PERSISTENT HILBERT-FUNCTION DEFECTS FOR PRODUCTS OF POWERS OF LINEAR FORMS

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ABSTRACT. Multiplying powers of general linear forms by additional factors need not restore the Hilbert series predicted for general forms. We give a quantitative criterion for this failure in terms of a hypersurface with prescribed multiplicities at the dual points. Under a strict degree–multiplicity inequality, the failure occurs in every sufficiently large degree, even when the degrees of the additional factors grow linearly with the degree of the generators. Moreover, the quotient has dimension at least a constant times d^{n-1} in a linearly growing interval where the predicted Hilbert function is zero. The argument applies to arbitrary additional factors, which may have different degrees on different generators. The Waldschmidt constant determines the sharp asymptotic threshold for this criterion. Conics and related plane curves give examples with five, six, seven, and eight generators in three variables; a quadric gives examples with nine generators in four variables. For five ternary generators the first non-pure failure occurs in degree 14, for the partition $(13, 1)$, and its Hilbert series is exactly the predicted series plus t^{24}. The obstruction is geometric; the assertion of minimality is supported by a reproducible exact certificate.

1. INTRODUCTION

Let $S = \mathbb{C}[x_1, \dots, x_n]$, where $n \geq 2$, and write

$$G_{n,d,r}(t) = \left[\frac{(1-t^d)^r}{(1-t)^n}\right]_+.$$

Here the brackets mean truncation at the first non-positive coefficient. Fröberg's conjecture predicts this Hilbert series for the quotient by r general forms of degree d [Fro85]. In three variables the prediction is a theorem of Anick [An86].

For a partition $\mu = (\mu_1, \dots, \mu_s) \vdash d$, a μ-power form is a product $L_1^{\mu_1} \cdots L_s^{\mu_s}$ of pairwise non-proportional linear forms. Problem F of Fröberg, Lundqvist, Oneto, and Shapiro [FLOS18] asks whether r general such forms have Hilbert series $G_{n,d,r}$ whenever $\mu \neq (d)$. The exclusion of the pure partition is necessary: ideals generated by powers of linear forms can have larger Hilbert functions, as reflected in the Fröberg–Iarrobino conjecture [Iar97].

The answer to Problem F is negative. More specifically, a large linear-factor multiplicity can force a defect that survives the introduction of additional factors. Suppose that

$$I = (L_1^{d-a_1} H_1, \dots, L_r^{d-a_r} H_r), \qquad \deg H_i = a_i. \tag{1}$$

The elementary containment $I \subseteq (L_1^{d-a_1}, \dots, L_r^{d-a_r})$, together with the inverse-system description of powers of linear forms [EI95], gives lower bounds for $\mathrm{HF}_{S/I}$. The issue is whether those bounds remain larger than the predicted Hilbert function after the degrees of the generators have increased. The main result gives a uniform answer, including when the a_i increase with d.

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Let $T = \mathbb{C}[X_1, \dots, X_n]$ be the dual polynomial ring. A linear form $L_i = \sum_j \lambda_{ij} x_j$ determines the point $P_i = [\lambda_{i1} : \cdots : \lambda_{in}]$ in the projective space with coordinates $X_1, \dots, X_n$. Put $N = n - 1$.

Theorem 1.1 (Persistence under additional factors)**.** *Suppose that a nonzero form $B \in T_k$ vanishes to order at least $m_0 \geq 1$ at each of $P_1, \dots, P_r$, and set*

$$\sigma = \frac{k}{m_0}, \qquad R = r^{1/N}.$$

Assume $2 \leq \sigma < R$. For every real number

$$0 \leq \varepsilon < \frac{R - \sigma}{\sigma(R-1)} \tag{2}$$

there exist constants d_0, $c > 0$, and $1 < \lambda_- < \lambda_+ < 2$ with the following property. For every integer $d \geq d_0$, every choice of integers $0 \leq a_i \leq \varepsilon d$, and every choice of nonzero $H_i \in S_{a_i}$, the ideal (1) *satisfies*

$$[t^q] G_{n,d,r} = 0, \qquad \mathrm{HF}_{S/I}(q) \geq c d^{n-1} \quad \textit{for } \lambda_- d \leq q \leq \lambda_+ d. \tag{3}$$

In particular, the sum of these Hilbert-function values over the indicated integer degrees is at least $c' d^n$ for some $c' > 0$ and all sufficiently large d.

