SOLUTIONS OF A GENERALISED STIELTJES EQUATION

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Abstract. We study a Stieltjes system of the form $\sum_{j\neq i}\frac{1}{x_i-x_j} = Q(x_i)$, $i=1,\dots,N$ with Q a monic polynomial of degree $M+1$.

When $M=0$ the system is the standard Stieltjes system. This has a unique – up to permutations – solution, which coincides – up to a transposition – with the roots of the Hermite polynomial of degree N.

We study the generalised Stieltjes system in the regime when the linear term of the potential is large. We prove that in this regime the generalised Stieltjes system effectively splits into $M+1$ weakly-coupled standard Stieltjes systems. As a corollary, we show that the number of inequivalent solutions of the generalised Stieltjes is $\binom{N+M}{N}$, namely $N+M$ choose N.

Contents

1. Introduction

In this short note we extend some aspects of the classical Heine-Stieltjes theory predicting the number of polynomial solutions in certain families of linear second order differential equations, see [2, 4]. A modern survey of this area together with its generalizations to higher order equations can be found in [3]. Our results are related to that of [1].

Let $M\geq 0, N\geq 1$ be natural numbers. For every $\underline{a}=(a_0,a_1,a_2,\dots,a_M)\subset\mathbb{C}^{M+1}$, the Stieltjes system is the following system of algebraic equations for the pairwise distinct complex unknowns $(x_1,\dots,x_N)\in\mathbb{C}^N\setminus\Delta_N$

$$\sum_{j\neq i}^{N}\frac{1}{x_i-x_j}=Q_{\underline{a}}(x_i),\quad i=1\dots N, \tag{1.1}$$

with $Q_{\underline{a}}(x)=x^{M+1}+a_Mx^M+\cdots+a_0$, and Δ_N is the *big - diagonal.* The polynomial $Q_{\underline{a}}$ is called the source.

Definition 1.1. We say that two solutions $\underline{x},\underline{y}$ are equivalent if they differ by a permutation, and we write $\underline{x}\approx\underline{y}$.

We say that a monic polynomial P of degree N solves the Stieltjes system with coefficients $\underline{a}$ if the roots of P solves the Stieltjes system with source $Q_{\underline{a}}$.

Equivalence classes of solutions are naturally in bijection with polynomial solutions.

Theorem 1.2. *For every monic polynomial of degree $M+1$, the number of inequivalent solutions of the Stieltjes system is less or equal than $\binom{N+M}{N}$.*

The above inequality is an equality on a dense and open (Zariski open) subset of polynomials.

Notation. Given $\underline{a} = (a_0, a_1, a_2, \dots, a_M) \subset \mathbb{C}^{M+1}$, we denote by $\underline{a}'$ the vector $(a_0, a_2, \dots, a_M) \subset \mathbb{C}^M$.

Conversely given $\underline{a}' = (a_0, a_2, \dots, a_M) \in \mathbb{C}^M$ and $a_1 \in \mathbb{C}$, we denote by $\underline{a}' \cup a_1$ the vector $\underline{a} = (a_0, a_1, a_2, \dots, a_M)$

Fixed $\underline{a}' \in \mathbb{C}^M$, we let $\mathcal{S}_{\underline{a}'} \subset \mathbb{C}^N \times \mathbb{C}$ consist of the ordered pair (P, a_1) where P is a polynomial solution of the Stieltjes equation with source $Q_{\underline{a}}$.

We also define $\overline{\mathcal{S}_{\underline{a}'}}$ as the closure (with respect to the euclidean topolgy) of $\mathcal{S}_{\underline{a}',N}$ in $\mathbb{C}^{N+1}$.

Given a real positive number $\rho > 0$, we let $\overline{C}_\rho = \{x \in \mathbb{C}, |x| > \rho\} \cup \{\infty\}$.

A partition $\lambda \vdash (N, M)$ of N into $M+1$ distinguishable boxes is is the datum of a vector $\lambda = (\lambda_0, \dots, \lambda_{M-1}, \lambda_O) \in \mathbb{N}^{M+1}$ such that $|\lambda| := \sum_{k=1}^{M-1} \lambda_k + \lambda_O = N$.

It is well known that there are $\binom{N+M}{N}$ partitions of (N, M).

Given a partition $\lambda = (\lambda_0, \dots, \lambda_{M-1}, \lambda_O) \vdash (N, M)$, we let $\bar{\lambda}_k = \sum_{l=0}^{k-1} \lambda_l$ and $\bar{\lambda}_O = \sum_{l=0}^{M-1} \lambda_l$.

