SWALLOWTAILS AND WHITNEY UMBRELLAS ARE HOMEOMORPHIC

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Abstract. We show that the surface of all degree $2n$ real polynomials with multiple real zeros is homeomorphic to the surface of all pairs of degree n real polynomials with common real zeros. This proves the simplest case of the Szücs' conjecture suggesting a homeomorphism of generalized swallowtails and Whitney umbrellas.

§1. Introduction

Swallowtails and Whitney umbrellas are naturally arising as the simplest singularities of smooth maps and functions. Increasing interest in the geometrical properties of these surfaces is connected with the study of the topology of various functional spaces (see [A],[V]).

Here we deal exclusively with the real situation. It should be mentioned that the correspondent complex objects are nonhomeomorphic: the "handles of Whitney umbrella and the swallow tail lie in different connected components. However it was proved recently that the homology of the complements to these complex objects are isomorphic (see [V1], [CCMM]).

Our main objects can be also defined as the following hypersurfaces in the spaces of polynomials.

1.1. Definition. Let $Pol_n \simeq \mathbf{R}^n$ be the space of all real degree n polynomials of the form $x^n + a_1x^{n-1} + \cdots + a_n$. The hypersurface of all polynomials with multiple real zeros is called the nth swallowtail $\Sigma_n \in Pol_n$.

Remark. Σ_n can be defined also as a 1-dimensional cylinder over the bifurcation diagram of the singularity A_{n-1} (see [AVG]).

1.2. Definition. Let $Pol_{2,n} \simeq \mathbf{R}^{2n}$ be the space of all pairs of polynomials of the form $(y^n + b_1y^{n-1} + \cdots + b_n,\ y^n + c_1y^{n-1} + \cdots + c_n)$. The hypersurface of all pairs with at least one common real zero is called the nth Whitney umbrella $W_{2,n} \in Pol_{2,n}$.

Remark. $W_{2,n}$ can be defined also as the 1-dimensional cylinder over the image of a map from $\mathbf{R}^{2n-1}$ to $\mathbf{R}^{2n}$ in a neighborhood of a point of rank $2n-2$ (see[AVG]).

The main result of this paper is as follows.

Theorem. For any n the pairs (Pol_{2n}, Σ_{2n}) and$(Pol_{2,n}, W_{2,n})$ are homeomorphic.

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Example. See case $n = 2$ shown on fig. 1 (more precisely there are shown homeomorphic 3-dimensional generic sections of the 4-dimensional spaces Pol_4 and $Pol_{2,2}$).

The proof of this theorem is based on the explicit cell decompositions of Pol_{2n} and $Pol_{2,n}$ including decompositions of Σ_{2n} and $W_{2,n}$ respectively. Then we establish a bijection between the cells of both decompositions preserving the adjacency diagrams of cells. Finally, we construct inductively a homeomorphism between Pol_{2n} and $Pol_{2,n}$ which includes a homeomorphism of the embedded Σ_{2n} and $W_{2,n}$.

One can also consider Whitney umbrellas in the spaces of pairs of polynomials of different degrees and swallowtails in the spaces of polynomials of odd degree. These objects are homeomorphic to the cylinders over the ordinary Whitney umbrellas and swallowtails defined above (see [AVG]). Therefore our results trivially extend to this case.

1.4. Remark. The theorem stated above confirms the following conjecture by A. Szücs.

Definition. The generalized swallowtail $\Sigma_{k,kn} \in Pol_{kn}$ is the set of all degree kn polynomials with at least one real zero of multiplicity $\geq k$.

Definition. Let $Pol_{k,n}$ be the kn-dimensional linear space of all k-tuples of polynomials of degree n. The set of all k-tuples with common real zeros is called the generalized Whitney umbrella $W_{k,n}$.

Conjecture (A. Szücs). The pairs $(Pol_{kn}, \Sigma_{k,kn})$ and $(Pol_{k,n}, W_{k,n})$ are homeomorphic.

We prove this conjecture for $k = 2$.

1.5. The following infinite-dimensional analog of the above conjecture is due to V. I. Arnold and was proved recently by V. A. Vassiliev, (see [V]).