The constants depend only on the numerical data and ε; no generality assumption on the H_i is needed. In particular, they may be products of linear forms with prescribed multiplicities. A fixed bound on the a_i is allowed, since it is eventually smaller than εd for any fixed positive ε satisfying (2).

The required form B is easy to construct in the following cases. The proof in Section 3 uses only interpolation and multiplication of forms; the familiar interpretation by curves on del Pezzo surfaces is not needed for existence.

Corollary 1.2. *For each of*

$$(n, r) = (3, 5), (3, 6), (3, 7), (3, 8), (4, 9)$$

and each fixed integer $a \geq 1$, every non-pure partition of d having a part at least $d - a$ gives a negative answer to Problem F for all sufficiently large d. The stronger conclusion of Theorem 1.1, allowing different partitions on different generators, also holds in each case.

For $n = 3$ these are failures of the actual Hilbert series of general forms, by Anick's theorem. In four variables the comparison throughout the paper is with $G_{4,d,r}$; no unproved case of Fröberg's conjecture is used to identify this series with the Hilbert series of general forms.

The first ternary example is quite small, although it lies beyond the range suggested by low-degree experiments.

Theorem 1.3 (The first five-generator failure)**.** *Five general μ-power forms in three variables have Hilbert series $G_{3,d,5}$ for every non-pure partition of every $2 \leq d \leq 13$. In degree 14 the same assertion holds for every non-pure partition except* $(13, 1)$. *For five general forms of this exceptional type,*

$$\mathrm{HS}_{S/(L_1^{13} M_1, \dots, L_5^{13} M_5)}(t) = G_{3,14,5}(t) + t^{24}.$$

The nonzero class in degree 24 is provided by the twelfth power of the conic through the five dual points. Its existence requires no computation. The exact Hilbert series and the assertion that no earlier non-pure example occurs use the finite certificate in Appendix A.

The surrounding positive results help delimit the problem. Coalescing the linear factors shows that every positive result for pure powers also applies to all partitions of the same degree. Stanley's strong Lefschetz theorem therefore settles $r \le n+1$ for every partition. More generally, Nenashev's maximal-rank results apply to any nonempty class of forms invariant under linear changes of coordinates [Nen17], hence directly to μ-power forms. The equivariant vector-bundle methods of Blomenhofer and Casarotti [BC25] give further results of this kind. Recent work of Boij, Dannetun, and Lundqvist [BDL26] proves additional cases for general forms in low degrees. The present results concern a complementary range: a fixed number of generators and increasing degree, with a large repeated linear factor.

Section 2 proves the comparison statements and Theorem 1.1. Section 3 constructs the families and gives explicit thresholds. Section 4 proves Theorem 1.3 and explains what the method leaves open.

2. Comparison with power ideals and persistence

We first record the two different ways in which a product of powers can be compared with a pure power. Specialization gives upper bounds on the generic Hilbert function, whereas divisibility gives lower bounds for every member of a family.

2.1. **Specialization and the positive range.**

A partition ν is a coarsening of μ if its parts are obtained by grouping and summing the parts of μ.