2. The case $M = 0$.

If $M = 0$, by a linear change of coordinate we can always reduce to the system of the form

$$\sum_{j \neq i} \frac{1}{x_i - x_j} = x_i, \ i = 1, \dots, N. \tag{2.1}$$

The latter is the standard Stieltjes system. It is well-known that the unique polynomial solution is N-th Hermite polynomial

$$H_N(x) = (-2)^N e^{x^2} \frac{d^N}{dx^N} e^{-x^2}.$$

The zeroes of the N-th Hermite polynomial are all distinct and real. We name them $u_i^{(N)}$, $i = 1 \dots N$ are the zeroes of $H_n(x)$ such and we assume that they are ordered, namely $u_i^{(N)} < u_{i+1}^{(N)}$, $i = 1, \dots, N-1$. Since the Stieltjes system is invariant under the transformatio $x \to -x$, it follows that $u_i^{(N)} = u_{N+1-i}^{(N)}$.

Lemma 2.1. *Now let $J^{(N)}$ be the Jacobian matrix of the (2.1) evaluated at the solution $u_i^{(N)}$, namely*

$$J_{ij}^{(N)} = \delta_{ij} \left(1 + \sum_{j \neq i} \frac{1}{(u_i^{(N)} - u_j^{(N)})^2} \right) - (1 - \delta_{ij}) \frac{1}{(u_i^{(N)} - u_j^{(N)})^2}. \tag{2.2}$$

The eigenvalues of $J^{(N)}$ are the integer numbers $1, \dots, N$.

Proof. Let

$$F_i(x_1, \dots, x_N) := x_i - \sum_{j \neq i} \frac{1}{x_i - x_j}, \qquad i = 1, \dots, N.$$

Then the standard Stieltjes system (2.1) is precisely

$$F_i(x_1, \dots, x_N) = 0, \qquad i = 1, \dots, N,$$

and $J^{(N)}$ is the Jacobian matrix of the map

$$F = (F_1, \dots, F_N) : \mathbb{C}^N \to \mathbb{C}^N$$

evaluated at the zeroes $u_1^{(N)},\dots,u_N^{(N)}$ of H_N.

We shall construct explicitly N linearly independent eigenvectors of $J^{(N)}$.

Step 1: a family of deformed polynomials.

Fix $n\in\{0,\dots,N-1\}$ and consider

$$P_\varepsilon(x) := H_N(x)+\varepsilon H_n(x).$$

Since the zeroes $u_i^{(N)}$ of H_N are simple, for ε small enough the polynomial P_ε has N simple roots $x_i(\varepsilon)$ depending analytically on ε, with

$$x_i(0)=u_i^{(N)},\qquad i=1,\dots,N.$$

Differentiating the identity $P_\varepsilon(x_i(\varepsilon))=0$ at $\varepsilon=0$ gives

$$H_n(u_i^{(N)})+H_N'(u_i^{(N)})\,x_i'(0)=0,$$

hence

$$x_i'(0)=-\frac{H_n(u_i^{(N)})}{H_N'(u_i^{(N)})},\qquad i=1,\dots,N. \tag{2.3}$$

Step 2: evaluate the Stieltjes residual along this family.

For any monic polynomial

$$P(x)=\prod_{k=1}^{N}(x-x_k)$$

with simple roots $x_1,\dots,x_N$, one has the classical identity

$$\frac{P''(x_i)}{2P'(x_i)}=\sum_{j\neq i}\frac{1}{x_i-x_j},\qquad i=1,\dots,N.$$

Therefore, for the roots $x_i(\varepsilon)$ of P_ε,

$$F_i(x_1(\varepsilon),\dots,x_N(\varepsilon))=x_i(\varepsilon)-\frac{P_\varepsilon''(x_i(\varepsilon))}{2P_\varepsilon'(x_i(\varepsilon))}=-\frac{P_\varepsilon''(x_i(\varepsilon))-2x_i(\varepsilon)P_\varepsilon'(x_i(\varepsilon))}{2P_\varepsilon'(x_i(\varepsilon))}. \tag{2.4}$$

Now recall the Hermite differential equation

$$H_m''(x)-2xH_m'(x)+2mH_m(x)=0,\qquad m\geq 0.$$

Applying this to H_N and H_n yields

$$P_\varepsilon''(x)-2xP_\varepsilon'(x)+2NP_\varepsilon(x)=2(N-n)\varepsilon\,H_n(x).$$

Since $x_i(\varepsilon)$ is a root of P_ε, we get

$$P_\varepsilon''(x_i(\varepsilon))-2x_i(\varepsilon)P_\varepsilon'(x_i(\varepsilon))=2(N-n)\varepsilon\,H_n(x_i(\varepsilon)).$$

Substituting into (2.4) gives

$$F_i(x_1(\varepsilon),\dots,x_N(\varepsilon))=-(N-n)\varepsilon\,\frac{H_n(x_i(\varepsilon))}{P_\varepsilon'(x_i(\varepsilon))}. \tag{2.5}$$

