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Linear ordinary differential equations and Schubert calculus

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Abstract. In this short survey we recall some basic results and relations between the qualitative theory of linear ordinary differential equations with real time and the reality problems in Schubert calculus. We formulate a few relevant conjectures.

1. Introduction

Questions asking under what conditions a given enumerative problem in geometry with all real initial data has all real solutions have a long history and appear often in engineering applications. (Below we refer to these as the questions about the total reality of the corresponding enumerative problem.) The most basic and classical of these questions is undoubtedly when a univariate polynomial with real coefficients has all roots real. It goes back to the times of R. Descartes and I. Newton and is very important in connection with the stability problems in control theory developed by (among others) J. C. Maxwell, C. Runge, M. W. Kutta, M. G. Krein.

Another natural test field for questions in total reality is Schubert calculus. In the early 80's W. Fulton revived the interest in these issues by asking whether each enumerative problem in Schubert calculus admits real initial data under which all its solutions will be real. At the same time V. I. Arnold and his school were developing a completely different area, namely, they were studying various generalizations of the classical Sturm theory about the properties of the zeroes of solutions to second order linear ordinary differential equations with real time. It turned out that a natural qualitative theory of linear ordinary differential equations of order greater than 2 is closely related to Schubert calculus.

The purpose of this short note is to review a connection (partially proven and partially conjectural) between the qualitative theory of linear ode of order greater than 2, transversality, and total reality in Schubert calculus.

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2. Sturm theory, disconjugate ODE, and transversality

2.1. Linear ordinary differential equations and curves in vector spaces

Consider a linear homogeneous differential equation (l.o.d.e.) of order n

$$L_n[y] = y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = 0, \tag{2.1}$$

defined on a given interval I where $a_i(x) \in C^0[I]$.

Denote by V the n-dimensional vector space of solutions of (2.1) considered on I. An arbitrary point $x \in I$ defines the linear evaluation functional on V by assigning to a solution $f \in V$ its value $f(x)$ at x. The image of this evaluation map is a curve $\ell : I \to V^*$ uniquely associated to the original l.o.d.e. Choosing a basis $y_1, \ldots, y_n$ in V we, therefore, identify V^* with $\mathbb{R}^n$. The latter curve will then have the form $\ell(x) = (y_1(x), \ldots, y_n(x)) \in \mathbb{R}^n$ in the standard coordinates. Since the Wronskian determinant does not vanish on I the curve ℓ is non-degenerate, i.e., it has a non-degenerate osculating frame $\{\ell(x), \ell'(x), \ldots, \ell^{(n-1)}(x)\}$ for all $x \in I$.

Note that any solution f of (2.1) is interpreted as a vector in V which, in its turn, defines the corresponding hyperplane $H_f \subset V^*$. The zeros of f correspond to the intersections of ℓ with H_f. The number of zeros of f on I equals the number of intersection points in $\ell \cap H_f$ (counting multiplicities).

For each $1 \leq k \leq n-1$ the curve ℓ induces its *osculating Grassmann curve* $\ell_{G_k(n)} : I \to G_k(n)$ in the Grassmannian $G_k(n)$ of k-dimensional subspaces of $\mathbb{R}^n$. Analogously, one obtains from ℓ its *osculating flag curve* $\ell_{\mathcal{F}} : I \to Fl(n)$ in the variety $Fl(n)$ of complete flags in $\mathbb{R}^n$. Namely, for each point x the corresponding element $\ell_{G_k(n)}(x)$ is the k-dimensional osculating subspace to ℓ at the point x, i.e., the subspace spanned by $\ell(x), \ell'(x), \ldots, \ell^{(k-1)}(x)$. Analogously, the complete flag $\ell_{\mathcal{F}}(x)$ is formed by these subspaces, i.e., it is given by:

$$\Big(\mathrm{span}\langle \ell(x)\rangle \subset \mathrm{span}\langle \ell(x), \ell'(x)\rangle \subset \ldots \subset \mathrm{span}\langle \ell(x), \ell'(x), \ldots, \ell^{(n-2)}(x)\rangle \subset \mathbb{R}^n\Big).$$

Definition 2.1. Two compete flags $F_\bullet$ and $G_\bullet$ (in the same linear space) are called *transversal* if all their corresponding subspaces are in general position w.r.t. each other, i.e., for all i and j, $1 \leq i, j \leq n$ one has

$$\dim(F_i \cap G_j) = \max(i + j - n, 0),$$

which is the minimal possible value.