Proposition 2.1. *Let $\mu \vdash d$ and let ν be a coarsening of μ. The generic Hilbert function for r μ-power generators is bounded above, degree by degree, by that for r ν-power generators. If the latter Hilbert series equals $G_{n,d,r}$, then so does the former. The same assertion holds when the partition is allowed to vary from one generator to another.*

Proof. Coalesce the factors in each group to obtain a ν-power form. Thus the ν-power locus is contained in the closure of the μ-power locus. In degree $q \ge d$, the quotient dimension is the corank of

$$\Phi_q : S_{q-d}^r \longrightarrow S_q, \qquad (A_1, \dots, A_r) \longmapsto \sum_{i=1}^{r} A_i F_i. \tag{4}$$

Matrix rank is lower semicontinuous, which proves the first assertion. Here we use the usual generic Hilbert series, constant on a nonempty open subset of the fixed-degree parameter space. Fröberg's general lower bound is lexicographic [Fro85]. If the specialized series is $G_{n,d,r}$, the generic series is coefficientwise at most $G_{n,d,r}$ and lexicographically at least $G_{n,d,r}$, so equality follows. The specialization can be made independently for each generator. □

Corollary 2.2. *If $r \le n+1$, then general products of powers of linear forms, each of total degree d, have Hilbert series $G_{n,d,r}$ for any prescribed partitions. For binary forms the conclusion holds for every r.*

Proof. Specialize each generator to a pure power. For $r \le n$ one obtains an initial segment of $x_1^d, \dots, x_n^d$, a regular sequence. For $r = n+1$ use $x_1^d, \dots, x_n^d, (x_1 + \dots + x_n)^d$ and Stanley's strong Lefschetz theorem for monomial complete intersections [Sta78]. In two variables the inverse-system description reduces the pure-power case to prescribed vanishing at distinct points of $\mathbb{P}^1$; in degree $q \ge d$ its dimension is $\max\{q+1-r(q-d+1), 0\}$. This gives $G_{2,d,r}$. Apply Proposition 2.1. □

A failure for a coarsening does not, by itself, imply a failure for a refinement. The next comparison supplies the lower bound needed for that direction.

2.2. **Divisibility and inverse systems.** The ring S acts on T by constant-coefficient differentiation. For a homogeneous ideal J, write $(S/J)_q^\vee$ for the orthogonal complement of J_q under the perfect pairing $S_q \times T_q \longrightarrow \mathbb{C}$. Let $\mathfrak{p}_i \subset T$ be the ideal of P_i. For $Z = \{P_1, \dots, P_r\}$, put

$$I_Z^{(m)} = \bigcap_i \mathfrak{p}_i^m, \qquad h_Z(q;m) = \dim [I_Z^{(m)}]_q \quad (m \geq 1).$$

We use the convention $h_Z(q;m) = \dim T_q$ when $m \leq 0$.

Proposition 2.3. *Let $I = (L_1^{b_1} H_1, \dots, L_r^{b_r} H_r)$, where $b_i \geq 1$ and the H_i are arbitrary homogeneous forms. In every degree q there is an inclusion*

$$\left[\bigcap_{i=1}^r \mathfrak{p}_i^{\max\{q-b_i+1,0\}}\right]_q \subseteq (S/I)_q^\vee. \tag{5}$$

In particular, if all generators have degree d, $b_i = d - a_i$, and $a = \max_i a_i$, then

$$\mathrm{HF}_{S/I}(q) \geq h_Z(q; q-d+a+1). \tag{6}$$

Proof. Set $J = (L_1^{b_1}, \dots, L_r^{b_r})$. Since $I \subseteq J$, we have $(S/J)_q^\vee \subseteq (S/I)_q^\vee$. The inverse-system identity of Emsalem and Iarrobino [EI95] identifies $(S/J)_q^\vee$ with the left side of (5). For completeness, take one factor $L = x_1$ and its dual point $P = [1 : 0 : \cdots : 0]$. A degree-q form is annihilated by $\partial_{X_1}^b$ precisely when every one of its monomials has total degree at least $q - b + 1$ in $X_2, \dots, X_n$. This is vanishing to that order at P. Intersecting these conditions proves the identity. The uniform multiplicity in (6) is at least each of the multiplicities in (5). □

This argument uses no condition on the factorization of H_i. It is also valid when their degrees differ. The apolarity identity is classical; the application here is the comparison between its vanishing conditions and the cutoff for generators of the larger degree d.