Differentiating at $\varepsilon=0$ we obtain

$$\sum_{j=1}^{N}J_{ij}^{(N)}\,x_j'(0)=-(N-n)\frac{H_n(u_i^{(N)})}{H_N'(u_i^{(N)})}.$$

Using (2.3), this becomes

$$\sum_{j=1}^{N}J_{ij}^{(N)}\,x_j'(0)=(N-n)\,x_i'(0),\qquad i=1,\dots,N.$$

Hence the vector

$$\underline{\xi}^{(n)} := \big(x_1'(0), \dots, x_N'(0)\big) = \left(-\frac{H_n(u_1^{(N)})}{H_N'(u_1^{(N)})}, \dots, -\frac{H_n(u_N^{(N)})}{H_N'(u_N^{(N)})} \right)$$

is an eigenvector of $J^{(N)}$ with eigenvalue $N-n$.

Step 3: linear independence.

As n ranges from 0 to $N-1$, the corresponding eigenvalues are

$$N, N-1, \dots, 1.$$

These are pairwise distinct, so the corresponding eigenvectors are automatically linearly independent.

Therefore $J^{(N)}$ has N linearly independent eigenvectors, and its spectrum is exactly

$$\{1, 2, \dots, N\}.$$

This proves the lemma. □

3. The generalised Stieltjes system when the linear term is large

From now on, we fix $\underline{a'} \in \mathbb{C}^M$ and study the solutions of the Stieltjes system with source $Q_{\underline{a'} \cup a_1}$ when $a_1 \to \infty$. The solutions will be obtained as a Puiseaux series in a_1 which depend parametrically on $\underline{a'}$.

As a first step, we study of the roots of $Q_{\underline{a'} \cup a_1}$ when $a_1 \to \infty$. A standard analysis based on the Newton polygon shows that there are M solutions that diverges as $(-a_1)^{\frac{1}{M}}$ and one solution which converges to zero as $O(a_1^{-1})$.

More precisely, we have the following

Lemma 3.1. *There exists a $\rho > 0$ – depending on $\underline{a'}$ – such that the following statements hold.*

1. There exists a unique analytic functions $Y : \bar{\mathbb{C}}_{\rho^{1/M}} \to \mathbb{C}$ which satisfies the functional equation

$$Q_{\underline{a'} \cup -\mu^M}(\mu Y(\mu)) = 0, \; Y(\infty) = 1. \tag{3.1}$$

The function Y has the following Laurent expansion

$$Y(\mu) = 1 + \sum_{l=1}^{\infty} y_l \mu^{-l}, \tag{3.2}$$

where the coefficient y_l is a polynomial in a_m for all $m \geq M+1-l, m \neq 1$.

2. Assuming $a_0 \neq 0$, there exists a unique analytic function $Y_0 : \bar{\mathbb{C}}_\rho \to \mathbb{C}$

$$Q_{\underline{a}}(-a_0 a_1^{-1} \bar{Y}(a_1)) = 0, \; Y_0(\infty) = 1. \tag{3.3}$$

Moreover, as $a_0 \to 0$, $\bar{Y}(a_1)$ converges uniformly to the zero function.

Proof. The proof is essentially trivial. We prove 1. for benefit of the readear.

With $a_1 = -\mu^M$, we get

$$F(Y, \mu) := \mu^{-M-1} Y^{-1} Q_{\underline{a}}(\mu Y(\mu)) = Y^M - 1 + \sum_{l=1, l \neq N}^{N+1} a_{N+1-l} \mu^{-l} Y^{M-l}. \tag{3.4}$$

Now $F(1, \infty) = 0$ and $\frac{\partial F}{\partial Y}_{|(1,\infty)} = M$. The implicit function theorem yields the thesis. □

The variable μ be that we have introduced in the above lemma is a M-th root of $-a_1$, namely $a_1 = -\mu^M$, hence the above proposition allows us to obtain all solutions of the algebraic equation $Q_{\underline{a'} \cup a_1}(x) = 0$ for a_1 large.

In fact,

- Given $a_1 \in \bar{\mathbb{C}}_\rho$ and a root μ of $-\mu^M = a_1$, M solutions are of the form $x_l = e^{\frac{2\pi\sqrt{-1}l}{M}}\mu Y(e^{\frac{2\pi\sqrt{-1}l}{M}}\mu)$, with $l = 0,\ldots,M_1$.
- One solution is of the form $x = -a_0 a_1^{-1}\bar{Y}(a_1)$.

The second step is the Puiseaux expansion of $Q_{\underline{a'}\cup a_1}$ at one of these roots.