Definition. Let Φ_e^k (Φ_o^k resp.) denotes the space of all C^∞-functions $f : \mathbf{R} \to \mathbf{R}$ such that

a) f does not have zeros of multiplicity $\geq k$;

b) f coincides with x^2 (x) outside some interval.

Consider the embedding $i : \Phi_e^k \to F_e^k$ ($i : \Phi_o^k \to F_o^k$ resp.) given by $i : \phi \to (\phi, \phi', \dots, \phi^{(k-1)})$ where F_e^k and F_o^k denote the following spaces. F_e^k is the space of k-tuples of functions $(f_1, f_2, \dots, f_k)$ without common zeros such that outside some interval $f_1 = x^2$; $f_2 = 2$; $f_i = 0$ $i \geq 3$. F_o^k is the space of k-tuples of functions $(f_1, f_2, \dots, f_k)$ such that outside some interval $f_1 = x$; $f_2 = 1$; $f_i = 0$ $i \geq 2$.

Theorem (V. A. Vassiliev) The embedding $i : \Phi_e^k \to F_e^k$ ($i : \Phi_o^k \to F_o^k$ resp.) is a weak homotopy equivalence.

We hope that Vassiliev's results have other finite dimensional analogs generalizing the above conjecture by A. Szücs.

§2. Cell decompositions

2.1. To begin with we consider the standard cell decomposition $\{T_\nu\}$ of Whitney umbrellas $(Pol_{2,n}, W_{2,n})$ (ν denotes the multiindex).

Definition. The common multiplicity of a real common zero for a pair of real polynomials is the minimum of their multiplicities at the considered point.

We use the obvious order of common real zeros from left to right on the real axis. A stratum is a connected component of the space of all pairs of polynomials with the same sequence of multiplicities $a_1, \dots, a_d$ of their common real zeros. Strata defined in such a way are contractible (see lemma 2 below) and hence are homeomorphic to open cells. This stratification coincides with the standard one corresponding to the types of singularities on the transversal sections at different points of Whitney umbrellas, (see, for example, [Kh]).

LEMMA 1. The total number of the strata T_ν equals $2^{n+1} - 1$. Strata can be enumerated by the sequences $\nu = (a_1, \ldots, a_d, m)$, $a_j \in \mathbf{N}$, $m \in \mathbf{Z}$ such that $a_1 + \cdots + a_d + |m| \leq n$.

PROOF. The sequence $a_1, \ldots, a_d$ coincides with the sequence of multiplicities while m is defined in the following more complicated way.

Given a pair of polynomials (p, q) let $x_1 < \cdots < x_d$ be the set of all their common real zeros considered with multiplicities $a_1, \ldots, a_d$. Let $\bar{p}$ and $\bar{q}$ be the quotients of division of p and q by $(x - x_1)^{a_1} \ldots (x - x_d)^{a_d}$. Then the polynomials $\bar{p}$ and $\bar{q}$ have no common real zeros. Let Δ be the union of all real zeros of $\bar{p}$ and $\bar{q}$. Now in order to simplify Δ we'll use the following procedure. If there is a pair of consecutive zeros of one of the polynomials not separated by the zeros of the other we'll remove this pair from the real axis (make these zeros coincide and then transform them into a complex conjugate pair). Iterating this procedure as many times as possible we will get the reduced set Δ' consisting of l zeros of one polynomial separated by the l zeros of the other. We define $m = l$ if the leftmost zero of Δ' belongs to $\bar{p}$ and $m = -l$ otherwise. One can easily see that $m \equiv n (\mod 2)$.

REMARK. According to our construction the stratum containing the given pair (p, q) depends on the set of real zeros of p and q only.

LEMMA 2. Each stratum is homeomorphic to a cell. The codimension of the stratum corresponding to the sequence $(a_1, \ldots, a_d, m)$ equals $2\sum_{i=1}^{d} a_i - d$.