For later use set

$$c_{n,d,r}(q) = \sum_{j=0}^r (-1)^j \binom{r}{j} \binom{q - jd + n - 1}{n-1}, \tag{7}$$

where $\binom{u}{v} = 0$ for integers $u < v$ and $v \geq 0$.

Corollary 2.4 (A finite-degree criterion)**.** *Let $q \geq d$ and $0 \leq a < d$. If*

$$h_Z(q; q-d+a+1) > 0 \quad \textit{and} \quad c_{n,d,r}(q) \leq 0,$$

then every ideal (1) with $a_i \leq a$ has $\mathrm{HF}_{S/I}(q) > [t^q] G_{n,d,r} = 0$. When $q < 2d$, the second condition is simply

$$\binom{q+n-1}{n-1} \leq r \binom{q-d+n-1}{n-1}. \tag{8}$$

Equivalently, writing $m = q - d + a + 1$, it says that the affine virtual dimension of the system of degree-q forms with multiplicity $m - a$ at the r points is non-positive.

Proof. Use (6). A non-positive coefficient of the untruncated series forces the corresponding coefficient of its positive truncation to be zero. For $q < 2d$ only the terms $j = 0, 1$ in (7) remain. □

2.3. **The quantitative argument.**

Proof of Theorem 1.1. The inequality on ε is equivalent to

$$\frac{R}{R-1} < \frac{\sigma(1-\varepsilon)}{\sigma-1}.$$

Since $R > \sigma \geq 2$, we may choose

$$\frac{R}{R-1} < \lambda_- < \lambda_+ < \frac{\sigma(1-\varepsilon)}{\sigma-1} \leq 2. \tag{9}$$

Put $\delta = \sigma(1-\varepsilon) - (\sigma-1)\lambda_+ > 0$. For an integer q in the specified interval and $a = \max_i a_i$, let

$$m = q - d + a + 1, \qquad t = \left\lceil \frac{m}{m_0} \right\rceil, \qquad u = q - kt.$$

For all sufficiently large d we have $m \geq 1$, and

$$u \geq q - \sigma m - k = \sigma(d-a-1) - (\sigma-1)q - k \geq \delta d - \sigma - k \geq \frac{\delta d}{2}.$$

Multiplication by B^t is injective on T_u and its image consists of degree-q forms vanishing to order at least m at every P_i. Thus Proposition 2.3 gives

$$\mathrm{HF}_{S/I}(q) \geq \dim(B^t T_u) = \binom{u+N}{N} \geq \frac{\delta^N}{2^N N!} d^N. \tag{10}$$

On the other hand, $d < q < 2d$. Uniformly for $s = q/d$ in the compact interval $[\lambda_-, \lambda_+]$,

$$c_{n,d,r}(q) = \frac{d^N}{N!}\big(s^N - r(s-1)^N\big) + O(d^{N-1}).$$

The expression in parentheses is strictly negative there, since $s > R/(R-1)$. Hence $c_{n,d,r}(q) < 0$ for all sufficiently large d, uniformly in q and in the a_i. This proves (3). The interval contains at least $(\lambda_+ - \lambda_-)d/2$ integers for large d, which gives the last assertion. □

Multiplying B^t by all forms of degree u gives the quantitative conclusion: a positive linear margin for u provides both a growing defect and the absence of a congruence restriction on d.

The geometric hypothesis can be expressed by the Waldschmidt constant

$$\widehat{\alpha}(Z) = \inf_{m \geq 1} \frac{\alpha(I_Z^{(m)})}{m} = \lim_{m \to \infty} \frac{\alpha(I_Z^{(m)})}{m},$$

where $\alpha(J)$ is the least degree of a nonzero form of J. The equality follows from subadditivity. This invariant measures the asymptotic degree needed for uniform vanishing; see [CHMR13] for its relation with Nagata's conjecture.