Lemma 3.2. *Let ρ and Y as in Lemma 3.1.*

1. The function

$$\mathcal{Q}_{\underline{a'}}(t,\nu) := \nu^{-M} Q_{\underline{a'}\cup -\nu^{2M}}(\nu^2 Y(\nu^2) + M^{\frac{1}{2}}\nu^{-M}t), \tag{3.5}$$

is analytic in the domain $\mathbb{C}\times\bar{\mathbb{C}}_{\rho^{1/2M}}$, and admits the following expansion at $\nu = \infty$

$$\mathcal{Q}_{\underline{a'}}(t,\nu) = t + \sum_{l=1}^{\infty}\sum_{k=0}^{N+1} c_{k,l}\nu^{-l}t^k,\ c_{k,l}\in\mathbb{C}. \tag{3.6}$$

Here, in case M is even and l is odd, $c_{k,l} = 0$.

2. The following identity holds

$$\mathcal{Q}^{(O)}_{\underline{a'}}(t,\nu) := \sqrt{-1}\nu^{-M} Q_{\underline{a'}\cup -\nu^{2M}}(\sqrt{-1}\nu^{-M}t) = t + \sum_{l=0,l\neq 1}^{N+1} c_l \nu^{-M(l+1)}t^l,\ c_l\in\mathbb{C}. \tag{3.7}$$

The third step of our approach is to make an a_1-dependent change of coordinates for the unknown variables $x_1,\ldots,x_N$ in such a way that for a_1 large the unknown $x_1,\ldots,x_N$ are localised at the M roots of the source $Q_{\underline{a'}\cup a_1}$. As it will turn-out, if the new variabel are correctly scaled, the generalised Stieltjes system is reduced to a small perturbation of $M+1$ weakly-coupled standard Stieltjes system.

To this aim we fix a partition $\lambda \vdash (N,M)$, and we define the following ν-dependent linear change of variables $\underline{x} = T^{(\lambda)}(\underline{t},\nu)$:

$$x_i = T_k(t^{(k)}_{i-\bar{\lambda}_k},\nu), 1+\bar{\lambda}_k \le i \le \bar{\lambda}_{k+1}, k = 0,\ldots,M-1 \tag{3.8}$$

$$x_i = T_O(t^{(O)}_{i-\bar{\lambda}_O},\nu), 1+\bar{\lambda}_O \le i \le N, \tag{3.9}$$

where

$$T_k(t,\nu) = e^{\frac{2\pi\sqrt{-1}k}{M}}\nu^2 Y(e^{\frac{2\pi\sqrt{-1}k}{M}}\nu^2) + M^{\frac{1}{2}}\nu^{-M}t \tag{3.10}$$

$$T_O(t,\nu) = \sqrt{-1}M^{\frac{1}{2}}\nu^{-M}t, \tag{3.11}$$

with Y is as in Lemma 3.1 [1].

Proposition 3.3. *Fix a vector $\underline{a'}\in\mathbb{C}^M$ and a partition $\lambda\vdash(N,M)$ of N into $M+1$ distinguishable boxes.*

1. After the change of variables $\underline{x} = T^{(\lambda)}(\underline{t},\nu)$, the generalised Stieltjes system with source $Q_{\underline{a'}\cup -\nu^{2M}}$ reads

$$\underline{F}(\underline{t},\nu) = 0, \qquad \underline{F} = (F^{(0)}_1,..,F^{(0)}_{\lambda_0},F^{(1)}_1,..,F^{(1)}_{\lambda_1},\ldots,F^{(O)}_1,\ldots,F^{(O)}_{\lambda_O}), \tag{3.12}$$

where $\underline{F}$ is analytic at $\nu=\infty$ and has the expansion

$$F^{(k)}_i(\underline{t},\nu) = t^{(k)}_i - \sum_{j=1,j\neq i}^{\lambda_k}\frac{1}{t^{(k)}_i - t^{(k)}_j} + O(\nu^{-1}), \qquad i = 1,\ldots,\lambda_k,\ k = 0,\ldots,M-1,$$

$$F^{(O)}_i(\underline{t},\nu) = t^{(O)}_i - \sum_{j=1,j\neq i}^{\lambda_O}\frac{1}{t^{(O)}_i - t^{(O)}_j} + O(\nu^{-1}), \qquad i = 1,\ldots,\lambda_O. \tag{3.13}$$

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[1] The reader may think that in the definition of the function T_k the term $\nu^M t$ should instead be $e^{\pi\sqrt{-1}k}\nu^M t$. However, since the standard Stieltjes system is invariant under $t\to -t$, the above modification is immaterial.

2. The system (3.12) admits a unique – up to the action of the permutation group $S_{\lambda_0}\times\cdots\times S_{\lambda_{M-1}}\times S_{\lambda_O}$ – solution $\underline{t}(\nu)$ analytic at $\nu=\infty$. This has the form

$$t_i^{(k)}(\nu)=u_i^{(\lambda_k)}+O(\nu^{-1}),\qquad i=1,\dots,\lambda_k,\tag{3.14}$$

$$t_i^{(O)}(\nu)=u_i^{(\lambda_O)}+O(\nu^{-1}),\qquad i=1,\dots,\lambda_O,\tag{3.15}$$

where $u_1^{(r)},\dots,u_r^{(r)}$ are the roots of the Hermite polynomial H_r.