PROOF. Each stratum T_ν corresponding to $\nu = (a_1, \ldots, a_d, m)$ is fibered over the contractible d-dimensional set of all d-tuples $x_1 < x_2 < \cdots < x_d$ where $x_1, \ldots, x_d$ are the common zeros of the pairs of polynomials from T_ν. The fiber F_ν of this bundle is the set of all polynomials with the fixed $d-$tuple $x_1, \ldots, x_d$. Up to homeomorphism we can consider the fiber F_ν over $x_i = i, \quad i = 1, \ldots, d$. Thus the set F_ν is homeomorphic to one of the connected components in the space of pairs of polynomials of degree $n - \sum_{i=1}^{d} a_i$ without common zeros. We just forget about all common zeros and it suffices to show that all connected components of the pairs of polynomials without common zeros are contractible.

By the above procedure these components are enumerated by the minimal number $l = |m|$ of the real zeros of polynomials in the pairs (p, q) belonging to T_ν with an appropriate sign. This sign is $+$ if the leftmost zero belongs to p and it is $-$ if the leftmost zero belongs q. It is obvious that the subset $\tilde{F}_\nu \in F_\nu$ of all pairs with exactly $|m|$ real zeros is contractible. To contract the whole F_ν to $\tilde{F}_\nu$ we apply the procedure of lemma 1 for the inductively decreasing number of real zeros. This operation can be made continuous on the choice of initial pair and gives the required contraction. $\square$

2.2. Let us now define the cell decomposition S_μ (μ is a multiindex) of the space (Pol_{2n}, Σ_{2n}) which includes the swallowtail. At first consider the standard cell decomposition of Pol_{2n} where each cell consists of all polynomials with fixed multiplicity sequence of real zeros (zeros are ordered from left to right). We enlarge this stratification by uniting into one stratum S_μ all the polynomials whose μ-sequences differ from each other just by the location of simple zeros and also add those polynomials in the closure of this set whose simple zeros coincide with some of the zeros of even degree.

For example we will unite all the polynomials with the multiplicity sequence $(1,1,2,4)$ with i) all polynomials with another location of simple zeros, namely $(1,2,1,4)$, $(1,2,4,1)$, $(2,1,1,4)$, $(2,1,4,1)$, $(2,4,1,1)$ and also ii) all the polynomials whose simple zeros can coincide with the zeros of multiplicity 2 and 4, namely $(1,3,4)$, $(3,1,4)$ $(3,4,1)$, $(1,2,5)$, $(2,1,5)$, $(2,5,1)$, and $(3,5)$. In particular this new decomposition ignores cuspidal lines of the ordinary swallowtail (see fig.1).

Lemma 3. The total number of strata of this decomposition equals $2^{n+1}-1$. These strata can be enumerated by the sequences $(a_1,\dots,a_d,l)$ of positive integers such that $a_1+\dots+a_d+l\le n$.

Proof. Let $d,\kappa_1,\dots\kappa_d$ be the number and the multiplicities of all nonsimple real zeros of the polynomial P according to their order ($\kappa_i\ge 2$ for $i=1,\dots,d$). Then P belongs to the stratum $S_{\{a_1,\dots,a_d,l\}}$ where $a_i=[\frac{\kappa_i}{2}]$ and $2l=\sum_{j=1}^d(\kappa_j-2a_j)+$number of simple real zeros.

Remark. A stratum containing a given polynomial depends only on the set of its real zeros.

Lemma 4. Each stratum is homeomorphic to a cell. The codimension of the stratum corresponding to the sequence $(a_1,\dots,a_d,l)$ is $2\sum_{i=1}^d a_i-d$.

Proof. Any polynomial belonging to S_ν can be connected within S_ν with some fixed polynomial whose real zeros are simple or of even multiplicity. This system of paths can be chosen continuosly depending on the initial point and thus defining a contraction of the stratum. The codimension of the stratum evaluated in a generic point (i.e. for a polynomial with even multiplicities $(2a_1,\dots,2a_d)$ of nonsimple zeros is equal to the difference of the sum of multiplicities and the number of the zeros $(2\sum_{i=1}^d a_i-d)$.