Remark 2.1. Flags $F_\bullet$ and $G_\bullet$ are transversal if and only if $\dim(F_i \cap G_{n-i}) = 0$ for all i, $1 \leq i \leq n$.

Definition 2.2. A k-dimensional vector subspace $W^k \subset \mathbb{R}^n$ is transversal to a flag $F_\bullet$ if it is in general position with all subspaces F_r of $F_\bullet$, i.e. $\dim(W^k \cap F_r) = max(k + r - n, 0)$.

Remark 2.2. Clearly, W^k is transversal to $F_\bullet$ if and only if it is transversal to F_{n-k}.

Definition 2.3. The *train* $Tr_{\mathcal{F}}F_\bullet$ (resp., *Grassmann train* $Tr_{G_k(n)}F_\bullet$) of a flag $F_\bullet$ in $\mathbb{R}^n$ is the set of all flags $G_\bullet$ (resp. all k-dimensional subspaces $W^k \in G_k(n)$) such that $G_\bullet$ (resp. W^k) and $F_\bullet$ are not transversal. We can say train in either case when the situation is clear from the context.

Example 2.3. Fix any complete flag $F_\bullet$ in V^* whose hyperplane H_f is dual to the line spanned by a solution f of (2.1). Then all moments x_i of non-transversality of the curve $\ell_{G_1(n)}$ with $F_\bullet$ are exactly the zeros $f(x_i) = 0$. More generally, all moments of non-transversality of the $\ell_{G_k(n)}$ or $\ell_{\mathcal{F}}$ with $F_\bullet$ correspond to intersections of the corresponding osculating Grassmann or flag curve with the corresponding train of flag $F_\bullet$. One can easily identify these moments of non-transversality with the zeros of the Wronskians of k-tuples of solutions of (2.1), see [12].

2.2. Sturm separation theorem

The classical Sturm separation theorem describes the relative position of the roots of two distinct solutions to a linear homogeneous second order differential equation. Namely, the following statement holds.

Theorem 2.1. Let y_1 and y_2 be two non-trivial real solutions of a second order ODE

$$y'' + p(x)y' + q(x)y = 0 \tag{2.2}$$

where $p(x)$ and $q(x)$ are continuous real-valued functions on I that are not multiples of each other. Denote by $\#_1$ (resp. $\#_2$) the number of zeros of y_1 (resp. of y_2) on I. Then between each pair of successive real roots of y_1 there is a root of y_2 and $|\#_1 - \#_2| \leq 1$.

V.I.Arnold generalized the above Sturm theorem to linear Hamiltonian systems with m degrees of freedom having a positive definite time-dependent Hamiltonian (see [1]). The role of zeros in this theory is played by the moments of non-transversality of the Grassmann curve in the Lagrangian Grassmannian to a fixed Lagrangian subspace. (They can also be interpreted as the zeros of certain Wronskians.)

However, at the moment no proven generalization of the Sturm separation theorem in the case of usual higher order l.o.d.e. is known.

Remark 2.4. Notice that by results of Kondratiev [7] for general l.o.d.e. of order greater than 2 no separation theorem can be obtained in terms of zeros of individual solutions.

Note also that we can interpret the zeros of solution f as non-transversality moments between curve $\ell_{G_1(n)}$ and some fixed flag $F_\bullet$ containing hyperplane H_f (see Example 2.3).

Our hope is to obtain a generalization of Sturm separation theorem for higher order l.o.d.e. in terms of the total number of non-transversality moments between $\ell_{\mathcal{F}}$ and some fixed flag $F_\bullet$.