3. Let $\underline{X}^{(\lambda)}(\nu)=T^{(\lambda)}(\underline{t}(\nu),\nu)$ denote the corresponding solution of the generalised Stieltjes system (1.1). Then, with $\zeta=e^{\pi i/M}$ and

$$\lambda_k'=\begin{cases}\lambda_{k+1}, & k=0,\dots,M-2,\\ \lambda_0, & k=M-1,\end{cases}\qquad \lambda_O'=\lambda_O,$$

one has

$$\underline{X}^{(\lambda)}(\zeta\nu)\approx\underline{X}^{(\lambda')}(\nu),\qquad \underline{X}^{(\lambda)}(-\nu)\approx\underline{X}^{(\lambda)}(\nu).$$

Proof. We write

$$\omega=e^{\frac{2\pi i}{M}},\qquad \alpha_k(\nu):=\omega^k\nu^2Y(\omega^k\nu^2),\qquad \beta(\nu):=M^{1/2}\nu^{-M}.$$

Then

$$T_k(t,\nu)=\alpha_k(\nu)+\beta(\nu)t,\qquad T_O(t,\nu)=i\,\beta(\nu)t.$$

We also set

$$I_k:=\{1+\bar\lambda_k,\dots,\bar\lambda_{k+1}\},\qquad I_O:=\{1+\bar\lambda_O,\dots,N\},$$

so that for $i\in I_k$ we write $x_i=T_k(t_i^{(k)},\nu)$, while for $i\in I_O$ we write $x_i=T_O(t_i^{(O)},\nu)$.

Step 1: formula for the transformed system.

Fix $k\in\{0,\dots,M-1\}$ and $i\in I_k$. The i-th equation of (1.1) reads

$$\sum_{\substack{j\in I_k\\ j\neq i}}\frac{1}{x_i-x_j}+\sum_{\ell\neq k}\sum_{j\in I_\ell}\frac{1}{x_i-x_j}+\sum_{j\in I_O}\frac{1}{x_i-x_j}=Q_{\underline{a}'\cup-\nu^{2M}}(x_i).\tag{3.16}$$

Since $x_i=\alpha_k+\beta t_i^{(k)}$ and $x_j=\alpha_k+\beta t_j^{(k)}$ for $j\in I_k$, we have

$$x_i-x_j=\beta\big(t_i^{(k)}-t_j^{(k)}\big),\qquad \frac{1}{x_i-x_j}=M^{-1/2}\nu^M\frac{1}{t_i^{(k)}-t_j^{(k)}}.$$

Multiplying (3.16) by $\beta=M^{1/2}\nu^{-M}$ we obtain

$$\sum_{\substack{j\in I_k\\ j\neq i}}\frac{1}{t_i^{(k)}-t_j^{(k)}}+\beta\sum_{\ell\neq k}\sum_{j\in I_\ell}\frac{1}{T_k(t_i^{(k)},\nu)-T_\ell(t_j^{(\ell)},\nu)}+\beta\sum_{j\in I_O}\frac{1}{T_k(t_i^{(k)},\nu)-T_O(t_j^{(O)},\nu)}$$
$$=M^{1/2}\nu^{-M}Q_{\underline{a}'\cup-\nu^{2M}}\big(T_k(t_i^{(k)},\nu)\big).\tag{3.17}$$

By Lemma 3.2, with the variable rescaled by the factor $M^{1/2}$,

$$M^{1/2}\nu^{-M}Q_{\underline{a}'\cup-\nu^{2M}}\big(T_k(t,\nu)\big)=M^{1/2}\mathcal{Q}_{\underline{a}'}(M^{1/2}t,\nu)=t+O(\nu^{-1}),$$

uniformly on compact subsets of the t-plane. Therefore we define

$$F_i^{(k)}(\underline{t},\nu):=M^{1/2}\nu^{-M}Q_{\underline{a}'\cup-\nu^{2M}}\big(T_k(t_i^{(k)},\nu)\big)-\sum_{\substack{j\in I_k\\ j\neq i}}\frac{1}{t_i^{(k)}-t_j^{(k)}}\tag{3.18}$$
$$-\beta\sum_{\ell\neq k}\sum_{j\in I_\ell}\frac{1}{T_k(t_i^{(k)},\nu)-T_\ell(t_j^{(\ell)},\nu)}-\beta\sum_{j\in I_O}\frac{1}{T_k(t_i^{(k)},\nu)-T_O(t_j^{(O)},\nu)}.$$

Then the equations (3.16) are exactly $F_i^{(k)}=0$.