Corollary 2.5. *If $r > 2^{n-1}$ and $\widehat{\alpha}(Z) < r^{1/(n-1)}$, then the conclusion of Theorem 1.1 holds for some $\varepsilon > 0$.*

Proof. Choose B with degree–multiplicity ratio strictly smaller than $r^{1/(n-1)}$. If this ratio is smaller than 2, multiply B by a form of degree $2m_0 - \deg B$. The resulting ratio is 2, still smaller than $r^{1/(n-1)}$. Apply Theorem 1.1. □

The Waldschmidt constant also gives the asymptotic boundary of the uniform fat-point criterion. This boundary concerns the criterion itself, not every possible source of Hilbert-function defect.

Proposition 2.6. *Suppose* $2 \le \omega = \widehat{\alpha}(Z) < R = r^{1/(n-1)}$, *and put*

$$\eta = \frac{R-\omega}{\omega(R-1)}.$$

If $d_j \to \infty$ *and triples* (d_j, a_j, q_j) *satisfy the hypotheses of Corollary 2.4, then*

$$\limsup_{j\to\infty} a_j/d_j \le \eta.$$

Conversely, for every $0 \le \varepsilon < \eta$ *and all sufficiently large* d*, the criterion applies with* $a = \lfloor \varepsilon d \rfloor$ *and a suitable* q*.*

Proof. For a triple in the first assertion, write $m = q - d + a + 1$. Nonemptiness and the definition of ω give $q \ge \omega m \ge 2m$, so $d \le q < 2d$ and

$$\frac{a}{d} \le 1 - \frac{1}{d} - \frac{\omega-1}{\omega}\frac{q}{d}.$$

Choose a subsequence realizing the limit superior of a/d, and then a further subsequence along which $q/d \to \lambda \in [1,2]$. Dividing (8) by d^{n-1} and taking limits gives $\lambda^{n-1} \le r(\lambda-1)^{n-1}$. Hence $\lambda \ge R/(R-1)$, and substitution in the preceding inequality yields the required bound.

For the converse, choose B whose ratio σ is sufficiently close to ω that $\varepsilon < (R-\sigma)/(\sigma(R-1))$. The definition of the infimum ensures such a choice, with $2 \le \sigma < R$. The construction in the proof of Theorem 1.1 gives a nonzero uniform fat-point system and a negative coefficient for every sufficiently large d. ☐

3. Plane curves and a spatial quadric

The following construction suffices for all the applications. At a point of $\mathbb{P}^{n-1}$, multiplicity at least v imposes at most $\binom{v+n-2}{n-1}$ linear conditions on forms of a fixed degree.

Lemma 3.1. *For each* $i = 1, \dots, r$*, suppose that there is a nonzero degree-*s *form* C_i *with multiplicity at least* u *at* P_i *and at least* v *at all the other points. Then* $B = \prod_i C_i$ *has degree* rs *and multiplicity at least* $u + (r-1)v$ *at every point. Such* C_i *exist whenever*

$$\binom{s+n-1}{n-1} > \binom{u+n-2}{n-1} + (r-1)\binom{v+n-2}{n-1},$$

where the summand for multiplicity zero is understood to be zero.

Proof. The displayed inequality leaves a nonzero solution to the homogeneous linear vanishing conditions. Multiplicities add under multiplication. ☐

For five points of $\mathbb{P}^2$, take a conic through them. For six points, multiply the six conics obtained by omitting one point in turn. For seven points, take cubics with a double point at one specified point and simple points at the other six. There are 10 coefficients and at most $3 + 6 = 9$ conditions. For eight points, take sextics with multiplicities $3, 2, \dots, 2$; there are 28 coefficients and at most $6 + 7 \cdot 3 = 27$ conditions. Multiplying the seven cubics or eight sextics gives the third and fourth plane rows of Table 1. Finally, a quadric in $\mathbb{P}^3$ through nine points exists because the space of quadrics has dimension 10.