Remark. Notice that Σ_{2n} and $W_{2,n}$ are the unions of all positive codimensional cells of the described decompositions of Pol_{2n} and $Pol_{2,n}$.

The adjacency of cells for the considered decomposition of the swallowtail can be described in terms of the following operation on the zeros of polynomial

a) a zero of multiplicity 2 splits into 2 real simple zeros;

b) a zero of multiplicity 2 splits into a complex conjugate pair;

c) a zero of even multiplicity greater than 2 splits into 2 zeros of even positive multiplicity each.

Notice that, as we described above, a nonsimple zero of odd multiplicity is splitted with its stratum into an even multiplicity zero and a simple zero.

For the Whitney umbrellas the corresponding adjacency rules for T_ν are as follows

a) a simple common zero of the pair (p,q) splits into a couple of zeros so that the zero of p is to the left of the zero of q;

b) a simple common zero of (p,q) splits so that the zero of p is to the right of the zero of q;

c) a common zero of the multiplicity greater than 1 splits into 2 common real zeros of positive multiplicity.

§3. Bijection of cell decompositions and their homeomorphism

First of all we construct a bijection of cells of the above cell complexes. For this purpose it suffices to define a 1-1 correspondence between the open cells of the above decompositions for all n (i.e. between the regions of pairs of polynomials without common zeros and regions of polynomials without multiple zeros) and make this correspondence respect degeneracies (see example on fig. 2).

LEMMA 5. There exists a 1-1 correspondence between the cells of complexes T_ν and S_ν respecting their adjacencies.

PROOF. Given n there exist $n+1$ connected components of degree $2n$ polynomials without nonsimple zeros. These components are enumerated by the number of zeros namely $2n$, $2n-2$, $\dots 0$. Analogously there exist $n+1$ possible sequences of interchanging zeros of two degree n polynomials (p,q) namely $\underbrace{ab \times \cdots \times ab}_{n \text{ times}}, \underbrace{ab \times \cdots \times ab}_{n-1 \text{ times}}, \dots \underbrace{ba \times \cdots \times ba}_{n \text{ times}}$, (here a-s denote zeros of p and b-s zeros of q). Each term of both sequences (an even number or a combination of a-s and b-s) corresponds to a certain cell of the highest dimension of decompositions T_ν and S_ν.

Now we establish the bijection between the sequences and therefore between the cells. For the stratum S_μ where $\mu = \{a_1, \dots, a_d, l\}$ we assign the stratum T_ν, where $\nu = \{a_1, \dots, a_d, 2l + \sum a_i - n\}$. Geometrically this bijection looks as follows. Let $\Delta = \{x_1 \le x_2 \le \dots \le x_k\}$ be the set of all real zeros of a polynomial $P \in Pol_{2n}$ belonging to a certain stratum of S_ν. We paint odd and even zeros into two different colors ($\times$ and $\circ$ on fig. 2). Then to the right of the set Δ we add $(2n-k)$ distinct points $x_1 \le x_2 \le \dots \le x_k < y_{k+1} < \dots < y_{2n}$. Let us paint the additional points $y_{k+1}, \dots, y_{2n}$ in the same way changing colors each time and so that the color of x_k coincides with the color of y_{k+1}. The set of points of each color will be considered as the set of the zeros of degree n polynomials. The corresponding pair of polynomials (p,q) with these zeros determines the stratum of T_ν. More precisely we apply the procedure of lemma 1 to exclude extra zeros and to make a sequence of interchanging zeros of $ab \times \dots \times ab-$ type.

In this geometric form it becomes evident that constructed bijection between strata of Σ_{2n} and $W_{2,n}$ preserves the rules of adjacency.

Now we costruct using induction by n the homeomorphism h_{2n} of (Pol_{2n}, Σ_{2n}) and $(Pol_{2,n}, W_{2,n})$ preserving described cell decomposition and satisfying the following two properties.