Below we formulate a conjectural generalization of Sturm separation theorem and give some motivation for its validity using the notion of a disconjugate l.o.d.e.

2.3. Disconjugate ODE

Definition 2.4. A l.o.d.e. of order n

$$z^{(n)} + p_1(x)z^{(n-1)} + \ldots + p_n(x)z = 0 \tag{2.3}$$

with real-valued continuous coefficients $p_i(x)$ is called *disconjugate* on an interval I if any of its nontrivial solutions has at most $(n-1)$ zeros on I counting multiplicities.

Example 2.5. The space of solutions of $z^{(n)} = 0$ consists of all polynomials in x of degree less than n, and, therefore, any solution has at most $n-1$ zeroes counting multiplicities on an arbitrary real interval I (as well as on an arbitrary subset of $\mathbb{C}$).

Remark 2.6. Any l.o.d.e. of order n has a nontrivial solution with at least $(n-1)$ zeros. (For example, consider the nontrivial solution of the initial value problem given by $z(x_0) = z'(x_0) = \ldots = z^{(n-2)}(x_0) = 0$, $z^{(n-1)}(x_0) = 1$.) Further, any l.o.d.e. with continuous $p_i(x)$ is locally disconjugate, i.e. for any $x_0 \in I$ there exists its neighborhood such that the above equation is disconjugate in this neighborhood, see e.g. [8].

Recalling the correspondence between nth order l.o.d.e. and non-degenerate curves in $\mathbb{R}^n$ we obtain a geometric interpretation of disconjugacy.

$$\{\text{a l.o.d.e. of order } n\} \;\leftrightarrow\; \begin{array}{l}\{\text{a non-degenerate curve in the } n\text{-dimensional space } V^*\\ \text{(dual to the space of all solutions)}\}\end{array}$$

with

$$\{\text{a disconjugate l.o.d.e.}\} \;\leftrightarrow\; \begin{array}{l}\{\text{a space curve in } V^* \text{ which intersects any hyperplane}\\ \text{in } V^* \text{ at most } n-1 \text{ times (counting multiplicities)}\}\,.\end{array}$$

Using this correspondence we will call a non-degenerate curve in $\mathbb{R}^n$ *disconjugate* if it intersects any hyperplane in $\mathbb{R}^n$ at most $n-1$ times (counting multiplicities).

The following two lemmas provide criteria of disconjugacy of linear ordinary differential equations (or, equivalently, disconjugate curve) on interval I, compare to [8].

Notation: For a curve $\gamma : I \to \mathbb{R}^n$ we denote by $\gamma_{G_k(n)}$ and $\gamma_{\mathcal{F}}$ the corresponding osculating Grassmann and flag curves, respectively.

Lemma 2.2. (see [12]) A non-degenerate curve γ is *disconjugate* on I if and only if for all $t_1 \neq t_2 \in I$ one has that $\gamma_{\mathcal{F}}(t_1)$ is transverse to $\gamma_{\mathcal{F}}(t_2)$.

Lemma 2.3. (see [12]) A non-degenerate curve γ is *conjugate* (i.e. **not disconjugate**) if and only if for any complete flag $G_\bullet$ there is some $t \in I$ such that $G_\bullet$ and $\gamma_{\mathcal{F}}(t)$ are non-transversal.

2.4. Conjectural multiplicative Sturm separation theorem

One can split the time interval I for any equation (2.1) into maximally disconjugate subintervals. Instead of individual solutions one should compare different fundamental solutions, i.e., count the number of moments of non-transverality of the flag curve to the trains of two different complete flags. Lemmata 2.3 and 2.2 then give some estimate of the number of non-transversalities on each of the disconjugate subintervals. This idea leads to the following conjecture which is a generalization of Sturm separation theorem to the case of higher order l.o.d.e.'s.