Proof of Corollary 1.2. All rows have $k \ge 2m_0$. The strict inequalities needed in Theorem 1.1 are

$$2^2 < 5, \quad 12^2 < 6 \cdot 5^2, \quad 21^2 < 7 \cdot 8^2, \quad 48^2 < 8 \cdot 17^2, \quad 2^3 < 9.$$

The constructions above work for general points, indeed for arbitrary distinct points. For a partition having a part at least $d - a$, extract that linear power and apply Theorem 1.1 with a fixed positive ε and d large enough that $a \le \varepsilon d$. ☐

TABLE 1. Forms supplying the persistence criterion. The last column is the strict upper bound for ε in (2), rounded for orientation; the exact value is given by that formula.

n	r	k	m_0	$\sigma = k/m_0$	ε_*
3	5	2	1	2	0.095492
3	6	12	5	12/5	0.014226
3	7	21	8	21/8	0.004803
3	8	48	17	48/17	0.000949
4	9	2	1	2	0.037073

For general plane points, the conics, cubics, and sextics in this construction are the familiar curves of self-intersection -1 on the corresponding blowups. The proof only requires a nonzero form with the indicated multiplicities. In particular, it does not infer uniqueness or independence of conditions from a dimension count.

3.1. **Explicit finite thresholds.** Although Theorem 1.1 applies in every sufficiently large degree, a single power of B gives convenient explicit examples.

Proposition 3.2. *For any row of Table 1, fix $a \geq 0$ and an integer $t \geq 1$ with $m_0 t > a$. Set*

$$q = kt, \qquad d = (k - m_0)t + a + 1.$$

If

$$\binom{kt + n - 1}{n - 1} \leq r \binom{m_0 t - a + n - 2}{n - 1}, \tag{11}$$

then every ideal generated by forms $L_i^{d-a} H_i$, with $\deg H_i = a$, has positive Hilbert function in degree q, while $[t^q] G_{n,d,r} = 0$.

Proof. The form B^t has degree q and multiplicity at least $m_0 t$ at the points. Moreover $q - d + a + 1 = m_0 t$ and $q < 2d$, since $k \geq 2m_0$. Apply Corollary 2.4. □

For $(n, r) = (3, 5)$ the threshold has a particularly simple form.

Corollary 3.3. *Let $a \geq 0$ and*

$$d \geq a + 1 + \left\lceil \frac{10a + 1 + \sqrt{80a^2 + 40a + 9}}{2} \right\rceil. \tag{12}$$

Then any five forms $F_i = L_i^{d-a} H_i \in \mathbb{C}[x, y, z]_d$, with $\deg H_i = a$, satisfy

$$\mathrm{HF}_{S/(F_1,\ldots,F_5)}(2d - 2a - 2) \geq 1, \qquad [t^{2d-2a-2}] G_{3,d,5} = 0.$$

Proof. Put $t = d - a - 1$. Twice the difference between the two sides of (11) is $-t^2 + (10a + 1)t + 2 - 5a(a - 1)$, which is non-positive under (12). The witness is C^t for a conic C through the five coefficient points. Such a conic exists even if the points are not in general position. □

For five general plane points, $\widehat{\alpha}(Z) = 2$. The conic gives the upper bound. If a curve of degree $q < 2m$ vanished to order m at the five points, Bézout would force the conic to be a component. Repeatedly removing it would give a curve of negative degree, a contradiction. Consequently Proposition 2.6 shows that

$$\eta = \frac{3 - \sqrt{5}}{8}$$

is the exact asymptotic boundary for this uniform fat-point criterion. The explicit bound (12) has the same limiting ratio a/d.

For $a = 1$ the first values furnished by the chosen forms B are as follows. These are thresholds for this construction, not claims of minimality for Problem F, except in the first row by Theorem 1.3.

n	r	t	d	q
3	5	12	14	24
3	6	12	86	144
3	7	18	236	378
3	8	36	1118	1728
4	9	26	28	52

Each row follows by substituting in (11); the accompanying certificate also verifies the integer thresholds. In the spatial example, Q^{26} survives in degree 52, and

$$\binom{55}{3} - 9\binom{27}{3} = 26235 - 26325 = -90.$$

3.2. **The limitation imposed by Nagata's inequality.** The same degree–multiplicity comparison explains why these particular constructions stop at eight plane points. It does not classify all possible failures of Problem F.