Now fix $i \in I_O$. The i-th equation of (1.1) is

$$\sum_{\substack{j\in I_O\\ j\neq i}} \frac{1}{x_i - x_j} + \sum_{k=0}^{M-1} \sum_{j\in I_k} \frac{1}{x_i - x_j} = Q_{\underline{a}'\cup -\nu^{2M}}(x_i), \qquad x_i = i\beta t_i^{(O)}. \tag{3.19}$$

For $j \in I_O$,

$$x_i - x_j = i\beta\big(t_i^{(O)} - t_j^{(O)}\big), \qquad \frac{1}{x_i - x_j} = -\,i\,M^{-1/2}\nu^{M} \frac{1}{t_i^{(O)} - t_j^{(O)}}.$$

Multiplying (3.19) by $-i\beta$ gives

$$\sum_{\substack{j\in I_O\\ j\neq i}} \frac{1}{t_i^{(O)} - t_j^{(O)}} - i\beta \sum_{k=0}^{M-1} \sum_{j\in I_k} \frac{1}{T_O(t_i^{(O)},\nu) - T_k(t_j^{(k)},\nu)}$$
$$= -\,i\,M^{1/2}\nu^{-M} Q_{\underline{a}'\cup -\nu^{2M}}\big(T_O(t_i^{(O)},\nu)\big). \tag{3.20}$$

Again by Lemma 3.2,

$$-\,i\,M^{1/2}\nu^{-M} Q_{\underline{a}'\cup -\nu^{2M}}\big(T_O(t,\nu)\big) = M^{1/2}\mathcal{Q}^{(O)}_{\underline{a}'}(M^{1/2}t,\nu) = t + O(\nu^{-1}),$$

so we define

$$F_i^{(O)}(\underline{t},\nu) := \,-\,i\,M^{1/2}\nu^{-M} Q_{\underline{a}'\cup -\nu^{2M}}\big(T_O(t_i^{(O)},\nu)\big) - \sum_{\substack{j\in I_O\\ j\neq i}} \frac{1}{t_i^{(O)} - t_j^{(O)}} \tag{3.21}$$
$$+ i\beta \sum_{k=0}^{M-1} \sum_{j\in I_k} \frac{1}{T_O(t_i^{(O)},\nu) - T_k(t_j^{(k)},\nu)}.$$

Then (3.19) is exactly $F_i^{(O)} = 0$.

Step 2: analyticity at $\nu = \infty$.

The only point to check is that the cross-interaction terms are analytic in ν^{-1} near $\nu^{-1} = 0$. Fix $k \neq \ell$. Since

$$\alpha_k(\nu) - \alpha_\ell(\nu) = \nu^2\Big(\omega^k Y(\omega^k\nu^2) - \omega^\ell Y(\omega^\ell\nu^2)\Big) = (\omega^k - \omega^\ell)\nu^2 + O(\nu),$$

we have

$$T_k(t,\nu) - T_\ell(s,\nu) = (\omega^k - \omega^\ell)\nu^2\Big(1 + O(\nu^{-1})\Big)$$

uniformly for t, s in compact sets. Hence

$$\frac{1}{T_k(t,\nu) - T_\ell(s,\nu)} = \frac{1}{(\omega^k - \omega^\ell)\nu^2}\Big(1 + O(\nu^{-1})\Big), \tag{3.22}$$

which is analytic in ν^{-1} near 0.

Likewise,

$$T_k(t,\nu) - T_O(s,\nu) = \alpha_k(\nu) + O(\nu^{-M}) = \omega^k\nu^2\big(1 + O(\nu^{-1})\big),$$

hence

$$\frac{1}{T_k(t,\nu) - T_O(s,\nu)} = \omega^{-k}\nu^{-2}\big(1 + O(\nu^{-1})\big), \tag{3.23}$$

again analytic in ν^{-1}.

Since each cross term in (3.18) or (3.21) is multiplied by $\beta = M^{1/2}\nu^{-M}$, all cross interactions are actually $O(\nu^{-M-2})$, in particular $O(\nu^{-1})$. Combining this with the source expansions above proves part 1:

$$F_i^{(k)}(\underline{t},\nu) = t_i^{(k)} - \sum_{j\neq i} \frac{1}{t_i^{(k)} - t_j^{(k)}} + O(\nu^{-1}), \qquad F_i^{(O)}(\underline{t},\nu) = t_i^{(O)} - \sum_{j\neq i} \frac{1}{t_i^{(O)} - t_j^{(O)}} + O(\nu^{-1}).$$

Step 3: the limiting system at $\nu = \infty$.