For a non-degenerate curve $\gamma : I \to \mathbb{R}^n$ and any pair of fixed flags $G_\bullet$ and $\hat{G}_\bullet$ we denote by $\#_{\mathcal{F}}$ (resp. $\widehat{\#_{\mathcal{F}}}$) the number of moments of non-transversality between $\gamma_{\mathcal{F}}(t)$ and $G_\bullet$ (resp. $\widehat{G_\bullet}$).

Conjecture 2.4. Let γ be a non-degenerate curve in $\mathbb{R}^n$, $n \geq 2$. Then

$$\widehat{\#_{\mathcal{F}}} \leq \frac{n^3 - n}{6} \left(\#_{\mathcal{F}} + 1\right). \tag{2.4}$$

Remark 2.7. Note that $\frac{n^3-n}{6} = \sum_k \dim G_k(n)$.

Kondratiev's results show that one can not hope to get similar estimates to (2.4) in terms of nontransversality moments for individual Grassmannians $G_k(n)$. However, Conjecture 2.4 would follow from Conjecture 2.5 below.

Definition 2.5. A curve in $G_k(n)$ is called *Grassmann convex* if it intersects the train of any flag at most $\dim G_k(n) = k(n-k)$ times.

Conjecture 2.5. If γ is a non-degenerate disconjugate curve in $\mathbb{R}^n$ then its osculating Grassmann curve $\gamma_{G_k(n)} \subset G_k(n)$ is Grassman convex.

Remark 2.8. Conjecture 2.5 is evident in $\mathbb{R}^3$, proven in $\mathbb{R}^4$ see [16], and open in $\mathbb{R}^n$ for $n \geq 5$.

2.5. Local geometry of osculating flag curves

Examples in low dimensions led us to the following conjectures on the local geometry of osculating flag curves.

Recall that the train $TrF_\bullet$ of any flag $F_\bullet$ is an algebraic hypersurface in the space of (complete) flags. If $x \in TrF_\bullet$ then $TrF_\bullet$ separates a sufficiently small open ball B in the space of flags centered at x into a finite number of connected components.

Conjecture 2.6. Let γ be the germ of a non-degenerate curve in $\mathbb{R}^n$ and $\gamma_{\mathcal{F}}(0) \in TrF_\bullet$. Then the germ of the osculating flag curve $\gamma_{\mathcal{F}}$ crosses $TrF_\bullet$ and goes from one connected component of $B \setminus TrF_\bullet$ to another one, i.e., for a sufficiently small $\delta > 0$ flags $\gamma_{\mathcal{F}}(-\delta)$ and $\gamma_{\mathcal{F}}(\delta)$ belong to different connected components of $B \setminus TrF_\bullet$.

Conjecture 2.7. If the osculating curve $\gamma_{\mathcal{F}}$ of a disconjugate non-degenerate curve γ passes from any connected component C of $B \setminus TrF_\bullet$ to another component then it never returns back to C.

Remark 2.9. Conjectures 2.6 and 2.7 would imply a weaker version of Conjecture 2.4.

Conjecture 2.8. Let γ be a non-degenerate curve in $\mathbb{R}^n$, $n \geq 2$. Then, there is a positive integer constant $K(n)$ depending on n only such that

$$\widehat{\#_{\mathcal{F}}} \leq K(n)\,(\#_{\mathcal{F}} + 1)\,. \tag{2.5}$$

3. Transversality and M-varieties

In this section we discuss a relation between transversality and the so-called M-property of intersections of Schubert varieties with the emphasis on intersections of trains of osculating flags to a non-degenerate curve γ. For the osculating flag $\gamma_{\mathcal{F}}(t)$ we call the time moment t its *reference point.*

Note first that (the proof) of Lemma 2.2 implies that the trains $Tr\gamma_{\mathcal{F}}(t_1)$ and $Tr\gamma_{\mathcal{F}}(t_2)$ of any two distinct osculating flags to a disconjugate curve γ are transversal. Therefore, the topology of their intersection does not change when we change reference points t_1 and t_2.