Proposition 3.4. *For nine general points of $\mathbb{P}^2$, the hypotheses of Corollary 2.4 cannot hold for any $a \geq 0$. For $r \geq 10$ very general points, the same is true for $a \geq 1$ if $\alpha(I_Z^{(m)}) \geq m\sqrt{r}$ for every $m \geq 1$. In particular, the latter conclusion follows from Nagata's conjecture, and is unconditional when $r \geq 10$ is a perfect square.*

Proof. For nine general points let C be their smooth cubic. A degree-q curve with multiplicity at least m at all nine points has $q \geq 3m$. Indeed, if $q < 3m$, Bézout forces C to be a component; removing it reduces (q, m) to $(q-3, m-1)$ and leads by induction to a negative degree. For the criterion put $m = q - d + a + 1$. The inequality $q \geq 3m$ implies $q < 2d$, and

$$\binom{q+2}{2} \geq \binom{3m+2}{2} = 9\binom{m+1}{2} + 1 > 9\binom{m-a+1}{2}.$$

Thus (8) fails.

Under the stated inequality for $r \geq 10$, nonemptiness gives $q \geq m\sqrt{r} > 2m$, again implying $q < 2d$. If $a \geq 1$, then

$$\binom{q+2}{2} > \frac{q^2}{2} \geq \frac{rm^2}{2} > r\binom{m-a+1}{2}.$$

The last assertion uses Nagata's theorem for square numbers of points [Nag59]; see also [CHMR13]. □

The proposition excludes only the comparison with a nonempty uniform fat-point system in a degree where the predicted series has vanished. It says nothing about other sources of defect, or about an excess in a degree where the predicted Hilbert function is positive.

4. The finite boundary and further questions

We separate the geometric part of Theorem 1.3 from its finite verification. This also isolates precisely what the computation proves.

Proof of Theorem 1.3. If $F_i = L_i^{13} M_i$, choose a nonzero conic C through the five coefficient points. Its twelfth power has degree 24 and multiplicity at least 12 at each point. Proposition 2.3 puts C^{12} in $(S/I)^{\vee}_{24}$. Meanwhile

$$c_{3,14,5}(24) = \binom{26}{2} - 5\binom{12}{2} = 325 - 330 = -5.$$

This proves the failure for every choice of the factors.

Appendix A gives a fixed integer specialization for which, modulo 32003, the maps (4) have ranks

$$\operatorname{rank}\Phi_{23} = 275, \qquad \operatorname{rank}\Phi_{24} = 324, \qquad \operatorname{rank}\Phi_{25} = 351.$$

Nonzero minors lift to characteristic zero. The first rank excludes any syzygy in degree at most 23; the last makes the quotient zero in every degree at least 25. The middle rank, together with the conic lower bound, gives dimension exactly one in degree 24. Semicontinuity proves the claimed generic Hilbert series.

The same certificate proves $G_{3,d,5}$ for every two-part partition $(d-a, a)$ with $2 \le d \le 13$ and $1 \le a \le \lfloor d/2 \rfloor$, and for $d = 14$, $2 \le a \le 7$. Proposition 2.1 then gives every non-pure partition in degrees at most 13. A non-pure partition of 14 other than $(13, 1)$ has a subset of parts with sum between 2 and 12, so it is covered by one of the positive two-part cases as well. □

Corollary 3.3 gives a failure for $(d-1, 1)$ for every $d \ge 14$. By Theorem 1.1, the excess eventually occupies an interval of degrees and becomes arbitrarily large.

There are two natural questions beyond these lower bounds. The first asks whether the elementary comparison already accounts for all the defect in the simplest family.