Setting $\nu = \infty$ in (3.18) and (3.21), the cross-interaction terms disappear and we obtain

$$F_i^{(k)}(\underline{t}, \infty) = t_i^{(k)} - \sum_{\substack{j=1 \\ j \neq i}}^{\lambda_k} \frac{1}{t_i^{(k)} - t_j^{(k)}}, \qquad F_i^{(O)}(\underline{t}, \infty) = t_i^{(O)} - \sum_{\substack{j=1 \\ j \neq i}}^{\lambda_O} \frac{1}{t_i^{(O)} - t_j^{(O)}}. \tag{3.24}$$

Thus the limiting system is the product of $M+1$ standard Stieltjes systems, one for each block.

For each $r \geq 0$, the standard Stieltjes system with r unknowns has, up to permutation, the unique solution given by the roots of H_r. Therefore (3.24) has, up to the natural action of $S_{\lambda_0} \times \cdots \times S_{\lambda_{M-1}} \times S_{\lambda_O}$, the unique solution

$$\underline{u}^{(\lambda)} := \left(u_1^{(\lambda_0)}, \dots, u_{\lambda_0}^{(\lambda_0)}; u_1^{(\lambda_1)}, \dots, u_{\lambda_1}^{(\lambda_1)}; \dots; u_1^{(\lambda_O)}, \dots, u_{\lambda_O}^{(\lambda_O)}\right).$$

Let J_∞ be the Jacobian matrix of $\underline{F}(\cdot, \infty)$ at $\underline{u}^{(\lambda)}$. By (3.24), J_∞ is block diagonal, with blocks

$$J^{(\lambda_0)}, \dots, J^{(\lambda_{M-1})}, J^{(\lambda_O)}.$$

By Lemma 2.1, each block is invertible. Hence J_∞ is invertible.

Since $\underline{F}(\underline{t}, \nu)$ is analytic in $(\underline{t}, \nu^{-1})$ near $(\underline{u}^{(\lambda)}, 0)$, the analytic implicit function theorem yields a unique analytic solution

$$\underline{t}(\nu) = \underline{u}^{(\lambda)} + O(\nu^{-1}), \qquad \nu \to \infty.$$

This proves part 2.

Step 4: symmetry under rotations of ν.

We first note that the source only depends on ν through $a_1 = -\nu^{2M}$, hence

$$Q_{\underline{a'} \cup -\nu^{2M}} = Q_{\underline{a'} \cup -(\zeta\nu)^{2M}} = Q_{\underline{a'} \cup -(-\nu)^{2M}}, \qquad \zeta = e^{\pi i/M}.$$

Now

$$\alpha_k(\zeta\nu) = \omega^k (\zeta\nu)^2 Y\left(\omega^k (\zeta\nu)^2\right) = \omega^{k+1} \nu^2 Y\left(\omega^{k+1} \nu^2\right) = \alpha_{k+1}(\nu),$$

where indices are taken modulo M. Moreover,

$$\beta(\zeta\nu) = M^{1/2} (\zeta\nu)^{-M} = -\beta(\nu).$$

Therefore

$$T_k(t, \zeta\nu) = \alpha_{k+1}(\nu) - \beta(\nu) t = T_{k+1}(-t, \nu).$$

Since the standard Stieltjes system is invariant under $t \mapsto -t$, the branch $\underline{t}(\zeta\nu)$ corresponds, up to permutation inside each block, to the branch attached to the shifted partition λ'. Hence

$$\underline{X}^{(\lambda)}(\zeta\nu) \approx \underline{X}^{(\lambda')}(\nu).$$

Similarly, since $(-\nu)^2 = \nu^2$ and $(-\nu)^{-M} = (-1)^M \nu^{-M}$, we have

$$\alpha_k(-\nu) = \alpha_k(\nu), \qquad T_k(t, -\nu) = \alpha_k(\nu) + (-1)^M \beta(\nu) t.$$

Likewise

$$T_O(t, -\nu) = (-1)^M T_O(t, \nu).$$

Thus, up to the involution $t \mapsto -t$ in each block, which preserves the limiting Hermite configuration up to permutation, the transformed system for $-\nu$ is the same as that for ν. By uniqueness of the analytic branch from part 2, we conclude that

$$\underline{X}^{(\lambda)}(-\nu) \approx \underline{X}^{(\lambda)}(\nu).$$

This proves part 3. □

4. Proof of Theorem 1.2 via the Heine–Stieltjes correspondence

In this section we complete the proof of Theorem 1.2 by relating the Stieltjes system to a second order differential equation and computing the degree of the resulting algebraic family, comp. [2, 4].

4.1. From the root system to a differential equation.

4.1. From the root system to a differential equation. Let $x_1, \dots, x_N$ be a solution of (1.1), and let

$$P(x) = \prod_{i=1}^{N} (x - x_i)$$

be the associated monic polynomial of degree N.

