^2$ for every $v$, but in general has additional graph-dependent relations.*

**Problem 5.20.** *Find deletion recurrences for* $\dim P_k(G)$*. Since matchings depend on vertex-disjointness, vertex deletion and deletion of the closed edge neighborhood are more natural here than ordinary matroid deletion–contraction.*

The results above leave a gap for $k \geq 3$:

$$k\text{-linked} \Longrightarrow k\text{-saturated} \Longrightarrow k\text{-connected}.$$

That is, we have a necessary combinatorial condition and a sufficient combinatorial condition for a graph to be $k$-saturated, but no condition that is both at once (except for $k \leq 2$). Perhaps this is too much to ask for, but perhaps not.

## 6. Cut quotients and zonotopal algebras

This section is independent of the matching relation that products of incident edges vanish. We temporarily work in the larger *square-zero edge algebra*

$$\widehat{\Phi}_G := \mathbf{k}[u_e : e \in E]/(u_e^2 : e \in E),$$

in which products of distinct incident edges are allowed. We use the same signed incidence sums $\theta_v$. Their subalgebra

$$\mathcal{C}_G^{\mathrm{ex}} := \mathbf{k}[\theta_v : v \in V] \subseteq \widehat{\Phi}_G$$

is the external graphical zonotopal algebra (the forest version of the Postnikov–Shapiro algebra); see [19, §11], as well as the zonotopal framework in [1, 11]. The point of this section is to explain what happens when one imposes the monomial relations coming from edge cuts.

For a nonempty proper subset $S \subset V$, let

$$\delta(S) := \{e \in E : |e \cap S| = 1\}, \qquad d_S := |\delta(S)|, \qquad \eta_S := \prod_{e \in \delta(S)} u_e.$$

Because internal edges cancel in the incidence sum,

$$\Theta_S := \sum_{v \in S} \theta_v = \sum_{e \in \delta(S)} \varepsilon_{S,e} u_e, \qquad \varepsilon_{S,e} \in \{\pm 1\}. \tag{13}$$

Since the $u_e$ are square-zero, the only surviving terms in the $d_S$th power use every cut edge exactly once. Therefore

$$\Theta_S^{d_S} = d_S! \left( \prod_{e \in \delta(S)} \varepsilon_{S,e} \right) \eta_S = \pm d_S!\, \eta_S. \tag{14}$$

In characteristic zero this shows, in particular, that every cut monomial $\eta_S$ already belongs to $\mathcal{C}_G^{\mathrm{ex}}$.

The relation with the usual external and central power ideals makes the next quotient transparent. Put

$$R_V := \mathbf{k}[z_v : v \in V] \Big/ \left( \sum_{v \in V} z_v \right), \qquad \ell_S := \sum_{v \in S} z_v.$$

The standard graphical power-ideal presentations are

$$\mathcal{C}_G^{\mathrm{ex}} \cong R_V / (\ell_S^{d_S+1} : \varnothing \neq S \subsetneq V), \tag{15}$$

$$\mathcal{C}_G^{\mathrm{central}} \cong R_V / (\ell_S^{d_S} : \varnothing \neq S \subsetneq V). \tag{16}$$

Here $z_v$ maps to $\theta_v$. These presentations are the graphical cases of the external and central zonotopal algebras; see [19, Secs. 9 and 11] and [1, 11].

**Proposition 6.1** (The cut quotient inside the incidence-sum algebra)**.** *For a connected graph $G$ in characteristic zero,*

$$\mathcal{C}_G^{\mathrm{ex}} / (\eta_S : \varnothing \neq S \subsetneq V) \cong \mathcal{C}_G^{\mathrm{central}}.$$

*Consequently, if $m = |E|$ and $r = |V| - 1$, then*

$$\mathrm{Hilb}\,(\mathcal{C}_G^{\mathrm{ex}} / (\eta_S), t) = t^{m-r} T_G(1, t^{-1}),$$

*where $T_G$ denotes the Tutte polynomial of $G$.*

*Proof.* Under the presentation (15), the element $\ell_S^{d_S}$ maps, by (14), to a nonzero scalar multiple of $\eta_S$. Thus quotienting $\mathcal{C}_G^{\mathrm{ex}}$ by all $\eta_S$ adds exactly the relations $\ell_S^{d_S} = 0$. The resulting presentation is (16). The Hilbert-series formula is the standard central zonotopal specialization of the Tutte polynomial [1, 11]. ☐

There is a second quotient by the same cut monomials, obtained by imposing them on the *whole* square-zero edge algebra rather than only on its incidence-sum subalgebra. This is larger, and it has an elementary basis. Let

$$I_{\mathrm{cut}} := (\eta_S : \varnothing \neq S \subsetneq V) \subseteq \widehat{\Phi}_G.$$

For $F \subseteq E$, write $u_F := \prod_{e \in F} u_e$.

**Proposition 6.2** (The ambient cut quotient)**.** *For connected $G$, the set*

$$\{u_F : G \setminus F \text{ is connected}\}$$

*is a basis of $\widehat{\Phi}_G/I_{\mathrm{cut}}$. Hence*

$$\begin{aligned}\mathrm{Hilb}(\widehat{\Phi}_G/I_{\mathrm{cut}}, t) &= \sum_{\substack{H\subseteq G;\\ H \text{ connected spanning}}} t^{m-|H|}\\ &= t^{m-r} T_G(1, 1+t^{-1}).\end{aligned}$$

*Proof.* The squarefree monomials $u_F$ form a basis of $\widehat{\Phi}_G$, and $I_{\mathrm{cut}}$ is a monomial ideal. Thus $u_F$ survives in the quotient exactly when it is not divisible by any $\eta_S$, that is, exactly when $\delta(S) \not\subseteq F$ for every nontrivial $S \subset V$. But a deletion set $F$ contains a whole cut $\delta(S)$ if and only if $G \setminus F$ has no edge from $S$ to $V \setminus S$, equivalently if and only if $G \setminus F$ is disconnected. This proves the basis statement and the first Hilbert-series formula.

For the second formula, use the corank–nullity expansion

$$T_G(1, y) = \sum_{\substack{H\subseteq G;\\ H \text{ connected spanning}}} (y-1)^{|H|-r}$$

and substitute $y = 1 + t^{-1}$. $\square$

Thus the same cut relations produce two different but closely related objects. They fit into the diagram

$$\begin{array}{ccc} \mathcal{C}_G^{\mathrm{ex}} & \subseteq & \widehat{\Phi}_G \\ \downarrow & & \downarrow \\ \mathcal{C}_G^{\mathrm{central}} & \subseteq & \widehat{\Phi}_G/I_{\mathrm{cut}}. \end{array}$$

The lower-left algebra has total dimension $T_G(1,1)$, the number of spanning trees. The lower-right algebra has total dimension $T_G(1,2)$, the number of connected spanning subgraphs; its grading records the number of deleted edges. More precisely, if

$$R_G(p) = \sum_{\substack{H\subseteq G;\\ H \text{ connected spanning}}} p^{|H|}(1-p)^{m-|H|}$$

is the all-terminal reliability polynomial, then

$$\mathrm{Hilb}(\widehat{\Phi}_G/I_{\mathrm{cut}}, t) = (1+t)^m R_G((1+t)^{-1}).$$

This is why the ambient algebra matters in this construction.

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