For any polynomial $p \in Pol_{2n}$ we will denote by $\bar{p}$ the greatest divisor of p with real zeros of even multiplicity only and call it the multiple part of p. The polynomial $\hat{p} = \frac{p}{\bar{p}}$ without multiple real zeros is called the simple part of p. For any pair of polynomials (q,r) we will denote by $\bar{q}$ the greatest real common divisor of both and call it the common part of (q,r). The pair of polynomials $(\hat{q}, \hat{r}) = (\frac{q}{\bar{q}}, \frac{r}{\bar{q}})$ is called the separate part of the pair (q,r).

The first property of the homeomorphism we are constructing is as follows. Let $h_{2n}(p) = (q,r)$ be the image of the polynomial p in the space of pairs $Pol_{2,n}$. We require that $\bar{p} = \bar{q}^2$ and the separate part $(\hat{q}, \hat{r})$ depends only on the simple part $\hat{p}$, i.e. if p_1 and p_2 have the same simple part $\hat{p}_1 = \hat{p}_2$ then $(\hat{q}_1, \hat{r}_1) = (\hat{q}_2, \hat{r}_2)$.

Introduce on the space Pol_{2n} ($Pol_{2,n}$ resp.) the natural quasihomogeneous structure by multiplying all the zeros of a polynomial (of both polynomials of the pair resp.) by any nonnegative scalar λ and denote by p_λ ((q_λ, r_λ) resp.) the orbit of p ((q,r) resp.) under this action.

The second property is $h_{2n}(p_\lambda) = (q_\lambda, r_\lambda)$, i.e. h_{2n} maps orbits onto orbits.

REMARK. One can easily see that Σ_{2n} ($W_{2,n}$ resp.) is a cone over the intersection $\Sigma_{2n} \cap \mathbf{S}^{2n+1}$ ($W_{2,n} \cap \mathbf{S}^{2n+1}$ resp.) with the unit sphere; this cone is formed by the quasihomogeneous orbits defined above, connecting the origin with the points of $\Sigma_{2n} \cap \mathbf{S}^{2n+1}$ ($W_{2,n} \cap \mathbf{S}^{2n+1}$ resp.). Each cell of the above decomposition except the one formed by $(x-\alpha)^{2n}$ $(((y-\alpha)^n, (z-\alpha)^n)$ resp.) is fibered with the fiber $\mathbf{R}^+$ over its intersection with $\mathbf{S}^{2n+1}$.

Base of induction. The pair (Pol_2, Σ_2) is homeomorphic to the pair $(Pol_{2,1}, W_{2,1})$. To begin with identify the 1-dimensional cell of all polynomials of type $(x-a)^2$ where $a \in \mathbf{R}$ with the 1-cell of pairs of polynomials $((x-a),(x-a))$. Then identify the 2-cell of the polynomials of type $(x-a)(x-b)$, $a<b$ with the 2-cell of pairs $((x-a),(x-b))$ where $a<b$. Finally identify the 2-cell of polynomials of type $(x-\alpha)^2+\beta^2 = (x-\alpha+i\beta)(x-\alpha-i\beta)$ with the 2-cell of pairs $((x-a),(x-b))$, $a>b$ where $\alpha = \frac{a+b}{2}$ and $\beta = \frac{a-b}{2}$. Both of the above properties in this case are obvious.

The inductive step. Assume that we have constructed the desired homeomorphism of pairs $h_{2n} : (Pol_{2n}, \Sigma_{2n}) \to (Pol_{2,n}, W_{2,n})$ satisfying the above properties. Let us construct the homeomorphism h_{2n+2}. At first define it on Σ_{2n+2} and $W_{2,n+1}$. Take a polynomial $p \in \Sigma_{2n+2}$ and take its leftmost multiple real zero α. The polynomial $\tilde{p} = \frac{p}{(x-\alpha)^2}$ belongs to Pol_{2n}. Denote by $(\tilde{q}, \tilde{r})$ the image of the already constructed $h_{2n}(\tilde{p})$. Now define that h_{2n+2} maps p onto the pair $((x-\alpha)\tilde{q}, (x-\alpha)\tilde{r})$. This map h_{2n+2} is 1-1 since for any pair $(q,r) \in W_{2,n}$ its leftmost common real zero α is uniquely defined, as well as the pair $(\tilde{q}, \tilde{r}) = (\frac{q}{y-\alpha}, \frac{q}{z-\alpha})$. Since h_{2n} is a homeomorphism we can find a unique inverse image of $(\tilde{q}, \tilde{r})$. Also both of the above properties are obvious for the partialy defined h_{2n+2}.