At a simple root x_i one has the classical identity

$$\frac{P''(x_i)}{2P'(x_i)} = \sum_{j \neq i} \frac{1}{x_i - x_j}.$$

Therefore the system (1.1) is equivalent to

$$P''(x_i) - 2Q(x_i)P'(x_i) = 0, \qquad i = 1, \dots, N,$$

hence

$$P \mid \big(P'' - 2QP'\big).$$

It follows that there exists a polynomial V of degree at most M such that

$$P'' - 2Q\,P' - V\,P = 0. \tag{4.1}$$

Conversely, if a monic polynomial P of degree N satisfies (4.1), then all its roots are simple, and they satisfy (1.1). Indeed, if $x = \xi$ is a root of multiplicity $m \geq 2$, then writing $P(x) = (x - \xi)^m \widetilde{P}(x)$ with $\widetilde{P}(\xi) \neq 0$, one checks that the coefficient of $(x - \xi)^{m-2}$ in the left-hand side of (4.1) equals $m(m-1)\widetilde{P}(\xi) \neq 0$, a contradiction.

Thus solutions of (1.1) are in bijection with monic polynomial solutions P of (4.1).

4.2. The algebraic family. Fix $\underline{a}' = (a_0, a_2, \dots, a_M)$ and consider the one-parameter family

$$Q_{a_1}(x) = x^{M+1} + a_M x^M + \dots + a_2 x^2 + a_1 x + a_0.$$

Write

$$P(x) = x^N + c_1 x^{N-1} + \dots + c_N, \qquad V(x) = v_M x^M + \dots + v_0.$$

Substituting into (4.1) and comparing coefficients of powers of x gives a system of $N + M + 1$ polynomial equations in the variables

$$(c_1, \dots, c_N, v_0, \dots, v_M, a_1).$$

Let $X_{\underline{a}'}$ be the corresponding affine algebraic set.

By the previous subsection, points of $X_{\underline{a}'}$ are in bijection with solutions of (1.1) for $Q = Q_{a_1}$.

Consider the projection

$$\pi : X_{\underline{a}'} \longrightarrow A^1, \qquad (P, V, a_1) \longmapsto a_1.$$

4.3. **Behaviour for large $|a_1|$.** By Proposition 3.3, for $a_1 = -\nu^{2M}$ with $|\nu| \gg 1$, and for every partition

$$\lambda = (\lambda_0, \dots, \lambda_{M-1}, \lambda_O) \vdash (N, M),$$

there exists a solution $(x_1, \dots, x_N)$ of (1.1) such that:

- for $k = 0, \dots, M-1$, exactly λ_k roots lie near the k-th large root of Q, at scale ν^2,
- λ_O roots lie near the small root, at scale ν^{-2M},
- after rescaling, each cluster converges to the unique Hermite configuration of the corresponding size.

In particular, for $|\nu| \gg 1$ the fiber $\pi^{-1}(a_1)$ contains at least

$$\binom{N+M}{N}$$

distinct points.

Lemma 4.1. *Every branch of solutions of* (1.1) *as* $|a_1| \to \infty$ *arises from a partition* $\lambda \vdash (N, M)$ *as above.*

Proof. Let $a_1^{(n)} \to \infty$ and let P_n be corresponding monic solutions of (4.1). Writing $P_n(x) = \prod_{i=1}^N (x - x_i^{(n)})$, we analyze the possible asymptotics of the roots $x_i^{(n)}$.

From the equation $P_n'' - 2Q_{a_1^{(n)}} P_n' - V_n P_n = 0$, balancing the highest degree terms shows that all roots must lie either at scale $|a_1|^{1/M}$ (near the M large roots of Q) or at scale $|a_1|^{-1}$ (near the small root). No intermediate scaling is compatible with the degree constraints in (4.1).

After passing to a subsequence, the roots split into $M+1$ clusters with multiplicities $\lambda_0, \dots, \lambda_{M-1}, \lambda_O$. Performing the same rescalings as in Proposition 3.3, one checks that each cluster converges to a solution of the standard Stieltjes system for the Hermite polynomial of the corresponding degree. By uniqueness of that system, each cluster is uniquely determined.

This shows that every branch is of the form constructed in Proposition 3.3. □

4.4. **Degree computation and conclusion.** Consider the closure $\overline{X}_{\underline{a}'}$ of $X_{\underline{a}'}$ over $\mathbb{C}P^1_{a_1}$. By Lemma 4.1 and Proposition 3.3, the fiber of π over $a_1 = \infty$ consists of exactly $\binom{N+M}{N}$ points.

Moreover, these points are reduced: after the rescalings used in Proposition 3.3, the Jacobian matrix becomes block diagonal, with blocks equal to the Jacobians of the classical Stieltjes systems, which are invertible by Lemma 2.1.

Therefore the degree of π equals

$$\deg \pi = \binom{N+M}{N}.$$

It follows that for every a_1 one has

$$\#\pi^{-1}(a_1) \le \deg \pi = \binom{N+M}{N},$$

and for generic a_1 equality holds and the fiber consists of reduced points.

This proves Theorem 1.2. □

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