Unfortunately, as it was shown in [10] for general disconjugate curves such transversality fails for intersections of the trains of more than two osculating flags.

However, the so-called dimensional transversality holds for intersection of Grassmann trains of osculating flags to the moment curve. The following result was proved by Eisenbud and Harris.

Lemma 3.1. (*Dimensional transversality*, see [6]) Suppose that $t_1 < t_2 < \ldots < t_r$ be the set of reference points on the moment curve ν in $\mathbb{R}^n$. Let $Sch_{\gamma_{\mathcal{F}}(t_i)}$ be the Schubert decomposition of Grassmannian $G_k(n)$with respect to the osculating flag $\nu_{\mathcal{F}}(t_i)$. Then flags $\nu_{\mathcal{F}}(t_i)$ have *dimensional transversality property*, i.e. the codimension of the intersection of an arbitrary set of cells $C_1, \ldots, C_r$, where C_i belongs to $Sch_{\nu_{\mathcal{F}}(t_i)}$, equals the maximum between $dim(G_k(n))$ and the sum of codimensions of C_is.

It implies, in particular, that intersections of $k(n-k)$-tuple of Grassmann trains in $G_k(n)$ of osculating flags to the moment curve is pure zero-dimensional, i.e., contains points only. Moreover, any intersection of more trains is empty.

The stronger transversality statement (see Conjecture 3.2) might be the actual reason why the total reality property holds for enumerative properties in Schubert calculus in Grassmannians (see [18]).

Below we discuss the total reality and the M-property of such problems. Let us first recall the notion of an M-variety.

Example 3.1. ***M*-curves.** Let $C = C^{\mathbb{C}}$ be a real (i.e., invariant under the standard complex conjugation) projective nonsingular algebraic curve of genus g embedded in $\mathbb{C}P^n$, and $C^{\mathbb{R}}$ be its real part, i.e., $C^{\mathbb{R}} = C^{\mathbb{C}} \cap \mathbb{R}P^n$. Then Harnack-Klein theorem claims that the number of connected components of $C^{\mathbb{R}}$ does not exceed $g+1$. The curve C is called an *M-curve* if the latter number of connected components *equals* $g+1$.

Furthermore, for any real (invariant under the complex conjugation) embedded projective algebraic variety $X = X^{\mathbb{C}} \subset \mathbb{C}P^n$, and its real part $X^{\mathbb{R}} = X \cap \mathbb{R}P^n$ the

Smith inequality claims that:

$$\sum b_i(X^{\mathbb{R}},\mathbb{Z}_2) \leq \sum b_i(X^{\mathbb{C}},\mathbb{Z}_2). \tag{3.1}$$

We say that X is an *M-variety* if $\sum b_i(X^{\mathbb{R}},\mathbb{Z}_2) = \sum b_i(X^{\mathbb{C}},\mathbb{Z}_2)$. (Roughly speaking, $X = X^{\mathbb{C}}$ is an M-variety if any cycle in $X^{\mathbb{C}}$ generates a cycle in $X^{\mathbb{R}}$.)

Example 3.2. The intersection of two open Schubert cells in general position in the space of complete flags is an M-variety [17].

Example 3.3. The same holds for intersections of two or more open real Schubert cells in the space of special incomplete two step flags consisting of a line in a hyperplane, see [15].

Conjecture 3.2. For the moment curve $\nu(t) = (1 : t : \ldots : t^{n-1})$ and any set of reference points $t_1 < t_2 < \ldots < t_m$ the corresponding Grassmann trains $Tr\nu_{G_k(n)}(t_i),\ 1 \leq i \leq m$ intersect transversally in $G_k(n)$.

F. Sottile in [19] proved that the special Schubert calculus is *totally real.* More exactly, he showed that there exist reference points

$$t_1 < t_2 < t_3 < \ldots < t_{k(n-k)} \tag{3.2}$$

on the moment curve ν in $\mathbb{R}^n$ such that all intersection points $\cap_{j=1}^{k(n-k)} Tr\nu_{G_k(n)}(t_j)$ are real.