Since the leftmost zero does not depend continuously on polynomial we need to prove the following statement.

LEMMA. The above map $h_{2n+2} : \Sigma_{2n+2} \to W_{2,n+1}$ is continuous. PROOF. Consider a polynomial $p \in \Sigma_{2n} \subset Pol_{2n+2}$ with several real multiple zeros $x_1, \ldots, x_d$ (i.e. p belongs to the so-called Maxwell stratum of Σ_{2n+2}). Denote by $\epsilon(p)$ some small ϵ-neighborhood of p in Σ_{2n+2}. The neighborhood $\epsilon(p)$ splits into d nonintersecting sets $\epsilon_1(p), \ldots, \epsilon_d(p)$ such that for any polynomial $p' \in \epsilon_j(p)$ its leftmost multiple real zero belongs to the neighborhood of x_j and thus it depends continuously on p' (see fig.3). Therefore by assumption that h_{2n} is continuous we have that h_{2n+2} is continuous on each of $\epsilon_1(p), \epsilon_2(p), \ldots, \epsilon_d(p)$. It is left to prove that h_{2n+2} is continuous on the closures of $\epsilon_j(p)$. The first of the above properties implies that if we have two 1-parameter families $p_1(t)$ and $p_2(t)$ $\quad t \in [0,1]$ of degree $2n$ polynomials such that the polynomials $p_1(1)$ and $p_2(1)$ have the same simple part and different multiple parts then their images $h_{2n}(p_1(1))$ and $h_{2n}(p_2(1))$ will have the same separate part and different common parts. Consider now two 1-parameter families $\tilde{p}_1(t)$ and $\tilde{p}_2(t)$ of degree $2n+2$ polynomials lying in different neighborhhods $\epsilon_{j1}(p)$ and $\epsilon_{j2}(p)$ respectively and having the same limit polynomial $P = \tilde{p}_1(1) = \tilde{p}_2(1)$. We must show that $\lim_{t\to 1} h_{2n+2}(\tilde{p}_1(t)) = \lim_{t \to 1} h_{2n+2}(\tilde{p}_2(t))$. Let $\alpha_1(t)$ and $\alpha_2(t)$ denote the leftmost real multiple zeros in the families $\tilde{p}_1(t)$ and $\tilde{p}_2(t)$ respectively. By the definition

$$h_{2n+2}(\tilde{p}_1(t)) = (x-\alpha_1(t)); (x-\alpha_1(t)) h_{2n}(\frac{\tilde{p}_1(t)}{(x-\alpha_1(t))^2}).$$

Let us compare 2 families $\frac{\tilde{p}_1(t)}{(x-\alpha_1(t))^2}$ and $\frac{\tilde{p}_2(t)}{(x-\alpha_2(t))^2}$ of degree $2n$ polynomials. The limit polynomials $\lim_{t\to 1}\frac{\tilde{p}_1(t)}{(x-\alpha_1(t))^2}$ and $\lim_{t\to 1}\frac{\tilde{p}_2(t)}{(x-\alpha_2(t))^2}$ coincide with $\frac{P}{(x-\alpha_1)^2}$ and $\frac{P}{(x-\alpha_2)^2}$ respectively . Polynomials $\frac{P}{(x-\alpha_1)^2}$ and $\frac{P}{(x-\alpha_2)^2}$ have the same simple parts $\hat{p}_1=\hat{p}_2$ and their multiple parts $\bar{p}_1$ and $\bar{p}_2$ satisfy the relation $(x-\alpha_1)^2\bar{p}_1=(x-\alpha_2)^2\bar{p}_2$. By the inductive argument and 2 properties of the map stated above h_{2n} is continuous and the separate part of $h_{2n}(\frac{P}{(x-\alpha_1)^2})=\lim_{t\to 1}h_{2n}(\frac{\tilde{p}_1(t)}{(x-\alpha_1(t))^2})$ is determined by the simple part $\hat{p}_1$ only. Analogously the separate part of $h_{2n}(\frac{P}{(x-\alpha_2)^2})=\lim_{t\to 1}h_{2n}(\frac{\tilde{p}_1(t)}{(x-\alpha_1(t))^2})$ is determined by $\hat{p}_2$ which coincide with $\hat{p}_1$. Therefore the separate parts of $h_{2n}(\frac{P}{(x-\alpha_1)^2})$ and $h_{2n}(\frac{P}{(x-\alpha_1)^2})$ coincide. Using the above relation for the multiple parts of $(\frac{P}{(x-\alpha_1)^2})$ and $(\frac{P}{(x-\alpha_2)^2})$ from the definition of h_{2n+2} we get