Question 4.1. For five general forms $L_i^{d-1} M_i \in \mathbb{C}[x, y, z]_d$ and $q \ge d$, is

$$\mathrm{HF}_{S/(L_1^{d-1}M_1,\ldots,L_5^{d-1}M_5)}(q) = \max\{[t^q]G_{3,d,5},\ h_Z(q; q-d+2)\}?$$

Both entries on the right are lower bounds: the first follows from Anick's theorem and semicontinuity, and the second from Proposition 2.3. Theorem 1.3 determines the first exceptional Hilbert series but does not resolve this question in arbitrary degree.

The second question concerns the size of the residual factor. For five general ternary generators, $(3-\sqrt{5})/8$ is the asymptotic boundary of the uniform fat-point criterion. Does the actual Hilbert-function defect persist for larger proportions a/d? Determining this would distinguish the obstruction studied here from effects of the complete factorization.

Appendix A. The exact finite certificate

We give the data and the reduction that make the finite verification reproducible. Its only role is the sharpness statement and the exact Hilbert series in Theorem 1.3.

For $d \ge 2$ let

$$Q_d = \min\left\{ q \ge d : \binom{q+2}{2} - 5\binom{q-d+2}{2} \le 0 \right\}.$$

We have $Q_d < 2d$, since the difference at $q = 2d-1$ is $-d(d+3)/2$. Furthermore the ratio $\binom{q+2}{2}/\binom{q-d+2}{2}$ decreases with $q \ge d$. Thus Q_d is the first zero degree of $G_{3,d,5}$.

Lemma A.1. *For five forms of degree d, injectivity of Φ_{Q_d-1} and surjectivity of Φ_{Q_d} imply Hilbert series $G_{3,d,5}$.*

Proof. A nonzero homogeneous syzygy in an earlier degree can be multiplied by a nonzero monomial to produce one in degree $Q_d - 1$. Hence there are no syzygies up to that degree, and the Hilbert function there is the difference of the source and target dimensions. If $I_{Q_d} = S_{Q_d}$, then multiplication by variables gives $I_q = S_q$ for every $q \geq Q_d$. $\square$

Use the following coefficient vectors over $\mathbb{Z}$, and form $F_i = L_i^{d-a} M_i^a$:

L_i	M_i
$(0, 7, 2)$	$(9, 8, 5)$
$(2, 6, 4)$	$(2, 10, 7)$
$(0, 8, 0)$	$(2, 0, 6)$
$(8, 1, 7)$	$(8, 7, 0)$
$(7, 9, 6)$	$(4, 6, 9)$

Rows of Φ_q are indexed by degree-q monomials; columns are the coefficient vectors of all AF_i for degree-$(q-d)$ monomials A. The file `verify_manuscript.py` constructs these matrices and performs Gaussian elimination modulo the prime 32003. No random data or floating-point rank decisions are used.

For every $2 \leq d \leq 13$ and $1 \leq a \leq \lfloor d/2 \rfloor$ the two boundary ranks are maximal. The same holds for $d = 14$, $2 \leq a \leq 7$. The cutoffs are

d	2	3	4	5	6	7	8	9	10	11	12	13	14
Q_d	3	4	6	8	10	12	13	15	17	19	21	23	24.

There are 96 maximal-rank checks for these positive cases. The three additional checks for $(d, a) = (14, 1)$ are

q	23	24	25
number of rows	300	325	351
number of columns	275	330	390
rank modulo 32003	275	324	351.

The file `verification/rank_checks.csv` records all 99 ranks and matrix sizes. The verifier runs with the Python standard library; optional NumPy acceleration gives the same results. Both implementations have been checked on all 99 matrices. Use `-pure-python` to disable acceleration.

A nonzero minor modulo 32003 remains nonzero over $\mathbb{C}$, so the maximal-rank checks prove generic characteristic-zero statements. For the exceptional rank 324, the conic argument supplies the matching upper bound: reduction modulo a prime alone would not certify a characteristic-zero rank defect.

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