Remark 3.4. Note that if a variety $X \subset \mathbb{C}P^n$ of pure dimension zero contains only real points then it is an M-variety.

The idea of transversality combined with the total reality of the special Schubert calculus leads to the following

Conjecture 3.3. (Total reality conjecture) Let $t_1 < \ldots < t_{k(n-k)}$ be an arbitrary $k(n-k)$-tuple of distinct reference points on the moment curve ν. Then the intersection $\cap_{j=1}^{k(n-k)} Tr\nu_{G_k(n)}(t_j)$ in any Grassmann manifold $G_k(n)$ is an M-variety.

Indeed, transversality means that all such intersections have the same topology for any positions of reference points. In particular, for the dimension zero the number of points would be the same.

The general transversality conjecture 3.2 remains widely open. But its zero-dimensional version and, in particular, conjecture 3.3 is proved. Conjecture 3.3 for $G_{n-2}(n)$ (or, by duality for $G_2(n)$) was proved by A. Eremenko and A. Gabrielov in [4]. The general case of $G_k(n)$ was proved by A. Varchenko, E. Mukhin, and V. Tarasov see [9].

As we mentioned above it was shown in [10] that if we replace the moment curve by a general disconjugate curve the analog of conjecture 3.3 fails. On the other hand, it is clear, that the set of curves for which conjecture 3.3 is valid is open in the C^n-topology on the space of smooth nondegenerate curves in $\mathbb{R}^n$.

Question: *Find an open neighborhood of the moment curve such that each curve in this neighborhood satisfies* total reality conjecture 3.3.

4. Total reality of meromorphic functions

In [4] the authors settled the total reality conjecture for $G_{n-2}(n)$ using its equivalent reformulation given below.

Namely, let $f = \dfrac{p(t)}{q(t)} = \dfrac{\sum_{i=0}^n p_i t^i}{\sum_{i=0}^n q_i t^i}$ be a rational function of degree n. Such a rational function determines a subspace $S_f \subset \mathbb{R}^{n+1}$ of codimension 2 given as follows. If $\{x_0, x_1, \dots, x_n\}$ are standard coordinates in $\mathbb{R}^{n+1}$ then S_f is the intersection of two hyperplanes given by the equations $\sum_{i=0}^n p_i x_i = 0$ and $\sum_{i=0}^n q_i x_i = 0$.

Remark 4.1. If $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0,\ a, b, c, d \in \mathbb{C}$ then we call functions $f(t)$ and $\dfrac{af(t)+b}{cf(t)+d}$ *Möbius-equivalent.*

Remark 4.2. Möbius-equivalent functions determine the same codimension 2 subspace.

Direct computation shows that the dimension two osculating subspace to the moment curve ν at the reference point t_0 intersects S_f if and only if $f'(t_0) = 0$.

Then the total reality conjecture can be formulated in the following way.

Equivalent reformulation: *Let* $f = \dfrac{p(t)}{q(t)}$ *be a rational function of degree* n *such that all its* $(2n-2)$ *critical points are real and distinct. Then* f *is Möbius-equivalent to a real function* g*, i.e., all coefficients of numerator and denominator of* g *are real.*

Inspired by the results of [4] for rational functions we want to ask whether total reality might hold for meromorphic functions on a real algebraic curve of a positive genus.

Question: *If* Σ *is a real curve of a positive genus and* ϕ *is a meromorphic function on* Σ *with all its critical points real, is it true that* ϕ *is Möbius-equivalent to a real meromorphic function?*

In conclusion, let us mention two partial cases when the above question has a positive answer.

Theorem 4.1. (See [5]) *A function of any prime degree* d *on any real curve of genus* $g > \frac{d^2-4d+3}{3}$ *whose critical points are real and distinct, is Möbius-equivalent to a real meromorphic function.*

Theorem 4.2. (See [3]) *Any function of degree at most four on any real curve whose critical points are real and distinct is Möbius-equivalent to a real meromorphic function.*

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