$$\lim_{t\to 1}h_{2n+2}(\tilde{p}_1(t))=\lim_{t\to 1}h_{2n+2}(\tilde{p}_2(t))$$

which gives the necessary continuity. Analogous arguments show that h^{-1}_{2n+2} is also continuous.

Now we extend our homeomorphism h_{2n+2} defined on Σ_{2n+2} and $W_{2,n+1}$ to the entire spaces Pol_{2n+2} and $Pol_{2,n+1}$. Recall that the complements $Pol_{2n+2}\setminus\Sigma_{2n+2}$ and $Pol_{2,n+1}\setminus W_{2,n+1}$ consists of the equal number of $2n+2$-dimensional cells. The construction of h_{2n+2} preserves the described cell decomposition and sends the intersection of Σ_{2n+2} with the standard unit sphere $\mathbf{S}^{2n+1}$ onto the intersection $W_{2,n+1}\cap\mathbf{S}^{2n+1}$. Moreover it defines their stratified homeomorphism. The section by $\mathbf{S}^{2n+1}$ of any (quasihomogeneous) cell from $Pol_{2n+2}\setminus\Sigma_{2n+2}$ is homeomorphic to a disk, and we extend h_{2n+2} from the boundary of the disk to its interior arbitrarily. Thus the homeomorphism h_{2n+2} is defined on the unit sphere in Pol_{2n+2} (sending it to the unit sphere in $Pol_{2,n+1}$) and (by quasihomogenuity) on the entire space Pol_{2n+2}. It should be mentioned that this extension of h_{2n+2} satisfies the second condition because outside Σ_{2n+2} polynomials do not have real multiple zeros as well as their images outside W_{2n+2} do not have common zeros. This extension completes the construction of the homeomorphism and thus the proof of the theorem.

Remark. It would be interesting to find a proof of the above theorem by constructing equivalent regular cell decompositions of $W_{2,n}$ and Σ_{2n} analogous to those used in the theory of arrangements of hyperplanes, see for example [BZ].

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REFERENCES

[A].V. I. Arnold, *The space of functions with moderate singularities*, Funct. Anal. and its Appl. **23** (1989), no. 3, 169-177.

[AVG].I. Arnold, A. N. Varchenko and S. M. Gusein-Zade, *Singularities of differentiable maps*, Boston, Birkhauser, 1985-1988.

[BZ]A. Bjorner and G. Ziegler, *Combinatorial stratification of complex arrangements*, preprint, Dept. of Math. of Royal Institute of Technology (1991).

[CCMM]F. Cohen, R. L. Cohen, B. M. Mann and R. J. Milgram, *The topology of the space of rational functions and divisors of surfaces*, Acta Math. **166**, no. 3, 163–221.

[Kh]B. A. Khesin, *Homogeneous vector fields and Whitney umbrellas*, Russ. Math. Surveys **42** (1987), no. 5, 171-172.

[V]. V. A. Vassiliev, *The topology of the spaces of functions without compounded singularities*, Funct. Anal. and its Appl. **23** (1989), no. 4, 277–286.

[V1] V. A. Vassiliev, *Spaces of rational functions and spaces of polynomials without multiple roots*, in preparation.